HW1 Quantum Foundation yuxiang@uchicago.edu

Problem 1

Consider a time–dependent Hamiltonian given by,

$$H(t) = \frac{\omega_0}{2}\sigma_z + \Omega\cos(\omega t + \phi)\sigma_x \tag{1.1}$$

Suppose the driving frequency ω is large compared to the driving strength Ω and the detuning $|\omega - \omega_0|$, i.e. $\omega \gg \Omega$, $\omega \gg |\omega - \omega_0|$. Apply the rotating wave approximation to obtain the effective Hamiltonian

$$H' = \frac{\omega_0 - \omega}{2}\sigma_z + \frac{\Omega}{2}\left(\sigma_x\cos(\phi) + \sigma_y\sin(\phi)\right) \tag{1.2}$$

[Hint: (i) You may want to write the interaction picture Hamiltonian in terms of $H_0 = \frac{\omega}{2}\sigma_z$. (ii) Consider coarse graining, i. e. $\langle e^{2i\omega t} + 1\rangle \to 1$ for time average over a time duration long compared to $1/\omega$.]

In a general case, when we apply unitary transformation for a state $|\psi(t)\rangle$, we get

$$|\phi(t)\rangle = \tilde{U}|\psi(t)\rangle.$$

$$\frac{d|\phi(t)\rangle}{dt} = \frac{d\tilde{U}}{dt}|\psi(t)\rangle + \tilde{U}\frac{d|\psi(t)\rangle}{dt}$$

$$= \frac{d\tilde{U}}{dt}\tilde{U}^\dagger|\phi(t)\rangle + \tilde{U}\frac{H}{i\hbar}\tilde{U}^\dagger|\phi(t)\rangle$$

$$i\hbar\frac{d|\phi(t)\rangle}{dt} = (\underbrace{\tilde{U}H\tilde{U}^\dagger + i\hbar\frac{d\tilde{U}}{dt}\tilde{U}^\dagger}_{H_{eff}})|\phi(t)\rangle.$$

with $\tilde{U} = e^{iH_0t/\hbar} = e^{\frac{i\omega t}{2}\sigma_z}$. we have

$$H_{eff} = \tilde{U}H(t)\tilde{U}^\dagger + i\hbar\frac{d\tilde{U}}{dt}\tilde{U}^\dagger$$

$$= e^{\frac{i\omega t}{2}\sigma_z}H(t)e^{-\frac{i\omega t}{2}\sigma_z} + i\hbar\cdot\frac{i\omega}{2}\sigma_z$$

$$H_{eff} = \begin{pmatrix} e^{\frac{i\omega t}{2}} & 0 \\ 0 & e^{\frac{-i\omega t}{2}} \end{pmatrix} \begin{pmatrix} \frac{\omega_0}{2} & \Omega\cos(\omega t+\phi) \\ \Omega\cos(\omega t+\phi) & -\frac{\omega_0}{2} \end{pmatrix} \begin{pmatrix} e^{-\frac{i\omega t}{2}} & 0 \\ 0 & e^{\frac{i\omega t}{2}} \end{pmatrix} - \frac{\hbar\omega}{2}\sigma_z$$

$$= \begin{pmatrix} e^{\frac{i\omega t}{2}} & 0 \\ 0 & e^{-\frac{i\omega t}{2}} \end{pmatrix} \begin{pmatrix} \frac{\omega_0}{2} e^{-\frac{i\omega t}{2}} & \Omega\cos(\omega t+\phi)\, e^{\frac{i\omega t}{2}} \\ \Omega\cos(\omega t+\phi)\, e^{-\frac{i\omega t}{2}} & -\frac{\omega_0}{2} e^{\frac{i\omega t}{2}} \end{pmatrix} - \frac{\hbar\omega}{2}\sigma_z$$

$$= \begin{pmatrix} \frac{\omega_0}{2} & \Omega\cos(\omega t+\phi)\, e^{i\omega t} \\ \Omega\cos(\omega t+\phi)\, e^{-i\omega t} & -\frac{\omega_0}{2} \end{pmatrix} - \frac{\hbar\omega}{2}\sigma_z \quad \xrightarrow{\hbar = 1}$$

$$= \frac{1}{2}\begin{pmatrix} \omega_0-\omega & \Omega\left(e^{2i\omega t+i\phi} + e^{-i\phi}\right) \\ \Omega\left(e^{i\phi} + e^{-2i\omega t-i\phi}\right) & -(\omega_0-\omega) \end{pmatrix} \quad \xrightarrow{e^{2i\omega t}+1 \,\to\, 1}$$

$$= \frac{1}{2}\begin{pmatrix} \omega_0-\omega & \Omega e^{-i\phi} \\ \Omega e^{i\phi} & -(\omega_0-\omega) \end{pmatrix} = \frac{\omega_0-\omega}{2}\sigma_z + \frac{1}{2}\begin{pmatrix} 0 & \Omega\cos\phi - i\Omega\sin\phi \\ \Omega\cos\phi + i\Omega\sin\phi & 0 \end{pmatrix}$$

$$= \frac{\omega_0-\omega}{2}\sigma_z + \frac{1}{2}\Omega\cos\phi\,\sigma_x + \frac{\Omega}{2}\sin\phi\,\sigma_y$$

$$= \frac{\omega_0-\omega}{2}\sigma_z + \frac{\Omega}{2}\left(\sigma_x\cos\phi + \sigma_y\sin\phi\right) = H'.$$

Problem 2

Consider an arbitrary (and unknown) pure state of two qubits given by

$$|\psi\rangle_{AB} = \alpha|00\rangle + \beta|11\rangle + \gamma|10\rangle + \delta|01\rangle \tag{2.1}$$

where α, β, γ and δ are complex numbers satisfying $|\alpha|^2+|\beta|^2+|\gamma|^2+|\delta|^2 = 1$. Answer the following questions:

(a) Suppose we trace out subsystem B, what is ρ_A?

(b) Calculate the von-Neumann entropy between the subsystems, which is given by $S_A = -\operatorname{tr}[\rho_A \log \rho_A]$. Is this the same as $S_{\mathrm{B}} = -\operatorname{tr}[\rho_B \log \rho_B]$? (Remark: Remember to diagonalize the reduced density matrices before the calculation.)

(c) Consider the special case when $\gamma = \delta = 0$ and $\alpha = \beta$. What are S_A and S_B for this case?

(a) $\rho_A = \mathrm{Tr}_B\left(|\psi\rangle_{AB}\langle\psi|_{AB}\right)$

$$\begin{pmatrix}\alpha\\ \delta\\ \gamma\\ \beta\end{pmatrix}\left(\alpha^*\ \ \delta^*\ \ \gamma^*\ \ \beta^*\right)$$

$$= \begin{pmatrix} |\alpha|^2+|\delta|^2 & \alpha\gamma^*+\beta^*\delta \\ \alpha^*\gamma+\beta\delta^* & |\gamma|^2+|\beta|^2 \end{pmatrix}$$

(b) $S = -\mathrm{Tr}\left[\rho\log\rho\right] = -\sum_i \lambda_i \log\lambda_i$.

For ρ_A, $|\rho_A - \lambda I| =$

$$\begin{vmatrix} |\alpha|^2+|\delta|^2-\lambda & \alpha\gamma^*+\beta^*\delta \\ \alpha^*\gamma+\beta\delta^* & |\gamma|^2+|\beta|^2-\lambda \end{vmatrix} = \lambda^2 - \underbrace{\left(|\alpha|^2+|\beta|^2+|\gamma|^2+|\delta|^2\right)}_{=1\ ?}\lambda$$

$$+ |\alpha|^2|\gamma|^2 + |\beta|^2|\delta|^2 + |\delta|^2|\gamma|^2 + |\alpha|^2|\beta|^2$$

$$- |\alpha|^2|\gamma|^2 - |\beta|^2|\delta|^2 - \alpha\beta\gamma^*\delta^* - \alpha^*\beta^*\gamma\delta$$

$$= \lambda^2 - \left(|\alpha|^2+|\beta|^2+|\gamma|^2+|\delta|^2\right)\lambda + \left(\alpha^*\beta^* - \gamma^*\delta^*\right)\left(\alpha\beta-\gamma\delta\right)$$

$|\rho_A - \lambda I| = 0$, we have

$$\Delta = \left(|\alpha|^2+|\beta|^2+|\gamma|^2+|\delta|^2\right)^2 - 4\left(|\alpha|^2|\beta|^2+|\gamma|^2|\delta|^2-\alpha\beta\gamma^*\delta^*-\alpha^*\beta^*\gamma\delta\right)$$

$$= \left(|\alpha|^2+|\beta|^2\right)^2+\left(|\gamma|^2+|\delta|^2\right)^2+2\left(|\alpha|^2+|\beta|^2\right)\left(|\gamma|^2+|\delta|^2\right)$$

$$-4|\alpha|^2|\beta|^2-4|\gamma|^2|\delta|^2+4\alpha\beta\gamma^*\delta^*+4\alpha^*\beta^*\gamma\delta$$

$$\lambda_1 = \frac{1}{2}\left(\sqrt{|\alpha|^2+|\beta|^2+|\gamma|^2+|\delta|^2}+\sqrt{\Delta}\right).$$

$$\lambda_2 = \frac{1}{2}\left(\sqrt{|\alpha|^2+|\beta|^2+|\gamma|^2+|\delta|^2}-\sqrt{\Delta}\right).$$

$$S_A = -\lambda_1\log\lambda_1-\lambda_2\log\lambda_2.$$

For ρ_B. $\rho_B = Tr_A(\rho_{AB}) = \begin{pmatrix} |\alpha|^2+|\gamma|^2 & \alpha\delta^*+\gamma\beta^* \\ \alpha^*\delta+\beta\gamma^* & |\beta|^2+|\delta|^2 \end{pmatrix}.$

$|\rho_B - I\lambda| = 0$. we have

$$\lambda^2-\left(|\alpha|^2+|\beta|^2+|\gamma|^2+|\delta|^2\right)\lambda+\left(|\alpha|^2+|\gamma|^2\right)\left(|\beta|^2+|\delta|^2\right)$$

$$-\left(|\alpha|^2|\delta|^2+|\beta|^2|\gamma|^2+\alpha^*\beta^*\gamma\delta+\alpha\beta\gamma^*\delta^*\right)=0$$

$$\Delta' = \left(|\alpha|^2+|\beta|^2+|\gamma|^2+|\delta|^2\right)^2-4\left(\alpha^*\beta^*-\gamma^*\delta^*\right)\left(\alpha\beta-\gamma\delta\right)=\Delta.$$

$$\lambda_1' = \lambda_1. \quad \lambda_2' = \lambda_2.$$

$$S_B = -\lambda_1' \log \lambda_1' - \lambda_2' \log \lambda_2' = S_A.$$

(c) $$\rho_A = \begin{pmatrix} |\alpha|^2 & 0 \\ 0 & |\alpha|^2 \end{pmatrix}.$$

$$(\lambda - |\alpha|^2)^2 = 0. \qquad \lambda = |\alpha|^2.$$

$$S_A = -0 - |\alpha|^2 \log |\alpha|^2 = -2|\alpha|^2 \log|\alpha|.$$

Problem 3

Parameterize the density matrix of a single qubit as:

$$\rho = \frac{1}{2}(1 + \vec{\lambda} \cdot \vec{\sigma}) \tag{3.1}$$

(a) Describe what happens to $\vec{\lambda}$ under the action of phase-damping channel.

(b) Describe what happens to $\vec{\lambda}$ under the action of amplitude damping channel defined by the Kraus operators,

$$M_0 = \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{1-p} \end{bmatrix}, M_1 = \begin{bmatrix} 0 & \sqrt{p} \\ 0 & 0 \end{bmatrix} \tag{3.2}$$

(c) The same for a "two–Pauli Channel" given by,

$$M_0 = \sqrt{1-p}I, \quad M_1 = \sqrt{\frac{p}{2}}\sigma_x, M_2 = \sqrt{\frac{p}{2}}\sigma_z \tag{3.3}$$

(a) Under phase damping channel

$$\rho \rightarrow \rho' = K_0 \rho K_0^{\dagger} + K_1 \rho K_1^{\dagger}.$$

with $K_0 = \begin{pmatrix} 1 & \\ & \sqrt{1-p} \end{pmatrix}$. $K_1 = \begin{pmatrix} & \\ & \sqrt{p} \end{pmatrix}$.

$$\rho = \frac{1}{2}\begin{pmatrix} 1+\lambda_z & \lambda_x - i\lambda_y \\ \lambda_x + i\lambda_y & 1-\lambda_z \end{pmatrix}.$$

$$\rho' = \begin{pmatrix} 1 & \\ & \sqrt{1-p} \end{pmatrix} \frac{1}{2}\begin{pmatrix} 1+\lambda_z & \lambda_x - i\lambda_y \\ \lambda_x + i\lambda_y & 1-\lambda_z \end{pmatrix}\begin{pmatrix} 1 & \\ & \sqrt{1-p} \end{pmatrix} + \begin{pmatrix} 0 & \\ & \sqrt{p} \end{pmatrix}\frac{1}{2}\begin{pmatrix} 1+\lambda_z & \lambda_x - i\lambda_y \\ \lambda_x + i\lambda_y & 1-\lambda_z \end{pmatrix}\begin{pmatrix} 0 & \\ & \sqrt{p} \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 1+\lambda_z & \sqrt{1-p}(\lambda_x - i\lambda_y) \\ \sqrt{1-p}(\lambda_x + i\lambda_y) & (1-p)(1-\lambda_z) \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 0 & \\ & p(1-\lambda_z) \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 1+\lambda_z & \sqrt{1-p}\,(\lambda_x - i\lambda_y) \\ \sqrt{1-p}\,(\lambda_x + i\lambda_y) & 1-\lambda_z \end{pmatrix} \Rightarrow \vec{\lambda}' = \left(\sqrt{1-p}\,\lambda_x,\ \sqrt{1-p}\,\lambda_y,\ \lambda_z\right).$$

The x.y component of λ become $\sqrt{1-p}$ times, while z component stays same.

(b) $\rho' = M_0 \rho M_0^\dagger + M_1 \rho M_1^\dagger$

$$= \begin{pmatrix} 1 & \\ & \sqrt{1-p} \end{pmatrix} \frac{1}{2}\begin{pmatrix} 1+\lambda_z & \lambda_x - i\lambda_y \\ \lambda_x + i\lambda_y & 1-\lambda_z \end{pmatrix}\begin{pmatrix} 1 & \\ & \sqrt{1-p} \end{pmatrix} + \begin{pmatrix} & \sqrt{p} \\ 0 & \end{pmatrix} \frac{1}{2}\begin{pmatrix} \frac{1}{2}+\lambda_z & \lambda_x - i\lambda_y \\ \lambda_x + i\lambda_y & \frac{1}{2}-\lambda_z \end{pmatrix}\begin{pmatrix} & 0 \\ \sqrt{p} & \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 1+\lambda_z & \sqrt{1-p}\,(\lambda_x - i\lambda_y) \\ \sqrt{1-p}\,(\lambda_x + i\lambda_y) & (1-p)(1-\lambda_z) \end{pmatrix} + \frac{1}{2}\begin{pmatrix} p(1-\lambda_z) & \\ & 0 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} (1+p)+(1-p)\lambda_z & \sqrt{1-p}\,(\lambda_x - i\lambda_y) \\ \sqrt{1-p}\,(\lambda_x + i\lambda_y) & (1-p)(1-\lambda_z) \end{pmatrix} \Rightarrow \begin{aligned} \lambda_x' &= \sqrt{1-p}\,\lambda_x. \\ \lambda_y' &= \sqrt{1-p}\,\lambda_y. \\ \lambda_z' &= p + (1-p)\lambda_z. \end{aligned}$$

So the x and y component of $\vec{\lambda}$ become $\sqrt{1-p}$ times.

z component becomes $(1-p)$ times plus p.

(c) $\rho' = M_0 \rho M_0^\dagger + M_1 \rho M_1^\dagger + M_2 \rho M_2^\dagger$

$$= (1-p)\rho + \frac{p}{2}\sigma_x \rho \sigma_x^\dagger + \frac{p}{2}\sigma_z \rho \sigma_z^\dagger$$

$$= \frac{1}{2}\left[\begin{pmatrix} (1-p)(1+\lambda_z) & (1-p)(\lambda_x - i\lambda_y) \\ (1-p)(\lambda_x + i\lambda_y) & (1-p)(1-\lambda_z) \end{pmatrix} + \frac{p}{2}\begin{pmatrix} 1-\lambda_z & \lambda_x + i\lambda_y \\ \lambda_x - i\lambda_y & 1+\lambda_z \end{pmatrix} + \frac{p}{2}\begin{pmatrix} 1+\lambda_z & -(\lambda_x - i\lambda_y) \\ -(\lambda_x + i\lambda_y) & 1-\lambda_z \end{pmatrix}\right]$$

$$= \frac{1}{2}\begin{pmatrix} 1+(1-p)\lambda_z & (1-p)\lambda_x + (1-2p)i\lambda_y \\ (1-p)\lambda_x + (1-2p)\lambda_y & 1-(1-p)\lambda_z \end{pmatrix} \Rightarrow \begin{aligned} P_x &= (1-p)\lambda_x . \\ P_y &= (1-2p)\lambda_y . \\ P_z &= (1-p)\lambda_z . \end{aligned}$$

So the x、z component of $\vec{\lambda}$ become $(1-p)$ times,

and y component become $(1-2p)$ times.

Problem 4 [Bonus]

Both Hamiltonian and density operators are Hermitian. Can we design a protocol to let a system evolve under some Hamiltonian that is proportional to another density matrix? The answer is yes. Here is a clever trick to achieve such a task.

(a) For two systems A and B, with the same dimension d, write down the unitary of the SWAP operator between the two systems, S_{AB}. Show that S_{AB} is a Hermitian operator.

(b) Prove that the exponential of S_{AB} has the following simple form:

$$e^{-iS_{AB}t} = I\cos t - iS_{AB}\sin t. \tag{4.1}$$

[Hint: Check S_{AB}^2.]

(c) For a small evolution time Δt, show that the quantum state of system A with density matrix σ will effectively evolve under a Hamiltonian equal to the density matrix ρ of system B up to the first order of Δt, after a unitary evolution with Hamiltonian $H = S_{AB}$ followed by tracing out system B:

$$\mathrm{tr}_B\left[e^{-iS_{AB}\Delta t}(\sigma\otimes\rho)e^{iS_{AB}\Delta t}\right] = e^{-i\rho\Delta t}\sigma e^{i\rho\Delta t} + O(\Delta t^2) \tag{4.2}$$

[Note: The above protocol was proposed for quantum principal component analysis in Nature Physics 10, 631 (2014).]

(a) The general form of operator in $\mathcal{H}^A\otimes\mathcal{H}^B$:

$$X = \sum_{ij\mu\nu} X_{i\mu,j\nu}\,|i\mu\rangle\langle j\nu|.$$

$$\langle\varphi|S_{AB}|\psi\rangle = \langle\varphi|\sum_{ij\mu\nu} S_{i\mu,j\nu}\,|i\mu\rangle\langle j\nu|\psi\rangle$$

$$= \sum_{ij\mu\nu} S_{i\mu,j\nu}\langle\varphi|i\mu\rangle\langle j\nu|\psi\rangle$$

$$|\psi\rangle = \sum_{j\nu}\langle j\nu|\psi\rangle|j\nu\rangle$$

$$|\varphi\rangle = \sum_{i\mu}\langle i\mu|\varphi\rangle|i\mu\rangle.$$

$$S_{AB}|\varphi\rangle = \sum_{ij\mu\nu} S_{i\mu,j\nu}\langle j\nu|\varphi\rangle|i\mu\rangle. \quad ①$$

$$S_{AB}^{\dagger}|\varphi\rangle = \sum_{ij\mu\nu} S^{*}_{i\mu,j\nu}\, |j\nu\rangle\langle i\mu|\varphi\rangle .$$

Redundant

$$\Rightarrow \quad \langle S_{AB}^{\dagger}\varphi| = \sum_{ij\mu\nu} S_{i\mu,j\nu}\langle\varphi|i\mu\rangle\langle j\nu| .$$

$$\langle S_{AB}^{\dagger}\varphi|\psi\rangle = \sum_{ij\mu\nu} S_{i\mu,j\nu}\langle\varphi|i\mu\rangle \langle j\nu| \cdot \sum_{j\nu}\langle j\nu|\psi\rangle|j\nu\rangle$$

$$= \sum_{ij\mu\nu} S_{i\mu,j\nu}\langle\varphi|i\mu\rangle\langle j\nu|\psi\rangle . \quad \checkmark$$

From ①,

$$\langle S_{AB}^{\dagger}\varphi| = \sum_{ij\mu\nu} S^{*}_{i\mu,j\nu}\langle\varphi|j\nu\rangle\langle i\mu| .$$

$$\langle S_{AB}^{\dagger}\varphi|\psi\rangle = \sum_{ij\mu\nu} S^{*}_{i\mu,j\nu}\langle\varphi|j\nu\rangle\langle i\mu| \sum_{j\nu}\langle j\nu|\psi\rangle|j\nu\rangle$$

$$= \sum_{ij\mu\nu} S^{*}_{i\mu,j\nu}\langle\varphi|j\nu\rangle\langle i\mu|\psi\rangle .$$

Also, for ① we know SWAP operation has:

$$S_{AB}|\varphi\rangle = \sum_{ij\mu\nu} S_{i\mu,j\nu}\langle j\nu|\varphi\rangle|i\mu\rangle = \sum_{i\mu}\langle i\mu|\varphi\rangle|\mu i\rangle = \sum_{i\mu}\langle \mu i|\varphi\rangle|i\mu\rangle$$

?

(b) Since S_{AB} is the SWAP operation. SWAP two times will give the same state. $S_{AB}^2 = I$.

The definition of the function of operator

$$f(\hat{A}) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} \hat{A}^n . \qquad f^{(n)}(0) = \left.\frac{d^n f(\hat{A})}{d\hat{A}^n}\right|_{\hat{A}=0} .$$

$$e^{-iS_{AB}t} = \sum_{n=0}^{\infty} \frac{1}{n!} (-iS_{AB}t)^n$$

$$= \sum_{\text{even}} \frac{(it)^n}{n!} S_{AB}^n + \sum_{\text{odd}} \frac{(it)^n}{n!} (-1) S_{AB}^n$$

$$= \sum_k \frac{(-1)^k t^{2k}}{(2k)!} I - i \sum_k \frac{(-1)^k t^{2k+1}}{(2k+1)!} S_{AB}$$

$$= I \cos t - i \sin t \cdot \hat{S}_{AB} .$$

(c) $\mathrm{Tr}_B \left[e^{-iS_{AB}\Delta t} (\sigma \otimes \rho) e^{iS_{AB}t} \right]$

$$= \mathrm{Tr}_B \left[(I \cos t - i \sin t \hat{S}_{AB}) (\sigma \otimes \rho) (I \cos t + i \sin t \hat{S}_{AB}) \right]$$

$$= \mathrm{Tr}_B \left[\cos^2 t (\sigma \otimes \rho) + \sin^2 t (\rho \otimes \sigma) - i \sin t \cos t [\hat{S}_{AB}, \sigma \otimes \rho] \right]$$

$$= (\cos^2 t) \rho + (\sin^2 t) \sigma - i \sin t \cos t \, \mathrm{Tr}_B [\hat{S}_{AB}, \sigma \otimes \rho] \qquad ?$$

HW 2 yuxiang@uchicago.edu

Problem 1 Kraus representation & Choi matrix

How to compute the Kraus operators for a complete positive and trace preserving (CPTP) map? Suppose $\mathcal{E}_A$ is a superoperator acting a d-dimensional system A. Here is a procedure to obtain a minimum set of Kraus operators for superoperator $\mathcal{E}_A$, where $\mathcal{E}_A(|m\rangle\langle k|) = \sum_{lj} C_{lmjk} |l\rangle\langle j|$.

(a) First, demonstrate that the elements C_{lmjk} can be retrieved as

$$\sum_{lmjk} C_{lmjk} |l\rangle\langle j| \otimes |m\rangle\langle k| = \mathcal{E}_A \otimes \mathcal{I}_B (|\Psi\rangle\langle\Psi|_{AB}), \tag{1.1}$$

with unnormalized maximal entangled state between systems A and B, $|\Psi\rangle_{AB} = \sum_{j=0}^{d-1} |j\rangle_A |j\rangle_B$. This matrix C is the Choi matrix of $\mathcal{E}_A$, and contains all information of the superoperator.

(b) The Choi matrix is a $d^2 \times d^2$ Hermitian matrix at most d^2 orthonormal eigenvectors with non-zero eigenvalues, $v^\mu = \sum_{j,k=0}^{d-1} v^\mu_{j,k} |j\rangle_A |k\rangle_B$ with eigenvalue λ^μ, for $\mu = 0, 1, \cdots, d^2-1$. If the superoperator is complete positive, all the eigenvalues will be positive. We can thus define a set of $d \times d$ matrices $(M_\mu)_{j,k} = (\lambda^\mu)^{1/2} v^\mu_{j,k}$. Prove that the set $\{M_\mu\}_{\mu=0,\cdots,d^2-1}$ provides a Kraus representation of $\mathcal{E}_A$:

$$\mathcal{E}_A(\rho_A) = \sum_\mu M_\mu \rho_A M_\mu^\dagger, \tag{1.2}$$

with $\sum_\mu M_\mu^\dagger M_\mu = I$ if $\mathcal{E}_A$ is trace-preserving.

(c) Show that the Kraus representation is not unique, because we can create other equivalent Kraus representations by unitary (more generally isometry) transformation

$$N_\alpha = \sum_\mu V_{\alpha\mu} M_\mu \tag{1.3}$$

where V is the isometry map, such that $V^\dagger V = I$.

(d) Prove that the rank of Choi matrix, $rank(C)$, is the the Kraus rank, which is the minimum number of non-zero Kraus operators needed for the Kraus representation.

(a) $\left(\mathcal{E}_A \otimes I_B\right)\left(|\Psi\rangle_{AB}\langle\Psi|_{AB}\right) = \left(\mathcal{E}_A \otimes I_B\right) \sum_{jj'} |i\rangle_A\langle i'|_A \, |i\rangle_B\langle i'|_B$

$$= \sum_{lj} C_{liji'} |l\rangle\langle j| \otimes \sum_{jj'} |i\rangle_B\langle i'|_B$$

$j \to m$

$j' \to k$

$$= \sum_{ljmk} C_{lmjk} |l\rangle\langle j| \otimes |m\rangle\langle k|.$$

(b) ρ_A could be written as mixed state ensemble $\{p_i, \varphi_i\}$.

$$\rho_A = \sum_i p_i |\varphi_i\rangle\langle\varphi_i| . \qquad |\varphi_i\rangle = \sum_n \alpha_n |n\rangle .$$

$$\mathcal{E}(\rho_A) = \sum_i p_i \, \mathcal{E}(|\varphi_i\rangle\langle\varphi_i|) .$$

$$\sum_\mu M_\mu \rho_A M_\mu^\dagger = \sum_\mu \sqrt{\lambda^\mu}\, U^\mu |\varphi_i\rangle\langle\varphi_i| \sqrt{\lambda^\mu}\, U^{\mu\dagger} .$$

$$= \sum_\mu \lambda^\mu \sum_{jk} U^\mu_{jk} |j\rangle\langle k|\, |\varphi_i\rangle\langle\varphi_i| \sum_{j'k'} U^{\mu\dagger}_{j'k'} |k'\rangle\langle j'|$$

$$= \sum_\mu \lambda^\mu \sum_{jj'kk'} U^\mu_{jk} U^{\mu\dagger}_{j'k'} \langle\varphi_i|k'\rangle\langle k|\varphi_i\rangle |j\rangle\langle j'|$$

Also $C = \sum_\mu \lambda^\mu U^\mu U^{\mu\dagger} = \sum_\mu \sum_{jj'kk'} \lambda^\mu U^\mu_{jk} U^{\mu\dagger}_{j'k'} |j\rangle|k\rangle\langle j'|\langle k'|$

$$= \sum_\mu \sum_{jj'kk'} C_{jkj'k'} \, |j\rangle\langle j'| \otimes |k\rangle\langle k'|$$

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(b) We know that

$$\sum_{lmjk} C_{lmjk} \, |l\rangle\langle j| \otimes |m\rangle\langle k| = \sum_{\mu} \lambda^{\mu} v_{\mu} v_{\mu}^{\dagger}$$

$$= \sum_{\mu} \lambda^{\mu} \sum_{jj'kk'} v^{\mu}_{jk} v^{\mu\dagger}_{j'k'} |j\rangle|k\rangle\langle j'|\langle k'|$$

$$= (\mathcal{E}_A \otimes I_B)(|\Psi\rangle_{AB}\langle\Psi|_{AB}).$$

And $\quad M_{\mu} = \sqrt{\lambda^{\mu}} \sum_{jk} v^{\mu}_{jk} |j\rangle\langle k|.$

Suppose $\rho_A = \sum_i p_i |\varphi_i\rangle_A \langle\varphi_i|_A.$

I will prove that

$$\mathcal{E}_A(|\varphi_i\rangle_A\langle\varphi_i|_A) = (\mathcal{E}_A \otimes I_B)\left[\langle\varphi_i^*|_B \Psi\rangle_{AB} \langle\Psi|_{AB} \varphi_i^*\rangle_B\right].$$

with $\quad |\varphi_i\rangle_A = \sum_n \alpha_n |n\rangle_A. \quad |\varphi_i^*\rangle_B = \sum_n \alpha_n^* |n'\rangle_B.$

$$(\mathcal{E}_A \otimes I_B)\left[\langle\varphi_i^*|_B \Psi\rangle_{AB} \langle\Psi|_{AB} \varphi_i^*\rangle_B\right].$$

$$= (\mathcal{E}_A \otimes I_B) \langle\varphi_i^*| \sum_{jj'} |j\rangle_A\langle j'|_A |j\rangle_B\langle j'|_B \varphi_i^*\rangle$$

$$= \mathcal{E}_A \sum_{jj'} \alpha_j \alpha_{j'}^* |j\rangle_A\langle j'|_A = \mathcal{E}_A(|\varphi_i\rangle_A\langle\varphi_i|_A)$$

Then $(\mathcal{E}_A \otimes I_B)\left[\langle \varphi_i^*|_B \, |\Psi\rangle_{AB} \langle \Psi|_{AB} \, |\varphi_i^*\rangle_B \right]$

$$= \langle \varphi_i^*|_B \left[(\mathcal{E}_A \otimes I_B)\left(|\Psi\rangle_{AB} \langle \Psi|_{AB} \right) \right] |\varphi_i^*\rangle_B$$

$$= \langle \varphi_i^*|_B \sum_{lmjk} C_{lmjk} \, |l\rangle\langle j| \otimes |m\rangle\langle k| \, |\varphi_i^*\rangle_B$$

$$= \langle \varphi_i^*|_B \sum_{\mu} \lambda^{\mu} \sum_{jj'kk'} U_{jk}^{\mu} U_{j'k'}^{\mu\dagger} \, |j\rangle|k\rangle\langle j'|\langle k'| \, |\varphi_i^*\rangle_B$$

$$= \sum_{\mu} \lambda^{\mu} \sum_{jj'kk'} U_{jk}^{\mu} U_{j'k'}^{\mu\dagger} \, \alpha_{k'}^{*} \alpha_k \, |j\rangle\langle j'|$$

$$= \sum_{\mu} \lambda^{\mu} \sum_{jj'kk'} U_{jk}^{\mu} U_{j'k'}^{\mu\dagger} \, |k\rangle\langle k| \, |k'\rangle\langle k'| \, \alpha_{k'}^{*} \, \alpha_k \, |j\rangle\langle j'|$$

$$= \sum_{\mu} \lambda^{\mu} \sum_{jj'kk'} U_{jk}^{\mu} U_{j'k'}^{\mu\dagger} \, |j\rangle\langle k| \, \langle k'|\alpha_{k'}^{*} \alpha_k|k\rangle \, |k'\rangle\langle j'|$$

$$= \sum_{\mu} M_{\mu} |\varphi_i\rangle_A \langle \varphi_i|_A M_{\mu}^{\dagger} .$$

So $\mathcal{E}_A(\rho_A) = \mathcal{E}_A\left(\sum_i p_i |\varphi_i\rangle_A \langle \varphi_i|_A \right)$

$$= \sum_i p_i \sum_{\mu} M_{\mu} |\varphi_i\rangle_A \langle \varphi_i|_A M_{\mu}^{\dagger}$$

$$= \sum_{\mu} M_{\mu} \rho M_{\mu}^{\dagger} .$$

(c) $N_\alpha = \sum_\mu V_{\alpha\mu} M_\mu$.

$$\sum_\alpha N_\alpha \rho_A N_\alpha^\dagger = \sum_\alpha \sum_\mu V_{\alpha\mu} M_\mu \rho_A \sum_{\mu'} M_{\mu'}^\dagger V_{\alpha\mu'}^\dagger$$

$$= \sum_{\alpha\mu\mu'} V_{\alpha\mu} V_{\alpha\mu'}^* M_\mu \rho_A M_{\mu'}^\dagger$$

$$= \sum_{\mu\mu'} \delta_{\mu\mu'} M_\mu \rho_A M_\mu^\dagger$$

$$= \sum_\mu M_\mu \rho_A M_\mu^\dagger .$$

So the Kraus representation is not unique.

(d) rank(C).

Problem 2 Master equation & steady states

In this problem you will explore the degree of freedom in the Lindblad master equation. The Lindblad master equation with Hamiltonian $\hat{H}$ and a single jump operator $\hat{J}$ is:

$$\frac{d\hat{\rho}}{dt} = -i[\hat{H}, \hat{\rho}] + \mathcal{D}[\hat{J}]\hat{\rho} \tag{2.1}$$

where $\mathcal{D}[\hat{J}]\hat{\rho} = \hat{J}\hat{\rho}\hat{J}^\dagger - \frac{1}{2}\{\hat{J}^\dagger\hat{J}, \hat{\rho}\}$.

(a) Show that an open system with jump operator $\hat{J} = (\hat{a} - \alpha)$ is equivalent to an open system with jump operator $\hat{J} = \hat{a}$ and Hamiltonian $\hat{H} = g\hat{a} + g^*\hat{a}^\dagger$, where g is some complex amplitude and g^* is its complex conjugate. Find the value of g for a given α. Also, verify that the coherent state $|\alpha\rangle$ is the only steady state, i.e. $\frac{d\hat{\rho}}{dt} = 0$ for $\hat{\rho}(0) = |\alpha\rangle\langle\alpha|$.

(b) Show that an open system with jump operator $\hat{J} = (\hat{a}^2 - \alpha^2)$ is equivalent to an open system with jump operator $\hat{J} = \hat{a}^2$ and Hamiltonian $\hat{H} = g\hat{a}^2 + g^*\hat{a}^{2\dagger}$. Again, find the value of g for a given α and verify that $|\alpha\rangle$ and $|-\alpha\rangle$ (and any superposition or mixture of the two) are steady states.

(a) $\hat{J} = \hat{a} - \alpha$.

$$D[\hat{J}]\hat{\rho} = \hat{J}\hat{\rho}\hat{J}^\dagger - \frac{1}{2}\{\hat{J}^\dagger\hat{J}, \hat{\rho}\}$$

$$= (\hat{a}-\alpha)\hat{\rho}(\hat{a}^\dagger-\alpha^*) - \frac{1}{2}\left\{(\hat{a}^\dagger-\alpha^*)(\hat{a}-\alpha), \hat{\rho}\right\}$$

$$= \hat{a}\rho\hat{a}^\dagger - \frac{1}{2}\hat{a}^\dagger\hat{a}\rho - \frac{1}{2}\rho\hat{a}^\dagger\hat{a} - \frac{1}{2}\alpha^*\hat{a}\rho - \frac{1}{2}\alpha\rho\hat{a}^\dagger + \frac{1}{2}\alpha^*\rho\hat{a} + \frac{1}{2}\alpha\hat{a}^\dagger\rho$$

$$= \hat{a}\rho\hat{a}^\dagger - \frac{1}{2}\{\hat{a}^\dagger\hat{a}, \rho\} - \frac{1}{2}\alpha^*\hat{a}\rho - \frac{1}{2}\alpha\rho\hat{a}^\dagger + \frac{1}{2}\alpha^*\rho\hat{a} + \frac{1}{2}\alpha\hat{a}^\dagger\rho$$

$$\frac{d\hat{\rho}}{dt} = -i[\hat{H}, \rho] + \hat{a}\rho\hat{a}^\dagger - \frac{1}{2}\{\hat{a}^\dagger\hat{a}, \rho\} - \frac{1}{2}\alpha^*\hat{a}\rho - \frac{1}{2}\alpha\rho\hat{a}^\dagger + \frac{1}{2}\alpha^*\rho\hat{a} + \frac{1}{2}\alpha\hat{a}^\dagger\rho$$

For $\hat{J} = \hat{a}$ and $\hat{H} = g\hat{a} + g^*\hat{a}^\dagger$.

$$\frac{d\hat{\rho}}{dt} = -i[g\hat{a} + g^*\hat{a}^\dagger, \rho] + \hat{a}\rho\hat{a}^\dagger - \frac{1}{2}\{\hat{a}^\dagger\hat{a}, \rho\}$$

$$= \hat{a}\rho\hat{a}^\dagger - \frac{1}{2}\{\hat{a}^\dagger a, \rho\} - ig\hat{a}\rho - ig^*\hat{a}^\dagger\rho + i\rho g\hat{a} + i\rho g^*\hat{a}^\dagger$$

Compare to get $g = -\frac{i}{2}\alpha^*$.

With $\rho(0) = |\alpha\rangle\langle\alpha|$ and

$$\frac{d\hat{\rho}}{dt} = -i\left[g\hat{a} + g^*\hat{a}^\dagger, \rho\right] + \hat{a}\rho\hat{a}^\dagger - \frac{1}{2}\{\hat{a}^\dagger\hat{a}, \rho\}$$

Go to interaction picture to get

$$\frac{d\hat{\rho}_I}{dt} = \hat{a}\rho_I\hat{a}^\dagger - \frac{1}{2}\hat{a}^\dagger\hat{a}\rho_I - \frac{1}{2}\rho_I\hat{a}^\dagger\hat{a}.$$

In number state basis

$$\langle n|\frac{d\hat{\rho}_I}{dt}|m\rangle = \langle n|\hat{a}\rho_I\hat{a}^\dagger|m\rangle - \frac{1}{2}n\langle n|\rho_I|m\rangle - \frac{1}{2}\langle n|\rho_I|m\rangle\cdot m$$

$$= \langle n+1|\sqrt{n+1}\,\rho_I\sqrt{m+1}\,|m+1\rangle - \frac{1}{2}(m+n)\langle n|\rho_I|m\rangle$$

$$\Rightarrow \frac{d\rho_{nm}}{dt} = \sqrt{(n+1)(m+1)}\,\rho_{n+1,m+1} - \frac{1}{2}(m+n)\,\rho_{n,m}.$$

With $\rho(0) = |\alpha\rangle\langle\alpha|$. $\quad g = -\frac{i}{2}\alpha^*$

$$\frac{d\hat{\rho}}{dt} = -i\left[g\hat{a} + g^*\hat{a}^\dagger, \rho\right] + \hat{a}\rho\hat{a}^\dagger - \frac{1}{2}\{\hat{a}^\dagger\hat{a}, \rho\}$$

$$= \underline{\frac{1}{2}\alpha\hat{a}^\dagger\rho} - \frac{1}{2}\rho\alpha\hat{a}^\dagger + \hat{a}\rho\hat{a}^\dagger - \underline{\frac{1}{2}\hat{a}^\dagger a\rho} - \frac{1}{2}\rho\hat{a}^\dagger\hat{a}$$

$$= 0.$$

So $|\alpha\rangle$ is a steady state.

(b) $\hat{J} = (\hat{a}^2 - \alpha^2)$.

$$\frac{d\rho}{dt} = -i[\hat{H}, \rho] + \hat{J}\rho\hat{J}^\dagger - \frac{1}{2}\{\hat{J}^\dagger\hat{J}, \rho\}$$

$$= -i[\hat{H}, \rho] + (\hat{a}^2 - \alpha^2)\rho(\hat{a}^{\dagger 2} - \alpha^{*2}) - \frac{1}{2}\left[(\hat{a}^{\dagger 2} - \alpha^{*2})(\hat{a}^2 - \alpha^2)\rho + \rho(\hat{a}^{\dagger 2} - \alpha^{*2})(\hat{a}^2 - \alpha^2)\right]$$

$$= -i[\hat{H}, \rho] + \hat{a}^2\rho\hat{a}^{\dagger 2} - \frac{1}{2}\{\hat{a}^{\dagger 2}\hat{a}^2, \rho\} - \frac{1}{2}\alpha^{*2}\hat{a}^2\rho + \frac{1}{2}\alpha^2\hat{a}^{\dagger 2}\rho$$

$$+ \frac{1}{2}\rho\alpha^{*2}\hat{a}^2 - \frac{1}{2}\rho\alpha^2\hat{a}^{\dagger 2}$$

$\hat{J} = \hat{a}^2$, $\hat{H} = g\hat{a}^2 + g^*\hat{a}^{\dagger 2}$.

$$\frac{d\rho}{dt} = -i[g\hat{a}^2 + g^*\hat{a}^{\dagger 2}, \rho] + \hat{a}^2\rho\hat{a}^{\dagger 2} - \frac{1}{2}\{\hat{a}^{\dagger 2}\hat{a}^2, \rho\}$$

$$= \hat{a}^2\rho\hat{a}^{\dagger 2} - \frac{1}{2}\{\hat{a}^{\dagger 2}\hat{a}^2, \rho\} - ig\hat{a}^2\rho - ig^*\hat{a}^{\dagger 2}\rho + ig\rho\hat{a}^2 + ig^*\rho\hat{a}^{\dagger 2}.$$

Compare to get $g = -\frac{i}{2}\alpha^{*2}$.

Steady state $\frac{d\rho}{dt} = 0$.

$$\frac{d\rho}{dt} = -i[g\hat{a}^2 + g^*\hat{a}^{\dagger 2}, \rho] + \hat{a}^2\rho\hat{a}^{\dagger 2} - \frac{1}{2}\{\hat{a}^{\dagger 2}\hat{a}^2, \rho\} = 0.$$

Go to interaction picture

$$\frac{d\rho_I}{dt} = \hat{a}^2\rho_I\hat{a}^{\dagger 2} - \frac{1}{2}\{\hat{a}^{\dagger 2}\hat{a}^2, \rho\}$$

In number state basis

$$\langle n|\frac{d\rho_I}{dt}|m\rangle = \langle n|\hat{a}^2\rho_I\hat{a}^{\dagger 2}|m\rangle - \frac{1}{2}\langle n|\hat{a}^{\dagger 2}\hat{a}^2\rho|m\rangle - \frac{1}{2}\langle n|\rho\hat{a}^{\dagger 2}\hat{a}^2|m\rangle$$

$$\Rightarrow \frac{d\rho_{nm}}{dt} = nm\,\rho_{nm} - \frac{1}{2}n^2\rho_{nm} - \frac{1}{2}\rho_{nm}\cdot m^2$$

$$= \left[nm - \frac{1}{2}(m^2+n^2)\right]\rho_{nm}.$$

$$\Rightarrow \rho_{nm}(t) = \rho_{nm}(0)\, e^{-\frac{1}{2}(m-n)^2 t}.$$

So, we have steady solution

$\rho_{nn}(t) = \rho_{nn}(0)$. $n = 0, 1, \cdots d-1$. that is $\rho(t) = \rho(0)$.

With $\rho(0) = |\alpha\rangle\langle\alpha|$ or $\rho(0) = |-\alpha\rangle\langle-\alpha|$.

Plug in $\frac{d\rho}{dt} = -i\left[g\hat{a}^2 + g^*\hat{a}^{\dagger 2}, \rho\right] + \hat{a}^2\rho\,\hat{a}^{\dagger 2} - \frac{1}{2}\left\{\hat{a}^{\dagger 2}\hat{a}^2, \rho\right\}$

We get $\frac{d\rho}{dt} = \left[\frac{1}{2}\alpha^2\hat{a}^{\dagger 2}, \rho\right] + \hat{a}^2|\alpha\rangle\langle\alpha|\hat{a}^{\dagger 2} - \frac{1}{2}\left(\hat{a}^{\dagger 2}\hat{a}^2|\alpha\rangle\langle\alpha| + |\alpha\rangle\langle\alpha|\hat{a}^{\dagger 2}\hat{a}^2\right)$

(For $\rho(0) = |\alpha\rangle\langle\alpha|$) $= \underline{\frac{1}{2}\alpha^2\hat{a}^{\dagger 2}\rho} - \frac{1}{2}\alpha^2\rho\,\hat{a}^{\dagger 2} + \hat{a}^2\rho\,\hat{a}^{\dagger 2} \underline{- \frac{1}{2}\hat{a}^{\dagger 2}\cdot\alpha^2\rho} - \frac{1}{2}\rho\,\hat{a}^{\dagger 2}\hat{a}^2$

$= 0.$

(For $\rho(0) = |-\alpha\rangle\langle-\alpha|$) $\frac{d\rho}{dt} = \left[\frac{1}{2}\alpha^2\hat{a}^{\dagger 2}, \rho\right] + \hat{a}^2|-\alpha\rangle\langle-\alpha|\hat{a}^{\dagger 2} - \frac{1}{2}\left(\hat{a}^{\dagger 2}\hat{a}^2|-\alpha\rangle\langle-\alpha| + |-\alpha\rangle\langle-\alpha|\hat{a}^{\dagger 2}\hat{a}^2\right)$

$= \underline{\frac{1}{2}\alpha^2\hat{a}^{\dagger 2}\rho} - \frac{1}{2}\alpha^2\rho\,\hat{a}^{\dagger 2} + \alpha^2\rho\,\hat{a}^{\dagger 2} \underline{- \frac{1}{2}\hat{a}^{\dagger 2}\cdot\alpha^2\rho} - \frac{1}{2}\rho\,\hat{a}^{\dagger 2}\hat{a}^2$

$= 0.$

So $|\alpha\rangle$ and $|-\alpha\rangle$ are steady states.

Problem 3 POVM

Due to power failure, components of two states $|\psi_1\rangle = |0\rangle$, $|\psi_2\rangle = |+\rangle$ became mixed, where $|\pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle)$. You are given a box containing both the components and two detectors as described by operators

$$E_1 = \frac{\sqrt{2}}{1+\sqrt{2}} |1\rangle\langle 1| \tag{3.1}$$

$$E_2 = \frac{\sqrt{2}}{1+\sqrt{2}} |-\rangle\langle -| \tag{3.2}$$

(a) Can you distinguish the two components?

(b) Does E_1 and E_2 form a POVM? If not, can you construct a POVM using E_1 and E_2?

(c) What can you deduce from the results of the POVM?

(a) $\langle \psi_1 | E_1 | \psi_1 \rangle = 0$.

$\langle \psi_2 | E_1 | \psi_2 \rangle = \frac{\sqrt{2}}{1+\sqrt{2}} \cdot \left(\frac{1}{\sqrt{2}}\right)^2 = \frac{1}{\sqrt{2}(1+\sqrt{2})}$.

$\langle \psi_1 | E_2 | \psi_1 \rangle = \frac{1}{\sqrt{2}(1+\sqrt{2})}$.

$\langle \psi_2 | E_2 | \psi_2 \rangle = 0$.

We cannot distinguish these two states. We can only distinguish them when the probability of a certain measwement gives 1.

(b) $E_1 + E_2 = \frac{\sqrt{2}}{1+\sqrt{2}} \begin{pmatrix} 0 & \\ & 1 \end{pmatrix} + \frac{1}{1+\sqrt{2}} \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix} \neq I$.

So it is not POVM. We should have

$$E_3 = I - E_1 - E_2 = \begin{pmatrix} 1 - \frac{1}{\sqrt{2}(1+\sqrt{2})} & \frac{1}{\sqrt{2}(1+\sqrt{2})} \\ \frac{1}{\sqrt{2}(1+\sqrt{2})} & 1 - \frac{3}{\sqrt{2}(1+\sqrt{2})} \end{pmatrix}.$$

Then $\{E_1, E_2, E_3\}$ constitute a POVM.

(c) $\langle\psi_1|E_1|\psi_1\rangle = 0$. $\langle\psi_2|E_2|\psi_2\rangle = 0$.

So when we get outcome E_1, we know the state is $|\psi_2\rangle$.

with outcome E_2, we know the state is $|\psi_1\rangle$.

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Problem 1

The no-cloning theorem shows that we can't build a unitary machine that will make a perfect copy of an unknown quantum state. But suppose we are willing to settle for an *imperfect* copy—what fidelity might we achieve?

Consider a machine that acts on three qubit states according to

$$|000\rangle_{ABC} \rightarrow \sqrt{\frac{2}{3}}|00\rangle_{AB}|0\rangle_C + \sqrt{\frac{1}{3}}|\psi^+\rangle_{AB}|1\rangle_C \tag{1.1}$$

$$|100\rangle_{ABC} \rightarrow \sqrt{\frac{2}{3}}|11\rangle_{AB}|1\rangle_C + \sqrt{\frac{1}{3}}|\psi^+\rangle_{AB}|0\rangle_C. \tag{1.2}$$

where $|\psi^+\rangle_{AB}$ is the Bell state $\frac{1}{\sqrt{2}}(|0\rangle_A|1\rangle_B + |1\rangle_A|0\rangle_B)$.

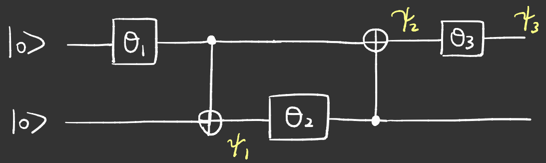
(a) Is such a device physically realizable, in principle?
If the machine operates on the initial state $|\psi\rangle_A|00\rangle_{BC}$, it produces a pure entangled state $|\Psi\rangle_{ABC}$ of the three qubits. But if we observe qubit A alone, its final state is the density opeator $\rho'_A = Tr_{BC}\left[|\Psi\rangle\langle\Psi|_{ABC}\right]$. Similarly, the qubit B observed in isolation, has the final state ρ'_B.It is easy to see that $\rho'_A = \rho'_B$ — these are identical, but imperfect, copies of the input pure state $|\psi\rangle_A$.

(b) The mapping from the initial state $|\psi\rangle\langle\psi|_A$ to the final state $\rho'_A$of qubit A defines a super-oiperator $\mathcal{E}$. Find an operator-sum representation of $\mathcal{E}$.

(c) For $|\psi\rangle_A = a|0\rangle_A + b|1\rangle_A$, find ρ'_A, and compute its fidelity $F \equiv \langle\psi|\rho'_A|\psi\rangle_A$.

(a) It is physically realizable.

We have circuit to give arbitrary state of 2 qubit basis:



$\psi_1 = \cos\theta_1|00\rangle + \sin\theta_1|11\rangle$

$\psi_2 = \cos\theta_1\cos\theta_2|00\rangle + \cos\theta_1\sin\theta_2|11\rangle - \sin\theta_1\sin\theta_2|10\rangle + \sin\theta_1\cos\theta_2|01\rangle.$

$\psi_3 = \cos\theta_1\cos\theta_2\cos\theta_3|00\rangle + \cos\theta_1\cos\theta_2\sin\theta_3|10\rangle - \cos\theta_1\sin\theta_2\sin\theta_3|01\rangle$

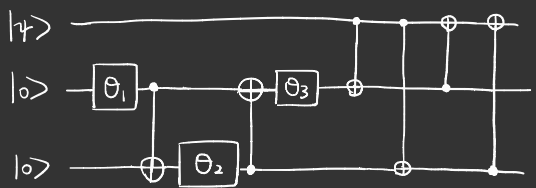
$+ \cos\theta_1\sin\theta_2\cos\theta_3|11\rangle + \sin\theta_1\sin\theta_2\sin\theta_3|00\rangle - \sin\theta_1\sin\theta_2\cos\theta_3|10\rangle$

$$+ \sin\theta_1 \cos\theta_2 \cos\theta_3 |01\rangle + \sin\theta_1 \cos\theta_2 \sin\theta_3 |11\rangle$$

$$= \underbrace{(\cos\theta_1 \cos\theta_2 \cos\theta_3 + \sin\theta_1 \sin\theta_2 \sin\theta_3)}_{A}|00\rangle + \underbrace{(\sin\theta_1 \cos\theta_2 \cos\theta_3 - \cos\theta_1 \sin\theta_2 \sin\theta_3)}_{B}|01\rangle$$

$$+ \underbrace{(\cos\theta_1 \cos\theta_2 \sin\theta_3 - \sin\theta_1 \sin\theta_2 \cos\theta_3)}_{C}|10\rangle + \underbrace{(\cos\theta_1 \sin\theta_2 \cos\theta_3 + \sin\theta_1 \cos\theta_2 \sin\theta_3)}_{D}|11\rangle$$

Let $A = \sqrt{\frac{2}{3}}$. $B = D = \sqrt{\frac{1}{6}}$. $C = 0$.



For $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$. the output state:

$$\alpha A|000\rangle + \alpha B|001\rangle + \alpha C|010\rangle + \alpha D|011\rangle + \beta A|100\rangle + \beta B|101\rangle + \beta C|110\rangle + \beta D|111\rangle$$

$$\Rightarrow \alpha A|000\rangle + \alpha B|001\rangle + \alpha C|010\rangle + \alpha D|011\rangle + \beta A|110\rangle + \beta B|111\rangle + \beta C|100\rangle + \beta D|101\rangle$$

$$\Rightarrow \alpha A|000\rangle + \alpha B|001\rangle + \alpha C|010\rangle + \alpha D|011\rangle + \beta A|111\rangle + \beta B|110\rangle + \beta C|101\rangle + \beta D|100\rangle$$

$$\Rightarrow \alpha A|000\rangle + \alpha B|001\rangle + \alpha C|110\rangle + \alpha D|111\rangle + \beta A|011\rangle + \beta B|010\rangle + \beta C|101\rangle + \beta D|100\rangle$$

$$\Rightarrow \alpha A|000\rangle + \alpha B|101\rangle + \alpha C|110\rangle + \alpha D|011\rangle + \beta A|111\rangle + \beta B|010\rangle + \beta C|001\rangle + \beta D|100\rangle$$

$$\Rightarrow \alpha\sqrt{\frac{2}{3}}|000\rangle + \alpha\sqrt{\frac{1}{6}}|101\rangle + \alpha\sqrt{\frac{1}{6}}|011\rangle + \beta\sqrt{\frac{2}{3}}|111\rangle + \beta\sqrt{\frac{1}{6}}|010\rangle + \beta\sqrt{\frac{1}{6}}|100\rangle.$$

With $|\psi\rangle_A = |0\rangle_A$. output state $\sqrt{\frac{2}{3}}|000\rangle + \sqrt{\frac{1}{3}}|\psi^+\rangle|1\rangle$.

$|\psi\rangle_A = |1\rangle_A$. output state $\sqrt{\frac{2}{3}}|111\rangle + \sqrt{\frac{1}{3}}|\psi^+\rangle|0\rangle$.

(b) $|\psi\rangle_{ABC} = \alpha\sqrt{\frac{2}{3}}|000\rangle + \alpha\sqrt{\frac{1}{6}}|101\rangle + \alpha\sqrt{\frac{1}{6}}|011\rangle + \beta\sqrt{\frac{2}{3}}|111\rangle + \beta\sqrt{\frac{1}{6}}|010\rangle + \beta\sqrt{\frac{1}{6}}|100\rangle$

$$\mathrm{Tr}_{BC}\left[|\psi\rangle_{ABC}\langle\psi|_{ABC}\right] = \begin{pmatrix} \frac{2}{3}|\alpha|^2+\frac{1}{6}|\alpha|^2+\frac{1}{6}|\beta|^2 & \frac{1}{3}\alpha\beta^*+\frac{1}{3}\alpha\beta^* \\ \frac{1}{3}\beta\alpha^*+\frac{1}{3}\beta\alpha^* & \frac{1}{6}|\alpha|^2+\frac{2}{3}|\beta|^2+\frac{1}{6}|\beta|^2 \end{pmatrix} = \begin{pmatrix} \frac{5}{6}|\alpha|^2+\frac{1}{6}|\beta|^2 & \frac{2}{3}\alpha\beta^* \\ \frac{2}{3}\beta\alpha^* & \frac{1}{6}|\alpha|^2+\frac{5}{6}|\beta|^2 \end{pmatrix}.$$

$$= \rho_A'.$$

$$\rho_A = |\psi\rangle_A\langle\psi|_A = \begin{pmatrix} |\alpha|^2 & \alpha\beta^* \\ \beta\alpha^* & |\beta|^2 \end{pmatrix}$$

$\rho_A' = \sum_\mu E_\mu \rho_A E_\mu^\dagger$. We know the quantum operation for a d-dimensional quantum system is $\mathcal{E}(\rho) = \frac{pI}{d} + (1-p)\rho$.

Compare to get $p = \frac{1}{4}$. And the operator-sum representation:

$$\rho_A' = \frac{3}{4}\rho + \frac{1}{12}(X\rho X + Y\rho Y + Z\rho Z).$$

(c) $\rho_A = \begin{pmatrix} |a|^2 & ab^* \\ ba^* & |b|^2 \end{pmatrix}$. $\quad \rho_A' = \begin{pmatrix} \frac{5}{6}|a|^2+\frac{1}{6}|b|^2 & \frac{2}{3}ab^* \\ \frac{2}{3}ba^* & \frac{1}{6}|a|^2+\frac{5}{6}|b|^2 \end{pmatrix}.$

$$F = \langle\psi|\rho_A'|\psi\rangle_A = (a^* \ b^*)\begin{pmatrix} \frac{5}{6}|a|^2+\frac{1}{6}|b|^2 & \frac{2}{3}ab^* \\ \frac{2}{3}ba^* & \frac{1}{6}|a|^2+\frac{5}{6}|b|^2 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix}$$

$$= a^*\left[\left(\frac{5}{6}|a|^2+\frac{1}{6}|b|^2\right)a + \frac{2}{3}a|b|^2\right] + b^*\left[\frac{2}{3}b|a|^2 + \left(\frac{1}{6}|a|^2+\frac{5}{6}|b|^2\right)b\right]$$

$$= \frac{5}{6}|a|^4 + \frac{1}{6}|b|^2|a|^2 + \frac{2}{3}|a|^2|b|^2 + \frac{2}{3}|b|^2|a|^2 + \frac{1}{6}|a|^2|b|^2 + \frac{5}{6}|b|^4$$

$$= \frac{5}{6}|a|^4 + \frac{5}{3}|a|^2|b|^2 + \frac{5}{6}|b|^4 = \frac{5}{6}\left(|a|^2+|b|^2\right)^2.$$

Problem 2

Compute the Schmidt decomposition for a pure state given by

$$|\phi'\rangle_{AB} = \frac{1}{\sqrt{6}}\left(\sqrt{3}\,|00\rangle_{AB} + \sqrt{2}\,|01\rangle_{AB} + |11\rangle_{AB}\right)$$

Note: Schmidt decomposition is essentially a restatement of singular value decomposition(SVD). The SVD of an $n \times m$ matrix (assume $n \geq m$) is a factorization of the form

$$M = U \begin{bmatrix} \Sigma \\ 0 \end{bmatrix} V^\dagger$$

where U is an $n \times n$ matrix, V is an $m \times m$ matrix and Σ is a $m \times m$ positive semi definite diagonal matrix. SVD can be calculated using MATLAB/MATHEMATICA or a similar software. Suppose, we write $U = [U_1, U_2]$, where U_1 is an $n \times m$ matrix, then we can write $M = U_1 \Sigma V^\dagger$. Try to construct the Schmidt decomposition using the first m column vectors of U_1 and V and the diagonal elements of Σ.

$$\rho_A = \mathrm{Tr}_B\left(|\phi'\rangle_{AB}\langle\phi|_{AB}\right) = \begin{pmatrix} \frac{5}{6} & \frac{1}{3\sqrt{2}} \\ \frac{1}{3\sqrt{2}} & \frac{1}{6} \end{pmatrix}$$

$$\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} \\ 0 \\ \frac{1}{\sqrt{6}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & 0 & \frac{1}{\sqrt{6}} \end{pmatrix}$$

$$\begin{matrix} \frac{1}{2} & \frac{1}{\sqrt{6}} & 0 & \frac{1}{2\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{1}{3} & 0 & \frac{1}{3\sqrt{2}} \\ 0 & 0 & 0 & 0 \\ \frac{1}{2\sqrt{3}} & \frac{1}{3\sqrt{2}} & 0 & \frac{1}{6} \end{matrix}$$

In[15]:= {u, σ, v} = SingularValueDecomposition$\left[\begin{pmatrix} \frac{5}{6} & \frac{1}{3\sqrt{2}} \\ \frac{1}{3\sqrt{2}} & \frac{1}{6} \end{pmatrix}\right]$
奇异值分解

Out[15]= $\left\{\left\{\left\{\frac{\frac{1}{3\sqrt{2}} + \frac{5}{6}(\sqrt{2}+\sqrt{3})}{\sqrt{\left(\frac{1}{3\sqrt{2}} + \frac{5}{6}(\sqrt{2}+\sqrt{3})\right)^2 + \left(\frac{1}{6} + \frac{\sqrt{2}+\sqrt{3}}{3\sqrt{2}}\right)^2}}, \frac{\frac{1}{3\sqrt{2}} + \frac{5}{6}(\sqrt{2}-\sqrt{3})}{\sqrt{\left(-\frac{1}{3\sqrt{2}} - \frac{5}{6}(\sqrt{2}-\sqrt{3})\right)^2 + \left(\frac{1}{6} + \frac{\sqrt{2}-\sqrt{3}}{3\sqrt{2}}\right)^2}}\right\},\right.\right.$

$\left\{\frac{\frac{1}{6} + \frac{\sqrt{2}+\sqrt{3}}{3\sqrt{2}}}{\sqrt{\left(\frac{1}{3\sqrt{2}} + \frac{5}{6}(\sqrt{2}+\sqrt{3})\right)^2 + \left(\frac{1}{6} + \frac{\sqrt{2}+\sqrt{3}}{3\sqrt{2}}\right)^2}}, \frac{\frac{1}{6} + \frac{\sqrt{2}-\sqrt{3}}{3\sqrt{2}}}{\sqrt{\left(-\frac{1}{3\sqrt{2}} - \frac{5}{6}(\sqrt{2}-\sqrt{3})\right)^2 + \left(\frac{1}{6} + \frac{\sqrt{2}-\sqrt{3}}{3\sqrt{2}}\right)^2}}\right\}\Bigg\},$

$\left\{\left\{\frac{1}{2}\sqrt{\frac{1}{3} \times (5+2\sqrt{6})}, 0\right\}, \left\{0, \frac{1}{2}\sqrt{\frac{1}{3} \times (5-2\sqrt{6})}\right\}\right\},$

$\left\{\left\{\frac{\sqrt{2}+\sqrt{3}}{\sqrt{1+(\sqrt{2}+\sqrt{3})^2}}, \frac{\sqrt{2}-\sqrt{3}}{\sqrt{1+(-\sqrt{2}+\sqrt{3})^2}}\right\}, \left\{\frac{1}{\sqrt{1+(\sqrt{2}+\sqrt{3})^2}}, \frac{1}{\sqrt{1+(-\sqrt{2}+\sqrt{3})^2}}\right\}\right\}\Bigg\}$

ρ_A has eigenvalues $\frac{\sqrt{3} \pm \sqrt{2}}{2\sqrt{3}}$ and eigenvectors $\frac{1}{\sqrt{6 \pm 2\sqrt{6}}}\left((\sqrt{2} \pm \sqrt{3})|0\rangle + |1\rangle\right)$.

let $|i_1\rangle_A = \frac{1}{\sqrt{6+2\sqrt{6}}}\left((\sqrt{2}+\sqrt{3})|0\rangle + |1\rangle\right)$.

$|i_2\rangle_A = \frac{1}{\sqrt{6-2\sqrt{6}}}\left((\sqrt{2}-\sqrt{3})|0\rangle + |1\rangle\right)$.

$$\rho_B = \mathrm{Tr}_A\left(|\phi'\rangle_{AB}\langle\phi'|_{AB}\right) = \begin{pmatrix} \frac{1}{2} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} & \frac{1}{2} \end{pmatrix}.$$

$$\{u, \sigma, v\} = \text{SingularValueDecomposition}\left[\begin{pmatrix} \frac{1}{2} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} & \frac{1}{2} \end{pmatrix}\right]$$

奇异值分解

Out[13]=
$$\left\{\left\{\left\{\frac{1}{\sqrt{2}}, \frac{-\frac{1}{2}+\frac{1}{\sqrt{6}}}{\sqrt{2}\left(\frac{1}{2}-\frac{1}{\sqrt{6}}\right)}\right\}, \left\{\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right\}\right\},\right.$$
$$\left\{\left\{\frac{1}{2}\sqrt{\frac{1}{3}\times\left(5+2\sqrt{6}\right)}, 0\right\}, \left\{0, \frac{1}{2}\sqrt{\frac{1}{3}\times\left(5-2\sqrt{6}\right)}\right\}\right\},$$
$$\left.\left\{\left\{\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right\}, \left\{\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right\}\right\}\right\}$$

ρ_B has eigenvalues $\frac{\sqrt{3}\pm\sqrt{2}}{2\sqrt{3}}$ and eigenvectors $\frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle)$.

Let $|i_1\rangle_B = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$.

$|i_2\rangle_B = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$.

We have $|\phi'\rangle = \sqrt{\frac{\sqrt{3}+\sqrt{2}}{2\sqrt{3}}}\,|i_1\rangle_A|i_1\rangle_B + \sqrt{\frac{\sqrt{3}-\sqrt{2}}{2\sqrt{3}}}\,|i_2\rangle_A|i_2\rangle_B$.

Verify:
$$\sqrt{\frac{\sqrt{3}+\sqrt{2}}{2\sqrt{3}}}\cdot\frac{1}{\sqrt{6+2\sqrt{6}}}\left((\sqrt{2}+\sqrt{3})|0\rangle + |1\rangle\right)\cdot\frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right)$$

$$+\sqrt{\frac{\sqrt{3}-\sqrt{2}}{2\sqrt{3}}}\cdot\frac{1}{\sqrt{6-2\sqrt{6}}}\left((\sqrt{2}-\sqrt{3})|0\rangle + |1\rangle\right)\cdot\frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right)$$

Problem 3

What transformations are possible for bipartite pure states? Suppose Alice and Bob share a bipartite pure state $|\Psi\rangle$. Using a LOCC protocol, they wish to transform it to another bipartite pure state $|\Phi\rangle$. Furthermore, the protocol must be *deterministic* — the state $|\Phi\rangle$ is obtained with probability one irrespective of the outcomes of the measurements that Alice and Bob perform.

Suppose that these initial state is the Bell state

$$|\Psi\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle|0\rangle + |1\rangle|1\rangle\right)$$

and the final state is

$$|\Phi\rangle = \left(\cos\theta|0\rangle|0\rangle + \sin\theta|1\rangle|1\rangle\right).$$

(a) Please construct a LOCC procedure to deterministically prepare the final state. (Hint: You may need to consider generalized measurement as defined by a set of operators $\{M_\mu\}$ such that $\sum_\mu M_\mu^\dagger M_\mu = I$. If the outcome μ occurs, the state $|\Psi\rangle$ evolves into $|\Psi_\mu\rangle = \frac{M_\mu|\Psi\rangle}{\sqrt{\langle\Psi|M_\mu^\dagger M_\mu|\Psi\rangle}}$. You need to find LOCC operations $\{M_\mu\}$ that ensures $|\Psi_\mu\rangle = |\Phi\rangle$).

(b) Bonus: More generally, suppose the initial and final states have Schmidt decompositions

$$|\Psi\rangle = \sum_i \sqrt{(p_\Psi)_i}\,|\alpha_i\rangle \otimes |\beta_i\rangle$$

$$|\Phi\rangle = \sum_i \sqrt{(p_\Phi)_i}\,\left|\alpha_i'\right\rangle \otimes \left|\beta_i'\right\rangle.$$

Show that if the deterministic transformation $|\Psi\rangle \to |\Phi\rangle$ is possible, then $p_\Psi \prec p_\Phi$ ("p_Ψ is majorized by p_Φ" [1]). (Hint: Preskill Notes, Problem 3.6.)

[1] WLOG $p_{\Phi,n} \geq p_{\Phi,n+1}, p_{\Psi,n} \geq p_{\Psi,n+1}$, then $p_\Psi \prec p_\Phi \Leftrightarrow \sum p_\Psi = \sum p_\Phi, \forall m \sum_{n=1}^m p_\Psi \leq \sum_{n=1}^m p_\Phi$

(a) With $M_1 = \begin{bmatrix} \cos\theta & \\ & \sin\theta \end{bmatrix}$. $M_2 = \begin{bmatrix} \sin\theta & \\ & \cos\theta \end{bmatrix}$.

$M_1^\dagger M_1 + M_2^\dagger M_2 = I$.

We have $|\psi_1\rangle = \frac{M_1|\psi\rangle}{\sqrt{\langle\psi|M_1^\dagger M_1|\psi\rangle}} = \frac{\cos\theta|00\rangle + \sin\theta|11\rangle}{\sqrt{\cos^2\theta + \sin^2\theta}} = \cos\theta|00\rangle + \sin\theta|11\rangle$.

$|\psi_2\rangle = \frac{M_2|\psi\rangle}{\sqrt{\langle\psi|M_2^\dagger M_2|\psi\rangle}} = \cos\theta|01\rangle + \sin\theta|10\rangle$.

$|\psi_1\rangle = |\Phi\rangle$. For $|\psi_2\rangle$ Alice could apply NOT operation, then $|\psi_2\rangle$ will be come $|\Phi\rangle$ after operation.

(b) I will follow the derivation on "Nielson & Chuang".

We could prove that the LOCC procedure could be realized by local measurement M_j by Alice, and local unitary operation U_j by Bob. (*)

From $|\psi\rangle$ to $|\Phi\rangle$. for Alice $\rho_\psi \longrightarrow \rho_\Phi$.

$$\rho_\psi = Tr_B(|\psi\rangle\langle\psi|)$$
$$\rho_\Phi = Tr_B(|\Phi\rangle\langle\Phi|).$$

So
$$M_j \rho_\psi M_j^\dagger = p_j \rho_\Phi . \quad p_j = \text{probability of outcome } j.$$

From polar decomposition we know matrix A can be expressed as $\sqrt{AA^\dagger}\,U$.

So
$$M_j \sqrt{\rho_\psi} = \sqrt{M_j \rho_\psi M_j^\dagger}\, V_j = \sqrt{p_j \rho_\Phi}\, V_j .$$

$$\Rightarrow \sqrt{\rho_\psi} M_j^\dagger M_j \sqrt{\rho_\psi} = V_j^\dagger p_j \rho_\Phi V_j$$

$$\Rightarrow \sqrt{\rho_\psi} \sum_j M_j^\dagger M_j \sqrt{\rho_\psi} = \sum_j p_j V_j^\dagger \rho_\Phi V_j .$$

$$\rho_\psi = \sum_j p_j V_j^\dagger \rho_\Phi V_j .$$

We could prove that $p_\psi < p_\Phi$. (**)

Prof of (*): Suppose Bob operate on $|\psi\rangle$ with M_j.

$$\sqrt{p_\psi}\,|\alpha_i\rangle|\beta_i\rangle$$

$$M_j = \sum_{kl} M_{j,kl} |\gamma_k\rangle\langle\beta_l| . \quad \mathcal{H}_B$$

Define $N_j = \sum_{kl} M_{j,kl} |\gamma_k\rangle\langle\beta_l|$. $\mathcal{H}_A$

State after Bob apply M_j:

$$|\psi_j\rangle \propto M_j |\psi\rangle = \sum_{kl} M_{j,kl} \sqrt{(p_\psi)}\, |\alpha_l\rangle |\gamma_k\rangle$$

after Alice apply N_j:

$$|\varphi_j\rangle \propto N_j |\psi\rangle = \sum_{kl} M_{j,kl} \sqrt{(p_\psi)}\, |\gamma_k\rangle |\alpha_l\rangle.$$

$|\psi_j\rangle$, $|\varphi_j\rangle$ is the same state. with same Schmidt number.

So we have unitary matrix V_j:

$$|\psi_j\rangle = (U_j \otimes V_j) |\varphi_j\rangle.$$

Proof of (**): Some math...

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Problem 1

Two prominent classes of entangled states according to LOCC for N-qubits are the GHZ and the W states which are given by

$$|GHZ\rangle_{12..N} = \frac{1}{\sqrt{2}}(|00...0\rangle_{1,2...N} + |11...1\rangle_{1,2...N})$$
$$|W\rangle_{12...N} = \frac{1}{\sqrt{N}}(|0..01\rangle_{1...N-1,N} + |0..10\rangle_{1...N-1,N} + ... + |1..00\rangle_{1...N-1,N}.$$

(a) Calculate the number of entangled bits (given by Von-Neumann entropy) between the subsystems (1|2,3,..N), (2|1,3,...N) , (3|1,2,...N) . . . (N|1,2,3,...$N-1$) for both states.

(b) Suppose, we consider a three-qubit GHZ state and distribute qubits (1,2) to Alice and qubit 3 to Bob, Design a quantum circuit that will show the deterministic teleportation of an arbitrary and unknown single qubit state from Alice to Bob using CNOT and Hadamard gates.

(c) If we distribute qubits (1,2) to Alice and 3 to Bob from the three-qubit GHZ state, how many classical bits can be transferred from Alice to Bob with superdense coding? Suppose, we distribute qubit (1) to Alice and qubits (2,3) to Bob, how many classical bits can be transferred from Alice to Bob with superdense coding?

(d) Can the three qubit W-state be used for the deterministic teleportation of an arbitrary single qubit state if we distribute qubits (1,2) to Alice and 3 to Bob? If yes, then give the scheme, if no, then explain why?

(e) If we consider the "asymmetric W state" given by

$$|W\rangle_{123} = a|001\rangle_{123} + b|100\rangle_{123} + c|010\rangle_{123}$$

where a, b, c are complex numbers satisfying $|a|^2 + |b|^2 + |c|^2 = 1$. For what choice of (a, b, c) is the state suitable for the teleportation of an arbitrary single qubit state between two parties for a suitable distribution of qubits?

(f) If we distribute qubits (1,2) to Alice and 3 to Bob from the asymmetric W-state with the values of a, b, c that you obtained above, how many classical bits can be transferred from Alice to Bob with superdense coding? Suppose, we distribute qubit (1) to Alice and qubits (2,3) to Bob, how many classical bits can be transferred from Alice to Bob with superdense coding?

Problem 2

Consider the following four-qubit entangled state

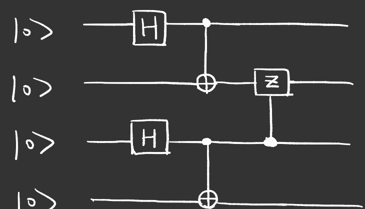
$$|\chi\rangle_{1234} = \frac{1}{2}(|0000\rangle_{1234} + |0011\rangle_{1234} + |1100\rangle_{1234} - |1111\rangle_{1234})$$

(a) Construct a quantum circuit with CNOT, CPHASE and (or) Hadamard gates for the creation of this entangled state.

(b) Show that using an appropriate qubit distribution of the above-entangled state, Alice can transmit an arbitrary and an unknown two-qubit state to Bob deterministically.

(c) Suppose, we distribute qubits (1,2) to Alice and (3,4) to Bob, how many classical bits can be transferred from Alice to Bob with superdense coding? Suppose, we distribute qubits (1,4) to Alice and (2,3) to Bob, how many classical bits can be transferred from Alice to Bob with superdense coding?

(a)



Verification: $|0000\rangle \Rightarrow \frac{1}{\sqrt{2}}(|0000\rangle + |1100\rangle) \Rightarrow \frac{1}{2}(|0000\rangle + |0011\rangle + |1100\rangle + |1111\rangle)$

$\Rightarrow \frac{1}{2}(|0000\rangle + |0011\rangle + |1100\rangle - |1111\rangle).$

(b)

Problem 3

Consider two-qubit system characterized by the following density matrix

$$\rho_{AB} = p_S \left|\Psi^-\right\rangle \left\langle\Psi^-\right| + a \left|\Psi^+\right\rangle \left\langle\Psi^+\right| + b\left|00\right\rangle \left\langle 00\right| + (m+b)\left|11\right\rangle \left\langle 11\right|,$$

where $|\Psi^{\pm}\rangle = \frac{1}{\sqrt{2}}\left(|01\rangle \pm |10\rangle\right)$ and $p_S + a + 2b + m = 1$ for $p_S, a, b, (m+b) \geq 0$.

(a) Compute the concurrence $C(\rho_{AB})$.

(b) Show that the system is entangled (i.e., $C(\rho_{AB}) > 0$) if

$$p_S > p_S^* = \frac{1-m^2}{2}.$$

(c) **[Bonus]** For arbitary density matrix of two-qubit system, we may define the singlet population $p_S = \langle\Psi^-|\,\rho_{AB}\,|\Psi^-\rangle$ and magnetization $m = \left|\mathrm{Tr}\left[\rho_{AB}\vec{S}\right]\right|$, with $\vec{S} = \frac{1}{2}\left(\vec{\sigma_A} + \vec{\sigma_B}\right)$. Show that for $p_S > p_S^*$, the system must be entangled (i.e., $C(\rho_{AB}) > 0$). (Hint: Use the fact that LOCC cannot generate entanglement).

(a) For 2-qubit states concurrence is defined as

$$C(\rho) = \max(0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4).$$

$\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \lambda_4.$ are ordered eigenvalues of

$$R = \sqrt{\sqrt{\rho}(Y\otimes Y)\rho^*(Y\otimes Y)\sqrt{\rho}}.$$

00
01
10
11

$$\rho_{AB} = \frac{p_S}{2}\begin{pmatrix}0\\1\\-1\\0\end{pmatrix}(0\ 1\ -1\ 0) + \frac{a}{2}\begin{pmatrix}0\\1\\1\\0\end{pmatrix}(0\ 1\ 1\ 0)$$

$$+\, b\begin{pmatrix}1\\0\\0\\0\end{pmatrix}(1\ 0\ 0\ 0) + (m+b)\begin{pmatrix}0\\0\\0\\1\end{pmatrix}(0\ 0\ 0\ 1)$$

$$= \begin{pmatrix} b & 0 & 0 & 0 \\ 0 & \frac{a+P_s}{2} & \frac{a-P_s}{2} & 0 \\ 0 & \frac{a-P_s}{2} & \frac{a+P_s}{2} & 0 \\ 0 & 0 & 0 & m+b \end{pmatrix},$$

$$Y \otimes Y = \begin{pmatrix} & -i \\ i & \end{pmatrix} \otimes \begin{pmatrix} & -i \\ i & \end{pmatrix} = \begin{pmatrix} & & & -1 \\ & & 1 & \\ & 1 & & \\ -1 & & & \end{pmatrix}.$$

$$R = \sqrt{\sqrt{\rho}(Y\otimes Y)\rho^*(Y\otimes Y)\sqrt{\rho}}$$ calculate to get

$$R = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\sqrt{(a-p)(a^*-p^*)}}{2} & 0 \\ 0 & -\frac{\sqrt{(a-p)(a^*-p^*)}}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

with eigenvalues

$$\underset{\lambda_1}{\frac{\sqrt{(a-p)(a^*-p^*)}}{2}},\ \underset{\lambda_2}{0},\ \underset{\lambda_3}{0},\ \underset{\lambda_4}{-\frac{\sqrt{(a-p)(a^*-p^*)}}{2}}$$

So $C(\rho_{AB}) = \max\left(0, \sqrt{(a-p)(a^*-p^*)}\right).$

(b) System is entangled if $C(\rho_{AB}) > 0$

$$\Rightarrow \quad |a|^2 + |P_s|^2 - a^* P_s - a P_s^* > 0.$$

$$a = 1 - P_s - m - 2b.$$

$$|a|^2 + |p_s|^2 - (1 - p_s^* - m - 2b)\, p_s - (1 - p_s - m - 2b)\, p_s^*$$

$$= |a|^2 - p_s + 3|p_s|^2 + (m+2b)\, p_s - p_s^* + (m+2b)\, p_s^*$$

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Problem 1

Suppose we have two copies of Bell states $|\psi_+\rangle$ which undergo the decoherence process and evolve into mixed states (also known as Werner states) given by

$$\rho_W = F|\psi_+\rangle\langle\psi_+| + \frac{1-F}{3}[|\psi_-\rangle\langle\psi_-| + |\phi_-\rangle\langle\phi_-| + |\phi_+\rangle\langle\phi_+|].$$

where F is the entanglement fidelity.

(a) Show how one entangled state of higher fidelity (F') could be obtained with two entangled states of lower fidelity (F) provided $F > 1/2$, where F' is given by

$$F_1' = \frac{F^2 + \frac{(1-F)^2}{9}}{F^2 + \frac{2}{3}F(1-F) + \frac{5}{9}(1-F)^2}$$

(b) One can carry out the above purification process N times and obtain the new fidelity F_N'. Assume that the initial fidelity of two Bell states is 0.6. Plot how the improved fidelity varies with the number of purification steps up to 20 steps.

(a) Reference: quant-ph/9511027

$|\psi_+\rangle \longrightarrow$ Werner states

$$\rho_w = F|\psi_+\rangle\langle\psi_+| + \frac{1-F}{3}\left[|\psi_-\rangle\langle\psi_-| + |\phi_-\rangle\langle\phi_-| + |\phi_+\rangle\langle\phi_+|\right]$$

Step 1: Perform σ_y rotation on each pair.

$$(\mathbb{1}\otimes Y)|\psi_+\rangle = (Y\otimes\mathbb{1})|\psi_+\rangle = \frac{1}{\sqrt{2}}(i|11\rangle - i|00\rangle) = -i|\phi_-\rangle$$

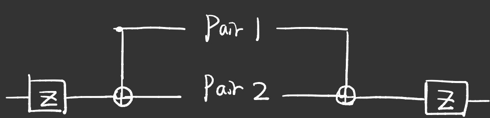
$$(\mathbb{1}\otimes Y)|\psi_-\rangle = (Y\otimes\mathbb{1})|\psi_-\rangle = \frac{1}{\sqrt{2}}(i|11\rangle + i|00\rangle) = i|\phi_+\rangle$$

$$(\mathbb{1}\otimes Y)|\Phi_+\rangle = (Y\otimes\mathbb{1})|\Phi_+\rangle = \frac{1}{\sqrt{2}}(i|01\rangle - i|10\rangle) = i|\psi_-\rangle$$

$$(\mathbb{1}\otimes Y)|\Phi_-\rangle = (Y\otimes\mathbb{1})|\Phi_-\rangle = \frac{1}{\sqrt{2}}(i|01\rangle + i|10\rangle) = i|\psi_+\rangle$$

$$\rho_w \rightarrow \rho_w' = F|\Phi_-\rangle\langle\Phi_-| + \frac{1-F}{3}\left(|\Phi_+\rangle\langle\Phi_+| + |\psi_-\rangle\langle\psi_-| + |\psi_+\rangle\langle\psi_+|\right)$$

Step 2: Apply CNOT operations for both pairs. measure in σ_z.



List a table of states after CNOT:

Pair 1	Pair 2	$\Rightarrow$	Pair 1	Pair 2
$\Phi_\pm$	$\Phi_\pm$ or $\Psi_\pm$		"	"
$\Phi_\pm$	Φ_- or Ψ_-		$\Phi_\mp$	"
$\Psi_\pm$	Φ_+ or Ψ_+		"	Ψ_+ or Φ_+
$\Psi_\pm$	Φ_- or Ψ_-		"	Ψ_- or Φ_-

A complete list with probability:

Prob	Pair 1	Pair 2	$\Rightarrow$	Pair 1	Pair 2	$\Rightarrow$	Pair 1	Pair 2
$\left(\frac{1-F}{3}\right)^2$	Ψ_-	Ψ_-		Φ_+	Φ_+		Φ_+	Φ_+
$\frac{F(1-F)}{3}$	Ψ_+	Ψ_-		Φ_-	Φ_+		Φ_-	Φ_+
$\left(\frac{1-F}{3}\right)^2$	Ψ_-	Φ_-		Φ_+	Ψ_+		Φ_+	Ψ_+
$\frac{F(1-F)}{3}$	Ψ_+	Φ_-		Φ_-	Ψ_+		Φ_-	Ψ_+
$\frac{F(1-F)}{3}$	Ψ_-	Ψ_+		Φ_+	Φ_-		Φ_-	Φ_-
F^2	Ψ_+	Ψ_+		Φ_-	Φ_-		Φ_+	Φ_-
$\left(\frac{1-F}{3}\right)^2$	Ψ_-	Φ_+		Φ_+	Ψ_-		Φ_-	Ψ_-
$\frac{F(1-F)}{3}$	Ψ_+	Φ_+		Φ_-	Ψ_-		Φ_+	Ψ_-

$\left(\frac{1-F}{3}\right)^2$	Φ_-	Ψ_-	Ψ_+	Φ_+	Ψ_+	Ψ_+
$\left(\frac{1-F}{3}\right)^2$	Φ_+	Ψ_-	Ψ_-	Φ_+	Ψ_-	Ψ_+
$\left(\frac{1-F}{3}\right)^2$	Φ_-	Φ_-	Ψ_+	Ψ_+	Ψ_+	Φ_+
$\left(\frac{1-F}{3}\right)^2$	Φ_+	Φ_-	Ψ_-	Ψ_+	Ψ_-	Φ_+
$\frac{F(1-F)}{3}$	Φ_-	Ψ_+	Ψ_+	Φ_-	Ψ_+	Ψ_-
$\frac{F(1-F)}{3}$	Φ_+	Ψ_+	Ψ_-	Φ_-	Ψ_-	Ψ_-
$\left(\frac{1-F}{3}\right)^2$	Φ_-	Φ_+	Ψ_+	Ψ_-	Ψ_+	Φ_-
$\left(\frac{1-F}{3}\right)^2$	Φ_+	Φ_+	Ψ_-	Ψ_-	Ψ_-	Φ_-

We could then apply σ_y to convert $|\Phi_-\rangle$ back to $|\Psi_+\rangle$.

Get fidelity
$$F' = \frac{F^2 + \frac{(1-F)^2}{9}}{F^2 + \frac{2F(1-F)}{3} + \frac{5(1-F)^2}{9}}.$$

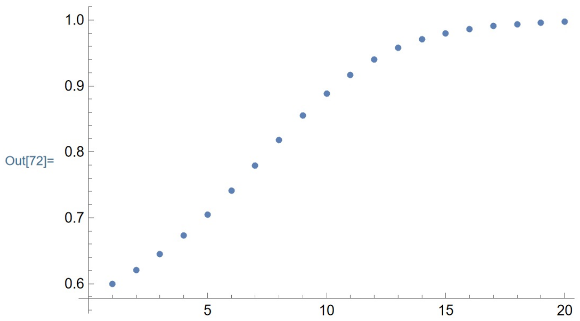
(b)

In[72]:= ListPlot[RecurrenceTable[
绘制点集 递归表

$$\left\{f[n+1] == \frac{f[n]^2 + \frac{(1-f[n])^2}{9}}{f[n]^2 + \frac{2}{3} f[n]\,(1-f[n]) + \frac{5\,(1-f[n])^2}{9}},\right.$$

f[1] == 0.6}, f, {n, 1, 20}]]

Out[72]=



Problem 2

In this problem you are required to verify/prove the deterministic teleportation of a CX gate between two parties. Suppose Alice and Bob share a bell pair $|\Psi^+\rangle = (|01\rangle + |10\rangle)/\sqrt{2}$. Prove that they can use the circuit shown in the right panel of Fig. 2.1 to deterministically implement a non-local CX gate between their data qubits D_1 and D_2. C_1, C_2 represent their shared bell pair.

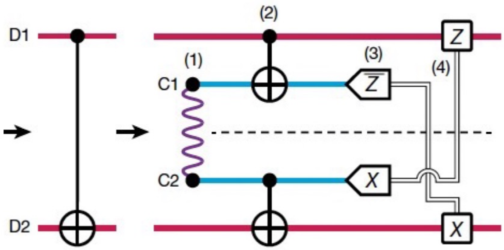


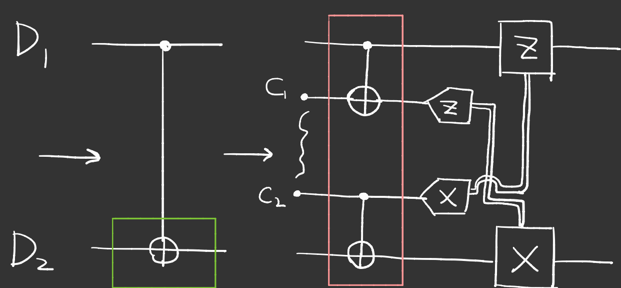
Figure 2.1: The circuit of the teleportation-based CX gate in [1].

References

[1] Chou, Kevin S., et al. "Deterministic teleportation of a quantum gate between two logical qubits." Nature 561.7723 (2018): 368-373.

Reference: Nature 561.7723 (2018)

Verify the deterministic teleportation of a CX gate between two parties.



non-local CX-gate between D_1 & D_2

Use $\{|0\rangle, |1\rangle\}$ for data qubit D_1, D_2, and $\{|g\rangle, |e\rangle\}$ for communication qubit C_1, C_2.

1. Initialization of system $\quad |a|^2+|b|^2+|c|^2+|d|^2 = 1.$

$D_1 D_2 \quad C_1 C_2$

$$|\psi_{12}\rangle|\psi^+\rangle = \left(a|00\rangle + b|01\rangle + c|10\rangle + d|11\rangle\right) \otimes \frac{1}{\sqrt{2}}\left(|ge\rangle + |eg\rangle\right)$$

2. Perform local CNOT (controlled module)

$$|\psi\rangle_{CNOT\ controll} = a|00ge\rangle + a|00eg\rangle + b|01ge\rangle + b|01eg\rangle$$
$$+ c|10ee\rangle + c|10gg\rangle + d|10ee\rangle + d|11gg\rangle$$

3. Perform local CNOT (Target module)

$$|\psi\rangle_{CNOT\ target} = a|01ge\rangle + a|00eg\rangle + b|00eg\rangle + b|01eg\rangle$$
$$+ c|11ee\rangle + c|10gg\rangle + d|10ee\rangle + d|11gg\rangle$$

4. Perform $\pi/2$-rotation on C_1, C_2

$$|\psi\rangle_{CNOT} = a\left(+|01gg\rangle - |01ge\rangle + |00eg\rangle + |00ee\rangle\right)$$
$$+b\left(+|00gg\rangle - |00ge\rangle + |01eg\rangle + |01ee\rangle\right)$$
$$+c\left(+|10gg\rangle + |10ge\rangle + |10ge\rangle - |11ee\rangle\right)$$
$$+d\left(+|11gg\rangle + |11ge\rangle + |10eg\rangle - |10ee\rangle\right)$$

Measure communication qubits, outcome $|g\rangle \rightarrow$ "0", $|e\rangle \rightarrow$ "1".

"00" : $|\psi_{00}\rangle = a|01\rangle + b|00\rangle + c|10\rangle + d|11\rangle$

"01" : $|\psi_{01}\rangle = -a|01\rangle + b|00\rangle + c|10\rangle + d|11\rangle$

"10" : $|\psi_{10}\rangle = a|00\rangle + b|01\rangle + c|11\rangle + d|10\rangle$

"11" : $|\psi_{11}\rangle = a|00\rangle + b|01\rangle - c|11\rangle - d|10\rangle$

The single-qubit operations for each state:

"00" : $IX|\psi_{00}\rangle = a|00\rangle + b|01\rangle + c|11\rangle + d|10\rangle = \hat{U}_{CNOT}|\psi\rangle$.

"01" : $ZX|\psi_{01}\rangle = \hat{U}_{CNOT}|\psi\rangle$.

"10" : $II|\psi_{10}\rangle = \hat{U}_{CNOT}|\psi\rangle$.

"11" : $ZI|\psi_{11}\rangle = \hat{U}_{CNOT}|\psi\rangle$.

Each implement a CNOT operation.

HW 6 yuxiang@uchicago.edu

Problem 1

A black box computes a function

$$f : \{0,1\}^n \to \{0,1\} \tag{1.1}$$

which can be represented by a binary string

$$X = X_{N-1}X_{N-2}\dots X_1X_0 \tag{1.2}$$

where $X_i = f(i), N = 2^n$. Our goal is to count the number r of states "marked" by the box, that is, to determine the Hamming weight $r = |X|$ of X. We can devise a quantum algorithm that counts the marked state by combining Grover's exhaustive search with the quantum Fourier transform.

(a) The black box performs an Unitary transformation U_f which acts according to

$$U_f(|x\rangle \otimes |y\rangle) = |x\rangle \otimes |y \oplus f(x)\rangle \tag{1.3}$$

where $|x\rangle$ is a n-qubit state and $|y\rangle$ is a single-qubit state. How can we achieve the unitary transformation

$$\tilde{U}_f|x\rangle = (-1)^{f(x)}|x\rangle \tag{1.4}$$

with the black box, Hadamard gate(s), and an auxiliary qubit?

(b) Let $|\psi_X\rangle = \frac{1}{\sqrt{r}}\sum_{j:X_j=1}|j\rangle$ denote the uniform superposition of the marked states and let U_{Grover} denote the "Grover iteration," which performs the rotation by the angle 2θ in the plane spanned by $|\psi_X\rangle$ and $|s\rangle = \frac{1}{\sqrt{N}}\sum_{j=0}^{N}|j\rangle$, where $\sin\theta = \langle s \mid \psi_X\rangle = \sqrt{\frac{r}{N}}$.

Consider a unitary transformation

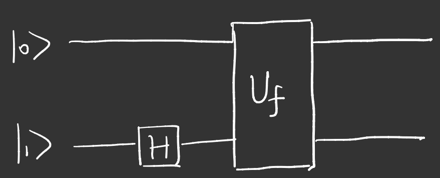
$$V : |t\rangle \otimes |\phi\rangle \to |t\rangle \otimes U^t_{\text{Grover}}\,|\phi\rangle \tag{1.5}$$

that reads a counter register taking values $t \in \{0, 1, 2, \dots, T-1\}$ (where $T = 2^m$), and then applies U_{Grover} t times. Explain how V can be applied calling the oracle T-1 times.

[Hint: Use the binary expansion $t = \sum_{k=0}^{m-1} t_k 2^k$ and the conditional oracle call from (a).

(c) Suppose that $r << N$. Show that, by applying V, performing the quantum Fourier transform on the counter register, and then measuring the counter register, we can determine θ to the accuracy $O(1/T)$, and hence we can find r with high success probability in $T = O(\sqrt{rN})$ queries.

(a)

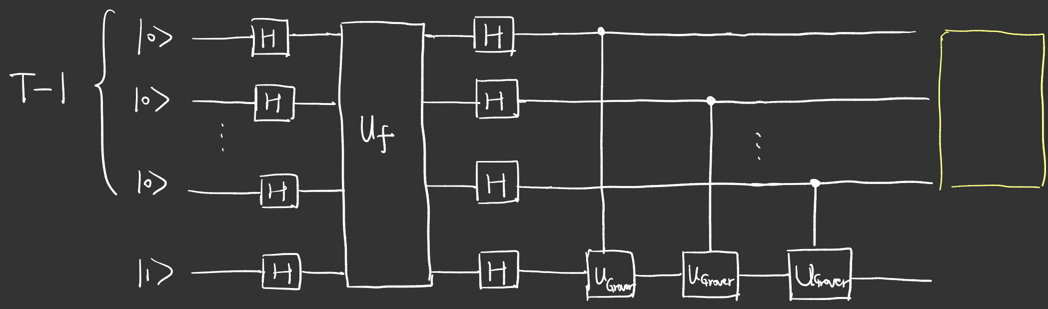


$$U_f|x\rangle\frac{1}{\sqrt{2}}(|0\rangle-|1\rangle)$$

$$=(-1)^{f(x)}|x\rangle\frac{1}{\sqrt{2}}(|0\rangle-|1\rangle).$$

$$\tilde{U}_f|x\rangle=(-1)^{f(x)}|x\rangle.$$

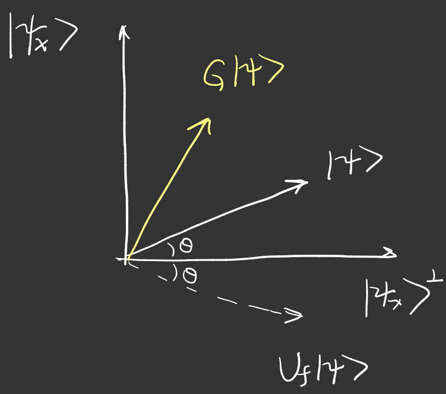
(b) $t = \sum_{k=0}^{m-1} t_k 2^k$. if $t_k = 1$ perform U_{Grover}.



(c) $V|t\rangle \otimes |\phi\rangle = |t\rangle \otimes U_{Grover}^{t} |\phi\rangle$

Then apply QFT and Measurement. $\square$

Reference: Nilsen & Chuang.



From graphic understanding of

Grover's Algorithm, we know

$$\theta \approx \sin\theta \approx 2\sqrt{\frac{r}{N}}$$

$$\varepsilon(\theta) = \sqrt{\frac{r}{N}} = O\left(\frac{1}{T}\right).$$

Number of query:

$$\sum_{t=0}^{T-1} |t\rangle \otimes \underbrace{\sum_{j=0}^{N-1} |j\rangle}_{|s\rangle}$$

$U_g \in SU(2)$

$\{|s\rangle, |\psi_x\rangle\}$

$e^{\pm i\theta}$

$|g_\pm\rangle$

$$\Big\downarrow V$$

$$\sum_{t=0}^{T-1} |t\rangle \otimes U_g^t |s\rangle$$

$\Big\downarrow$ QFT^{-1} on counter register

Problem 2

Consider a complete graph with N vertices 1,2,3,...,N, in which each vertex is coupled to every other vertex with a coupling constant γ. Suppose there exists a "defect-site" in the Hamiltonian at the site w, so that the Hamiltonian of the system can be written as

$$H = -\gamma L - |w\rangle\langle w| \tag{2.1}$$

where $L = \sum_{i \neq j} |i\rangle\langle j|$ for the complete graph.

(a) If we start with a state

$$|s\rangle = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} |i\rangle \tag{2.2}$$

compute the probability of reaching the target state as a function of time.

Hint: Consider the subspace spanned by $|s\rangle$ and $|w\rangle$.

(b) Find the time taken for which the probability of reaching the target state is maximum.

(c) With the optimized choice of time, find the optimal value of γ for which the probability of reaching the target state is maximum.

(d) With the optimal value of γ, show that the time taken to search for the target state scales as $O(\sqrt{N})$.

(a) In subspace spanned by $|s\rangle$ and $|w\rangle$.

$$\langle w|s\rangle = \langle w| \sum_{i=1}^{N} \frac{1}{\sqrt{N}} |i\rangle = \frac{1}{\sqrt{N}} ?$$

$$|\psi\rangle = \alpha|s\rangle + \beta|w\rangle.$$

?

$$H|\psi\rangle = (-\gamma L - |w\rangle\langle w|)\,\alpha|s\rangle + (-\gamma L - |w\rangle\langle w|)\,\beta|w\rangle$$

$$= -\alpha\gamma L|s\rangle - \beta\gamma L|w\rangle - \beta|w\rangle$$

$$= -\alpha\gamma \sum_{i\neq j} |i\rangle\langle j| \cdot \frac{1}{N} \sum_{m=1}^{N} |m\rangle - \beta\gamma \sum_{i \neq w} |i\rangle - \beta|w\rangle$$

$$= -\frac{\alpha\gamma}{N} \sum_{m=1}^{N} \sum_{i \neq m} |i\rangle - \gamma\beta \sum_{i=1}^{N} |i\rangle + (\gamma-1)\beta|w\rangle.$$

$$= -\gamma(N-1)\alpha|s\rangle - \gamma\sqrt{N}\beta|s\rangle + (\gamma-1)\beta|w\rangle$$

$$i\hbar\frac{d}{dt}\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = \begin{pmatrix}-\gamma(N-1)\alpha - \gamma\sqrt{N}\beta\\ (\gamma-1)\beta\end{pmatrix}.$$

$$\Rightarrow \begin{cases} i\hbar\frac{d\alpha}{dt} = -\gamma(N-1)\alpha - \gamma\sqrt{N}\beta \\ i\hbar\frac{d\beta}{dt} = (\gamma-1)\beta. \end{cases}$$

$$\Rightarrow \beta(t) = e^{-i(\gamma-1)t/\hbar}.$$ ✗ not right

$$\frac{d\alpha}{dt} = -\frac{\gamma(N-1)}{i\hbar}\alpha - \gamma\sqrt{N}e^{-i(\gamma-1)t/\hbar} \quad ?$$

Try again.

(a) In subspace spanned by $|s\rangle$ and $|w\rangle$.

$$|\psi\rangle = \alpha|s\rangle + \beta|w\rangle.$$

$$H|\psi\rangle = (-\gamma L - |w\rangle\langle w|)\alpha|s\rangle + (-\gamma L - |w\rangle\langle w|)\beta|w\rangle$$

$$= -\alpha\gamma L|s\rangle - \frac{\alpha}{\sqrt{N}}|w\rangle - \beta\gamma L|w\rangle - \beta|w\rangle$$

$$= -\alpha\gamma \sum_{i\neq j} |i\rangle\langle j| \cdot \frac{1}{\sqrt{N}} \sum_{m=1}^{N} |m\rangle - \frac{\alpha}{\sqrt{N}}|\omega\rangle - \beta\gamma \sum_{i\neq\omega} |i\rangle - \beta|\omega\rangle$$

$$= -\frac{\alpha\gamma}{\sqrt{N}} \sum_{i\neq m} |i\rangle - \frac{\alpha}{\sqrt{N}}|\omega\rangle - \gamma\beta \sum_{i=1}^{N} |i\rangle + (\gamma-1)\beta|\omega\rangle.$$

$$= -\alpha\gamma(N-1)|s\rangle - \frac{\alpha}{\sqrt{N}}|\omega\rangle - \gamma\sqrt{N}\beta|s\rangle + (\gamma-1)\beta|\omega\rangle$$

$$i\hbar \frac{d}{dt}\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = \begin{pmatrix} -\alpha\gamma(N-1) - \gamma\beta\sqrt{N} \\ -\frac{\alpha}{\sqrt{N}} + (\gamma-1)\beta \end{pmatrix}$$

$$\boxed{\begin{aligned} i\hbar\frac{d\alpha}{dt} &= -\gamma(N-1)\alpha - \gamma\sqrt{N}\beta. \\ i\hbar\frac{d\beta}{dt} &= -\frac{1}{\sqrt{N}}\alpha + (\gamma-1)\beta. \end{aligned}}$$

solution?

$$i\hbar\frac{d(\alpha+k\beta)}{dt} = \left(-\gamma(N-1) - \frac{k}{\sqrt{N}}\right)\alpha + \left(-\gamma\sqrt{N} + k(\gamma-1)\right)\beta$$

$$\Rightarrow k^2 + \sqrt{N}k(\gamma N-1) - \gamma N = 0$$

$$k = \frac{-\sqrt{N}(\gamma N-1) \pm \sqrt{N^2(\gamma N-1)^2 + 4\gamma N}}{2}.$$

$$i\hbar\frac{d(\alpha+k\beta)}{dt} = \left(-\gamma(N-1) - \frac{k}{\sqrt{N}}\right)(\alpha+k\beta).$$

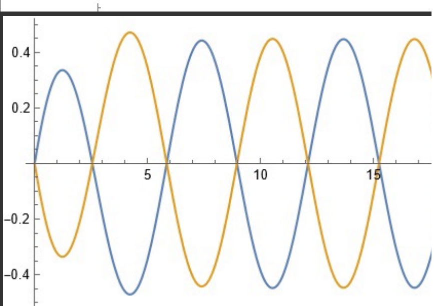
$$\alpha + k\beta = c\, e^{-\left(\gamma(N-1) + \frac{k}{\sqrt{N}}\right)t}.$$

In[26]:=
```
eqns = {x'[t] == i γ (N - 1) x[t] + i γ √N y[t], y'[t] == i/√N x[t] + (γ - 1) y[t]}

sol = DSolve[{eqns, x[0] == 1, y[0] == 0}, {x[t], y[t]}, t]
Plot[{x[t] /. psol, y[t] /. psol}, {t, 0, 20}]
```

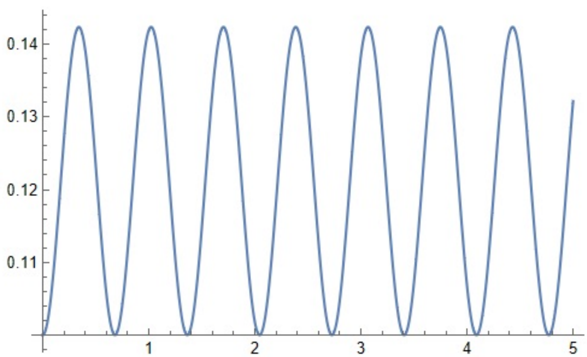
Out[26]= $\left\{x'[t] == i\,(-1+N)\,\gamma\,x[t] + i\,\sqrt{N}\,\gamma\,y[t],\; y'[t] == \frac{i\,x[t]}{\sqrt{N}} + (-1+\gamma)\,y[t]\right\}$

Out[27]=
$$\Bigg\{\Bigg\{x[t] \to \Bigg(-e^{\frac{t\left(-\sqrt{N}+(1-i)\sqrt{N}\gamma+iN^{3/2}\gamma-\sqrt{-N\left(-1+(6+2i)\gamma-2iN\gamma-2i\gamma^2-(2-2i)N\gamma^2+N^2\gamma^2\right)}\right)}{2\sqrt{N}}}\sqrt{N} + e^{\frac{t\left(-\sqrt{N}+(1-i)\sqrt{N}\gamma+iN^{3/2}\gamma+\sqrt{-N\left(-1+(6+2i)\gamma-2iN\gamma-2i\gamma^2-(2-2i)N\gamma^2+N^2\gamma^2\right)}\right)}{2\sqrt{N}}}\sqrt{N} +$$
$$(1+i)\,e^{\frac{t\left(-\sqrt{N}+(1-i)\sqrt{N}\gamma+iN^{3/2}\gamma-\sqrt{-N\left(-1+(6+2i)\gamma-2iN\gamma-2i\gamma^2-(2-2i)N\gamma^2+N^2\gamma^2\right)}\right)}{2\sqrt{N}}}\sqrt{N}\,\gamma -$$
$$(1+i)\,e^{\frac{t\left(-\sqrt{N}+(1-i)\sqrt{N}\gamma+iN^{3/2}\gamma+\sqrt{-N\left(-1+(6+2i)\gamma-2iN\gamma-2i\gamma^2-(2-2i)N\gamma^2+N^2\gamma^2\right)}\right)}{2\sqrt{N}}}\sqrt{N}\,\gamma - i\,e^{\frac{t\left(-\sqrt{N}+(1-i)\sqrt{N}\gamma+iN^{3/2}\gamma-\sqrt{-N\left(-1+(6+2i)\gamma-2iN\gamma-2i\gamma^2-(2-2i)N\gamma^2+N^2\gamma^2\right)}\right)}{2\sqrt{N}}}$$
$$N^{3/2}\gamma + i\,e^{\frac{t\left(-\sqrt{N}+(1-i)\sqrt{N}\gamma+iN^{3/2}\gamma+\sqrt{-N\left(-1+(6+2i)\gamma-2iN\gamma-2i\gamma^2-(2-2i)N\gamma^2+N^2\gamma^2\right)}\right)}{2\sqrt{N}}}N^{3/2}\gamma +$$
$$e^{\frac{t\left(-\sqrt{N}+(1-i)\sqrt{N}\gamma+iN^{3/2}\gamma-\sqrt{-N\left(-1+(6+2i)\gamma-2iN\gamma-2i\gamma^2-(2-2i)N\gamma^2+N^2\gamma^2\right)}\right)}{2\sqrt{N}}}\sqrt{-N\left(-1+(6+2i)\gamma-2iN\gamma-2i\gamma^2-(2-2i)N\gamma^2+N^2\gamma^2\right)} +$$
$$e^{\frac{t\left(-\sqrt{N}+(1-i)\sqrt{N}\gamma+iN^{3/2}\gamma+\sqrt{-N\left(-1+(6+2i)\gamma-2iN\gamma-2i\gamma^2-(2-2i)N\gamma^2+N^2\gamma^2\right)}\right)}{2\sqrt{N}}}\sqrt{-N\left(-1+(6+2i)\gamma-2iN\gamma-2i\gamma^2-(2-2i)N\gamma^2+N^2\gamma^2\right)}\Bigg)\Bigg/$$
$$\left(2\sqrt{-N\left(-1+(6+2i)\gamma-2iN\gamma-2i\gamma^2-(2-2i)N\gamma^2+N^2\gamma^2\right)}\right),$$
$$y[t] \to -\frac{i\left(e^{\frac{t\left(-\sqrt{N}+(1-i)\sqrt{N}\gamma+iN^{3/2}\gamma-\sqrt{-N\left(-1+(6+2i)\gamma-2iN\gamma-2i\gamma^2-(2-2i)N\gamma^2+N^2\gamma^2\right)}\right)}{2\sqrt{N}}} - e^{\frac{t\left(-\sqrt{N}+(1-i)\sqrt{N}\gamma+iN^{3/2}\gamma+\sqrt{-N\left(-1+(6+2i)\gamma-2iN\gamma-2i\gamma^2-(2-2i)N\gamma^2+N^2\gamma^2\right)}\right)}{2\sqrt{N}}}\right)}{\sqrt{-N}}\Bigg\}\Bigg\}$$



In[360]:=
```
P = Abs[x[t] /. sol/√N + y[t] /. sol]^2 /. {N → 10, γ → 1};
Plot[P, {t, 0, 5}]
```

Out[361]=

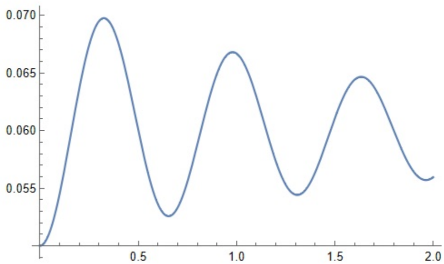


$$\text{Probability} = \left|\langle w|\left(\alpha|s\rangle + \beta|w\rangle\right)\right|^2 = \left|\frac{\alpha}{\sqrt{N}} + \beta\right|^2$$

(b) $P = \left| \dfrac{\alpha}{\sqrt{N}} + \beta \right|^2 = \dfrac{|\alpha|^2}{N} + |\beta|^2 + \dfrac{\alpha^* \beta + \alpha \beta^*}{\sqrt{N}}$.

```
In[392]:=
P = (Abs[(x[t] /. sol)/√N + y[t] /. sol])^2 /. {N → 20, γ → 0.5};
Plot[P, {t, 0, 2}]
```

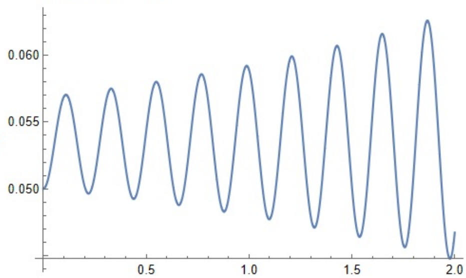
Out[393]=



Something went wrong

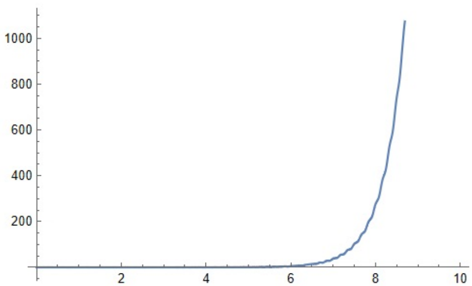
```
In[388]:= P = (Abs[(x[t] /. sol)/√N + y[t] /. sol])^2 /. {N → 20, γ → 1.5};
Plot[P, {t, 0, 2}]
```

Out[389]=




```
In[390]:= P = (Abs[(x[t] /. sol)/√N + y[t] /. sol])^2 /. {N → 20, γ → 2};
Plot[P, {t, 0, 10}]
```

Out[391]=



HW 7 yuxiang@uchicago.edu

Problem 1

In the period finding algorithm we prepared the "periodic state"

$$\frac{1}{\sqrt{A}}\sum_{j=0}^{A-1}|x_0 + jr\rangle \tag{1.1}$$

where $A = \lceil (N - x_0)/r \rceil$; then we perform the quantum Fourier transform with base N and measure. The probability distribution governing the measurement outcome y is

$$\Pr(y) = \frac{1}{NA}\left(\frac{\sin^2 \pi Ayr/N}{\sin^2 \pi yr/N}\right) \tag{1.2}$$

Let δ be the deviation of the rational number y/N from the nearest integer multiple of $1/r$,

$$\delta = \left|\frac{y}{N} - \frac{k}{r}\right| \tag{1.3}$$

This probability can be expressed as

$$\Pr(y) = \frac{1}{NA}\left(\frac{\sin^2 \pi Ar\delta}{\sin^2 \pi r\delta}\right) \tag{1.4}$$

Note that since there is a multiple of $1/r$ within the distance $1/2r$ from any real number, we may assume that $\delta \leq 1/2r$.

(a) Show that

$$\Pr(y) \leq \frac{1}{4NAr^2\delta^2} \tag{1.5}$$

(b) Let us say that the measurement outcome y is "ϵ-bad". If the distance to the nearest multiple of $1/r$ is larger than ϵ. Show that the probability $\text{Prob}(\delta > \epsilon)$ of a δ-bad outcome satisfies

$$\text{Prob}(\delta > \epsilon) < \frac{1}{N\epsilon} \tag{1.6}$$

Therefore, the probability of a ϵ-bad outcome is small for $N \ggg 1/\epsilon$.

Problem 2

(a) Plot the probability distribution of a quantum walk with a "Hadamard coin" starting in state $|\downarrow\rangle \otimes |0\rangle$ and after T=100 steps. Plot only the probability at even points.

(b) Plot the probability distribution of a quantum walk with a "Hadamard coin" starting in state

$$\frac{1}{\sqrt{2}}(|\uparrow\rangle + i|\downarrow\rangle) \otimes |0\rangle \tag{2.1}$$

after T=100 steps. Plot only the probability at even points.

(c) Plot the variance of the position of the walker with respect to time for the quantum walk and the classical random walk with an unbiased coin.

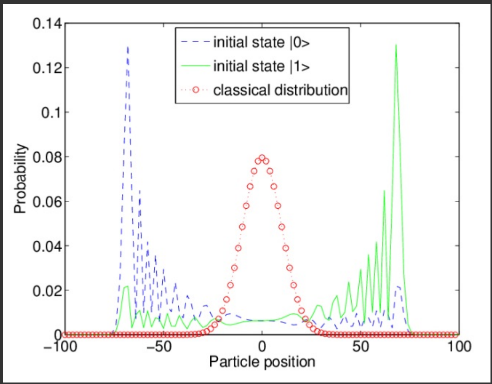
$$S = \sum_x \left(|x+1\rangle\langle x| \otimes |\uparrow\rangle\langle\uparrow| + |x-1\rangle\langle x| \otimes |\downarrow\rangle\langle\downarrow| \right).$$

$$U = S(I \otimes H). \qquad |x(t=0)\rangle = |0\rangle. \qquad |c(t=0)\rangle = |\downarrow\rangle.$$

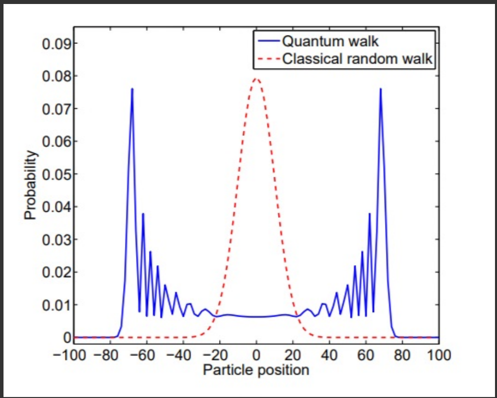
$$P(t=T, x') = \left| \langle x' | \mathrm{Tr}_c \left(U^T |x\rangle |c\rangle \right) \right|^2.$$

My code did not work. Below are plots from arxiv 1001.5326 and srep 19864.

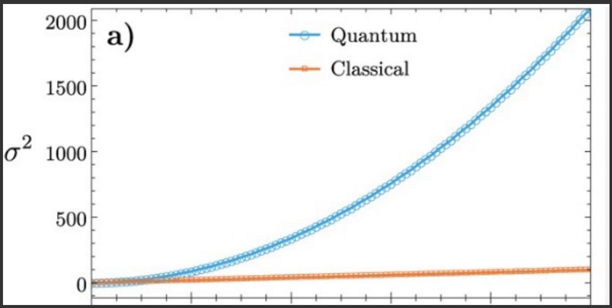
(a)



(b)



(c)



$\sigma^2 \sim T^2$.

HW 8 yuxiang @uchicago.edu

Problem 1

Consider the [[5,1,3]] code with stabilizers

$$\begin{aligned} S_1 &= XZZXI \\ S_2 &= IXZZX \\ S_3 &= XIXZZ \\ S_4 &= ZXIXZ. \end{aligned} \tag{1.1}$$

Show that this code can correct an arbitrary single qubit error.

Problem 2

To correct photon loss errors, we may consider using superposition of photon number states to encode information.

(a) We may encode quantum states as

$$\begin{aligned} |0\rangle_L &= \frac{1}{2}(|0\rangle + \sqrt{2}|2\rangle + |4\rangle) \\ |1\rangle_L &= \frac{1}{2}(|0\rangle - \sqrt{2}|2\rangle + |4\rangle) \end{aligned} \tag{2.1}$$

which are orthognal states with average photon number $\langle n \rangle = 2$. Use the quantum error correction criterion to show that the above encoding can correct the errors from the set $\mathcal{E} = \{I, \hat{a}\}$, with anhiliation operator defined as $\hat{a}|n\rangle = \sqrt{n}|n-1\rangle$.

(b) We may use a slightly more complicated encoding

$$\begin{aligned} |0\rangle_L &= \frac{1}{2\sqrt{2}}(|0\rangle + \sqrt{3}|3\rangle + \sqrt{3}|6\rangle + |9\rangle) \\ |1\rangle_L &= \frac{1}{2\sqrt{2}}(|0\rangle - \sqrt{3}|3\rangle + \sqrt{3}|6\rangle - |9\rangle) \end{aligned} \tag{2.2}$$

with average photon number 9/2. Use the quantum error correction criterion to show that the above encoding can correct the errors from the set $\{I, \hat{a}, \hat{a}^2\}$.

(c) Can you generalize the above encoding scheme to design a code that can correct up to L photon loss errors, i.e., $\mathcal{E} = \left\{I, \hat{a}, \hat{a}^2, \cdots, \hat{a}^L\right\}$. Hint: The square of the amplitudes are binomial coeffcients.