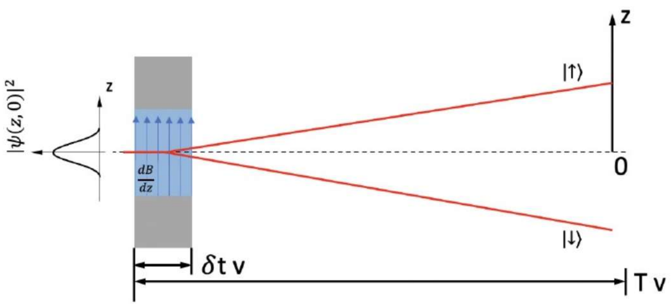
1 Problem 1 – Stern-Gerlach Measurement

An atom with spin $s = 1/2$ moves along the x axis. At time $t = 0$ the atom experiences a momentum kick of the form $H' = \lambda\sigma_z \otimes \hat{z}$ for a short time δt.



In the Stern-Gerlach experiment this is realized by a local magnetic field gradient $\lambda = (g\mu_B/2)\frac{\partial B}{\partial z}$. After having experienced a momentum kick, the atom continues free evolution for a time T.

Part (a)

Assume the atom starts in the following state.

$$|\psi(z, m_s, t = 0)\rangle = \frac{1}{\sqrt{\sqrt{2\pi}\sigma}} \exp\left\{-\frac{z^2}{4\sigma^2}\right\}\Big(\alpha_\downarrow |\downarrow\rangle + \alpha_\uparrow |\uparrow\rangle\Big) \tag{1.1}$$

(i) Write down the system Hamiltonian.

(ii) For now, assume $\lambda = 0$. Find the atom's wavefunction as a function of time $T + \delta t$.

Part (b)

Now find a solution for the general case $\lambda \neq 0$. Assume the atom is heavy, or rather that $\lambda\delta t/m \to 0$. This assumption will allow you to simplify the evolution operator in the field interaction region as follows.

$$U(T) = \exp\left\{-i\left(\frac{T}{\hbar}H_0 + \frac{\delta t}{\hbar}H'\right)\right\} \approx \exp\left\{-\frac{iT}{\hbar}H_0\right\}\exp\left\{-\frac{i\delta t}{\hbar}H'\right\} \tag{1.2}$$

(i) Find the evolution operator and simplify as far as possible.

(ii) Find the full wavefunction $|\psi(z, m_s, T)\rangle$. (Recall, $\exp\{-i\alpha p\}|\psi(k)\rangle = |\psi(k-\alpha)\rangle$.) What is the atom's position $\langle z(T)\rangle$ for $\left|\psi(z, m_s = \frac{1}{2}, T)\right\rangle$, and what is the variance $\langle\Delta z^2\rangle = \langle z^2\rangle - \langle z\rangle^2$?

Part (c)

Using our results from part (b), we look now at a measurement process. If you have not found a result in (b), you may continue with the following wavefunction.

$$|\psi(z, m_s, T)\rangle = N\sum_{m_s}\alpha_{m_s}\exp\left\{-\frac{(z - 2i\sigma^2\lambda\delta t m_s/\hbar)^2}{4(\sigma^2 + i\hbar T/(2m))}\right\}|m_s\rangle \tag{1.3}$$

N is a normalization constant as follows.

$$N = \frac{1}{\sqrt{\sqrt{2\pi}\sigma(1 + iT\hbar/(2m\sigma^2))}}\exp\left\{-\left(\frac{\sigma\lambda\delta t}{\hbar}\right)^2\right\} \tag{1.4}$$

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1 Stern-Gerlach Measurement

(a)

(i) $$H = \frac{\vec{p}^{\,2}}{2m} + g\mu_B \vec{S}\cdot\vec{B} + H' \quad (0<t\le \delta t)$$

$$= \frac{\vec{p}^{\,2}}{2m} + g\mu_B S_z B_z(x,z) + \lambda\, \sigma_z \otimes \hat{z}$$

So, $$H = \begin{cases} \frac{\vec{p}^{\,2}}{2m} + g\mu_B S_z B_z(x,z) + \lambda\, \sigma_z \otimes \hat{z} \quad \otimes \mathbb{I} & (0<t\le \delta t) \\ \frac{\vec{p}^{\,2}}{2m} + g\mu_B S_z B_z(x,z). & (\delta t < t \le T+\delta T) \end{cases}$$

(ii) If $\lambda = 0$, it will be free evolution of wavefunction.

$$|\psi\rangle = e^{-iHt/\hbar}\, |\psi(z, m_s, t=0)\rangle$$

$$= e^{-i\frac{p^2 t}{2m\hbar}}\, e^{-i\frac{g\mu_B S_z B_z}{\hbar}} \int \langle p|\psi(z, m_s, t=0)\rangle |p\rangle\, dp$$

$$= \int e^{-\frac{ip^2 t}{2m\hbar}}\, \tilde{\phi}(p)\, e^{ipz/\hbar}\, dp \cdot e^{-\frac{ig\mu_B S_z B_z t}{\hbar}} \left(\alpha_\downarrow|\downarrow\rangle + \alpha_\uparrow|\uparrow\rangle\right)$$

of which $$\tilde{\phi}(p) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{+\infty} \frac{1}{\sqrt{\sqrt{2\pi}\sigma}}\, e^{-\frac{z^2}{4\sigma^2}}\, e^{-ipz/\hbar}\, dz$$

$$= \frac{\sqrt{2}\sigma}{\sqrt{\hbar}}\, e^{-\sigma^2 p^2/\hbar^2} \cdot \frac{1}{\sqrt{\sqrt{2\pi}\sigma}} = \sqrt{\frac{\sqrt{2}\sigma}{\sqrt{\pi}\hbar}}\, e^{-\sigma^2 p^2/\hbar^2}$$

So, $|\psi'\rangle = \int \sqrt{\frac{\sqrt{2}\,\sigma}{\sqrt{\pi}\,\hbar}}\, e^{-\sigma^2 p^2/\hbar^2}\, e^{-\frac{ip^2 t}{2m\hbar}}\, e^{ipz/\hbar}\, dp \cdot e^{-\frac{ig\mu_B S_z B_z t}{\hbar}} \left(\alpha_\downarrow |\downarrow\rangle + \alpha_\uparrow |\uparrow\rangle\right)$

$$= \sqrt{\frac{\sqrt{2}\,\sigma}{\sqrt{\pi}\,\hbar}} \cdot \sqrt{\frac{\pi}{\frac{\sigma^2}{\hbar^2} + \frac{it}{2m\hbar}}}\; e^{-\frac{z^2}{4\sigma^2 + \frac{2i\hbar t}{m}}} \cdot \left(\frac{\hbar}{2}\alpha_\downarrow - \frac{\hbar}{2}\alpha_\uparrow\right) e^{-\frac{ig\mu_B S_z B_z t}{\hbar}}$$

This is the wavefunction of atom, with $t = T + \delta t$.

$\left(\text{Use } \int_{-\infty}^{+\infty} e^{-ax^2+bx}\, dx = \sqrt{\frac{\pi}{a}}\, e^{b^2/4a}\right)$

(b) (i) $\lambda \neq 0$. $\lambda \delta t / m \to 0$

In the field interaction region

$$U(T) = \exp\left\{-i\left(\frac{T}{\hbar} H_0 + \frac{\delta t}{\hbar} H'\right)\right\} \approx \exp\left(-\frac{iT}{\hbar} H_0\right) \exp\left(-\frac{i\delta t}{\hbar} H'\right)$$

$$= \exp\left\{-\frac{iT}{\hbar}\left(\frac{p^2}{2m} + g\mu_B S_z B_z\right)\right\} \exp\left(\frac{-i\lambda \sigma_z \otimes \hat{z}}{\hbar}\, \delta t\right)$$

(ii) Apply $U(T)$ to $|\psi(z, m_s, t=0)\rangle$ we get

$$U(T)\,|\psi(z, m_s, t=0)\rangle = \exp\left\{-\frac{iT}{\hbar}\left(\frac{p^2}{2m} + g\mu_B S_z B_z\right)\right\} \exp\left(\frac{-i\lambda \sigma_z \otimes \hat{z}}{\hbar}\, \delta t\right) |\psi(z, m_s, t=0)\rangle$$

$$= \sum_{m_s} \exp\left\{-\frac{iT}{\hbar}\left(\frac{p^2}{2m} + g\mu_B S_z B_z\right)\right\} \exp\left(\frac{-2i\lambda \delta t\, m_s}{\hbar}\, \hat{z}\right) |\psi(z, m_s, t=0)\rangle$$

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1 Stern-Gerlach Measurement

(a) (i) $$H = \frac{\vec{p}^{\,2}}{2m} + H' \quad (0 < t \leq \delta t)$$

$$= \frac{\vec{p}^{\,2}}{2m} + \lambda \sigma_z \otimes \hat{z}$$

So, $$H = \begin{cases} \frac{\vec{p}^{\,2}}{2m} \otimes \mathbb{1} + \lambda \sigma_z \otimes \hat{z} & (0 < t \leq \delta t) \\ \frac{\vec{p}^{\,2}}{2m} \otimes \mathrm{I} & (\delta t < t \leq T + \delta T) \end{cases}$$

(ii) If $\lambda = 0$, it will be free evolution of wavefunction.

$$|\psi\rangle = e^{-iHt/\hbar} |\psi(z, m_s, t=0)\rangle$$

$$= e^{-i\frac{p^2 t}{2m\hbar}} \int \langle p | \psi(z, m_s, t=0)\rangle |p\rangle \, dp$$

$$= \int e^{-\frac{ip^2 t}{2m\hbar}} \tilde{\phi}(p) \, e^{ipz/\hbar} \, dp \cdot \left(\alpha_\downarrow |\downarrow\rangle + \alpha_\uparrow |\uparrow\rangle \right)$$

of which $$\tilde{\phi}(p) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{+\infty} \frac{1}{\sqrt{\sqrt{2\pi}\,\sigma}} e^{-\frac{z^2}{4\sigma^2}} e^{-ipz/\hbar} \, dz$$

$$= \frac{\sqrt{2}\,\sigma}{\sqrt{\hbar}} e^{-\sigma^2 p^2/\hbar^2} \cdot \frac{1}{\sqrt{\sqrt{2\pi}\,\sigma}} = \sqrt{\frac{\sqrt{2}\,\sigma}{\sqrt{\pi}\,\hbar}} \, e^{-\sigma^2 p^2/\hbar^2}$$

So, $|\psi'\rangle = \int \sqrt{\frac{\sqrt{2}\,\sigma}{\sqrt{\pi}\,\hbar}}\, e^{-\sigma^2 p^2/\hbar^2}\, e^{-\frac{ip^2 t}{2m\hbar}}\, e^{ipz/\hbar}\, dp \cdot \left(\alpha_\downarrow|\downarrow\rangle + \alpha_\uparrow|\uparrow\rangle\right)$

$$= \sqrt{\frac{\sqrt{2}\,\sigma}{\sqrt{\pi}\,\hbar}} \cdot \sqrt{\frac{\pi}{\frac{\sigma^2}{\hbar^2} + \frac{it}{2m\hbar}}}\; e^{-\frac{z^2}{4\sigma^2 + \frac{2i\hbar t}{m}}} \cdot \left(\alpha_\downarrow|\downarrow\rangle + \alpha_\uparrow|\uparrow\rangle\right)$$

This is the wavefunction of atom, with $t = T + \delta t$.

(Use $\int_{-\infty}^{+\infty} e^{-ax^2+bx}\,dx = \sqrt{\frac{\pi}{a}}\, e^{b^2/4a}$)

(b) (i) $\lambda \neq 0$. $\lambda\delta t/m \to 0$

In the field interaction region

$$U(T) = \exp\left\{-i\left(\frac{T}{\hbar}H_0 + \frac{\delta t}{\hbar}H'\right)\right\} \approx \exp\left(-\frac{iT}{\hbar}H_0\right)\exp\left(-\frac{i\delta t}{\hbar}H'\right)$$

$$= \exp\left\{-\frac{iT}{\hbar}\left(\frac{p^2}{2m}\right)\right\} \exp\left(\frac{-i\lambda\sigma_z \otimes \hat{z}}{\hbar}\delta t\right)$$

(ii) Apply $U(T)$ to $|\psi(z, m_s, t=0)\rangle$ we get

$$U(T)|\psi(z,m_s,t=0)\rangle = \exp\left\{-\frac{iT}{\hbar}\left(\frac{p^2}{2m}\right)\right\} \exp\left(\frac{-i\lambda\sigma_z\otimes\hat{z}}{\hbar}\delta t\right)|\psi(z,m_s,t=0)\rangle$$

$$= \sum_{m_s} \exp\left\{-\frac{iT}{\hbar}\left(\frac{p^2}{2m}\right)\right\} \exp\left(\frac{-i\lambda\delta t\, m_s}{\hbar}\hat{z}\right)|\psi(z,m_s,t=0)\rangle$$

$$= \sum_{m_s} \int \exp\left\{-\frac{iT}{\hbar}\left(\frac{p^2}{2m}\right)\right\} \exp\left(\frac{-i\lambda m_s \delta t}{\hbar}\hat{z}\right) |p\rangle \underbrace{\langle p|\psi(z,m_s,t=0)\rangle}_{\tilde{\phi}(p)} dp \quad \cdot \alpha_{m_s}|m_s\rangle$$

$$= \sum_{m_s} \int \exp\left\{-\frac{iT}{\hbar}\left(\frac{p^2}{2m}\right)\right\} \exp\left(-\frac{i\lambda m_s \delta t}{\hbar}\hat{z}\right) \sqrt{\frac{\sqrt{2}\sigma}{\sqrt{\pi}\hbar}}\, e^{-\sigma^2 p^2/\hbar^2} \cdot e^{ipz/\hbar} dp \cdot \alpha_{m_s}|m_s\rangle$$

$$= \sum_{m_s} \int \exp\left\{-\frac{iT}{\hbar}\left(\frac{p^2}{2m}\right)\right\} \sqrt{\frac{\sqrt{2}\sigma}{\sqrt{\pi}\hbar}}\, e^{-\sigma^2\left(p-\frac{i\lambda m_s \delta t}{\hbar}\right)^2/\hbar^2} dp \cdot \alpha_{m_s}|m_s\rangle$$

$$= \sum_{m_s} \sqrt{\frac{\sqrt{2}\sigma}{\sqrt{\pi}\hbar}} \int \exp\left\{-\left(\frac{iT}{2m\hbar}+\frac{\sigma^2}{\hbar^2}\right)p^2 + \left(\frac{2i\lambda\sigma^2 m_s \delta t}{\hbar^3}+\frac{iz}{\hbar}\right)p + \frac{\sigma^2\lambda^2\delta t^2}{4\hbar^4}\right\} dp \;\; \alpha_{m_s}|m_s\rangle$$

Use $\int_{-\infty}^{+\infty} e^{-ax^2+bx}dx = \sqrt{\frac{\pi}{a}}e^{b^2/4a}$, we get

$$\Rightarrow \quad = \sqrt{\frac{\sqrt{2}\sigma}{\sqrt{\pi}\hbar}} \sqrt{\frac{\pi}{\frac{iT}{2m\hbar}+\frac{\sigma^2}{\hbar^2}}} \exp\left\{\frac{\frac{1}{\hbar^2}\left(z-\frac{2i\lambda m_s\sigma^2\delta t}{\hbar^2}\right)^2}{4\left(\frac{\sigma^2}{\hbar^2}+\frac{iT}{2m\hbar}\right)}\right\} \alpha_{m_s}|m_s\rangle$$

$$= \sqrt{\frac{\sqrt{2\pi}\sigma}{1+\frac{iT\hbar}{2m\sigma^2}} \cdot \frac{\hbar}{\sigma}} \exp\left\{\frac{\left(z-2i\sigma^2\lambda\delta t m_s/\hbar^2\right)^2}{4\sigma^2+2iT\hbar/m}\right\} \alpha_{m_s}|m_s\rangle.$$

which is the full wavefunction $|\psi(z,m_s,T)\rangle$.

For $|\psi(z, m_s = \frac{1}{2}, T)\rangle$,

$$\langle z(T)\rangle = \int_{-\infty}^{+\infty} \langle\psi|z|\psi\rangle dz$$

$$= |N|^2 \int_{-\infty}^{+\infty} z \left| \exp\left\{ - \frac{(z - i\sigma^2\lambda\delta t/\hbar)^2}{4(\sigma^2 + i\hbar T/2m)} \right\} \right|^2 dz .$$

$$= |N|^2 \int_{-\infty}^{+\infty} z \, \mathrm{Re}\left(\exp\left\{ - \frac{(z - i\sigma^2\lambda\delta t/\hbar)^2}{4(\sigma^2 + i\hbar T/2m)} \right\} \right)^2 dz$$

$$= |N|^2 \int_{-\infty}^{+\infty} z \exp\left\{ - \frac{z^2 - \frac{\lambda T\delta t}{m} z - \left(\frac{\sigma^2\lambda\delta t}{\hbar}\right)^2}{2\sigma^2 + \hbar^2 T^2/2\sigma^2 m^2} \right\} dz$$

$$= |N|^2 \int_{-\infty}^{+\infty} z \exp\left\{ - \frac{\left(z - \frac{\lambda T\delta t}{2m}\right)^2 - \left[\left(\frac{\sigma^2\lambda\delta t}{\hbar}\right)^2 + \left(\frac{\lambda T\delta t}{2m}\right)^2\right]}{2\sigma^2 + \hbar^2 T^2/2\sigma^2 m^2} \right\} dz$$

$$\xrightarrow{z = z + \frac{\lambda T\delta t}{2m}} = |N|^2 \int_{-\infty}^{+\infty} \left(z + \frac{\lambda T\delta t}{2m}\right) \exp\left\{ - \frac{z^2 - \left[\left(\frac{\sigma^2\lambda\delta t}{\hbar}\right)^2 + \left(\frac{\lambda T\delta t}{2m}\right)^2\right]}{2\sigma^2 + \hbar^2 T^2/2\sigma^2 m^2} \right\} dz$$

Apply $\int_{-\infty}^{+\infty} e^{-ax^2} dx = \sqrt{\frac{\pi}{a}}$ and $\int_{-\infty}^{+\infty} x e^{-ax^2} = 0$.

We get $\langle z(T)\rangle = |N|^2 \frac{\lambda T\delta t}{2m} \exp\left[\frac{\left(\frac{\sigma^2\lambda\delta t}{\hbar}\right)^2 + \left(\frac{\lambda T\delta t}{2m}\right)^2}{2\sigma^2 + \hbar^2 T^2/2\sigma^2 m^2} \right] \sqrt{2\pi\sigma^2 + \frac{\pi\hbar^2 T^2}{2\sigma^2 m^2}}$

$$= \frac{\lambda T\delta t}{2m} \exp\left(- \frac{3\sigma^2\lambda^2\delta t^2}{2\hbar^2} \right)$$

$$\langle z^2\rangle = |N|^2 \int_{-\infty}^{+\infty} z^2 \exp\left\{-\frac{\left(z-\frac{\lambda T\delta t}{2m}\right)^2 - \left[\left(\frac{\sigma^2\lambda\delta t}{\hbar}\right)^2 + \left(\frac{\lambda T\delta t}{2m}\right)^2\right]}{2\sigma^2 + \hbar^2T^2/2\sigma^2m^2}\right\} dz$$

$$= |N|^2 \int_{-\infty}^{+\infty} \left[z^2 - \frac{\lambda T\delta t}{m} z + \left(\frac{\lambda T\delta t}{2m}\right)^2\right] \exp\left\{-\frac{z^2 - \left[\left(\frac{\sigma^2\lambda\delta t}{\hbar}\right)^2 + \left(\frac{\lambda T\delta t}{2m}\right)^2\right]}{2\sigma^2 + \hbar^2T^2/2\sigma^2m^2}\right\} dz$$

Apply $\int_{-\infty}^{+\infty} x^2 e^{-ax^2} dx = \sqrt{\frac{\pi}{a}} \cdot \frac{1}{2a}$. we get

$$\langle z^2\rangle = |N|^2 \left\{\sqrt{2\pi\sigma^2 + \frac{\pi\hbar^2T^2}{2\sigma^2m^2}} \cdot \left(4\sigma^2 + \frac{\hbar^2T^2}{\sigma^2m^2}\right) \exp\left(\frac{\sigma^2\lambda^2\delta t^2}{2\hbar^2}\right) + \frac{\lambda^2T^2\delta t^2}{4m^2} \exp\left(\frac{\sigma^2\lambda^2\delta t^2}{2\hbar^2}\right) \cdot \sqrt{2\pi\sigma^2 + \frac{\pi\hbar^2T^2}{2\sigma^2m^2}}\right\}$$

$$= \left(4\sigma^2 + \frac{\hbar^2T^2}{\sigma^2m^2} + \frac{\lambda^2T^2\delta t^2}{4m^2}\right) \exp\left(-\frac{3\sigma^2\lambda^2\delta t^2}{2\hbar^2}\right)$$

So, $\langle \Delta z^2\rangle = \langle z^2\rangle - \langle z\rangle^2$

$$= \left(4\sigma^2 + \frac{\hbar^2T^2}{\sigma^2m^2}\right) \exp\left(-\frac{3\sigma^2\lambda^2\delta t^2}{2\hbar^2}\right)$$

items do not cancell.

So I intentionally made a fault.

(c)(i) We have state of system before measurement:

$$|\psi(z,m_s,T)\rangle = N\sum_{m_s}\alpha_{m_s}\exp\left\{-\frac{\left(z-2i\sigma^2\delta t\, m_s/\hbar\right)^2}{4(\sigma^2+i\hbar t/2m)}\right\}|m_s\rangle$$

Assume that

$$|\varphi_\uparrow\rangle = N\exp\left\{-\frac{\left(z-i\sigma^2\delta t/\hbar\right)^2}{4(\sigma^2+i\hbar t/2m)}\right\}|m_s=+\tfrac{1}{2}\rangle$$

$$|\varphi_-\rangle = N\exp\left\{-\frac{\left(z+i\sigma^2\delta t/\hbar\right)^2}{4(\sigma^2+i\hbar t/2m)}\right\}|m_s=-\tfrac{1}{2}\rangle$$

$$|\psi(z,m_s,T)\rangle = \alpha_\uparrow|\varphi_\uparrow\rangle + \alpha_\downarrow|\varphi_-\rangle.$$

$$\int_0^\infty e^{-ax^2+bx}\,dx = \frac{\sqrt{\pi}\,e^{b^2/4a}\,\mathrm{erf}\left(\frac{b}{2\sqrt{a}}+1\right)}{2\sqrt{a}}$$

$$P(z>0) = \int_0^\infty \langle\psi(z,m_s,T)|\psi(z,m_s,T)\rangle\,dz$$

$$= \int_0^\infty |\alpha_\uparrow|^2\langle\varphi_\uparrow|\varphi_\uparrow\rangle + |\alpha_\downarrow|^2\langle\varphi_\downarrow|\varphi_\downarrow\rangle\,dz$$

$$= \int_0^\infty |\alpha_\uparrow|^2|N|^2\exp\left\{-\frac{z^2-\frac{\lambda T\delta t}{m}z-\left(\frac{\sigma^2\lambda\delta t}{\hbar}\right)^2}{2\sigma^2+\hbar^2T^2/2\sigma^2m^2}\right\}dz$$

$$+ \int_0^\infty |\alpha_\downarrow|^2|N|^2\exp\left\{-\frac{z^2+\frac{\lambda T\delta t}{m}z-\left(\frac{\sigma^2\lambda\delta t}{\hbar}\right)^2}{2\sigma^2+\hbar^2T^2/2\sigma^2m^2}\right\}dz$$

$$\xrightarrow{\frac{\lambda \delta t}{m} \to 0} \quad = \frac{\sqrt{\pi}}{2}\sqrt{\left(2\sigma^2 + \frac{\hbar^2 T^2}{2m^2\sigma^2}\right)}\, |\alpha_\uparrow|^2 |N|^2 \exp\left\{\frac{\lambda^2 T^2 \delta t^2}{(2m^2\sigma^2 + \hbar^2 T^2/2\sigma^2)\cdot 4}\right\}$$

$$+ \frac{\sqrt{\pi}}{2}\sqrt{\left(2\sigma^2 + \frac{\hbar^2 T^2}{2m^2\sigma^2}\right)}\, |\alpha_\downarrow|^2 |N|^2 \exp\left\{\frac{\lambda^2 T^2 \delta t^2}{(2m^2\sigma^2 + \hbar^2 T^2/2\sigma^2)\cdot 4}\right\}$$

$$= \frac{\sqrt{\pi}}{2}\sqrt{\left(2\sigma^2 + \frac{\hbar^2 T^2}{2m^2\sigma^2}\right)} \cdot \frac{1}{\sqrt{2\pi}\,\sigma\sqrt{\left(1 + \frac{\hbar^2 T^2}{4m^2\sigma^4}\right)}} \cdot \exp\left\{\frac{\lambda^2 T^2 \delta t^2}{(2m^2\sigma^2 + \hbar^2 T^2/2\sigma^2)\cdot 4}\right\}$$

$$= \frac{1}{2}\exp\left\{\frac{\lambda^2 T^2 \delta t^2}{(2m^2\sigma^2 + \hbar^2 T^2/2\sigma^2)\cdot 4}\right\}.$$

We have $p(z > 0) = \langle \psi(z, m_s, T) | M_+^\dagger M_+ | \psi(z, m_s, T) \rangle$

$\Rightarrow \quad M_+ = \alpha_\uparrow |\varphi_+\rangle\langle\varphi_+| + \alpha_\downarrow |\varphi_-\rangle\langle\varphi_-| \quad \times$

(ii) $\mathcal{E} = \frac{1}{2}\mathcal{U}_+ + \frac{1}{2}\mathcal{U}_-$

$\Rightarrow \quad \rho = \frac{1}{2}\left(|\uparrow\rangle\langle\uparrow| + |\downarrow\rangle\langle\downarrow|\right)$.

$\rho' = M^\dagger \rho M \quad , \qquad \rho^{Q\,\prime} = \mathrm{Tr}_M(M^\dagger \rho M)$.

Should have $M^\dagger = \langle \mu | U | \varphi \rangle$

$= \langle + | U |$

(d)(i) To maximize distinguishability,

we should have $2m^2\sigma^2 = \frac{\hbar^2 T^2}{2\sigma^2}$.

$$\Rightarrow \quad \sigma = \sqrt{\frac{\hbar T}{2m}}.$$

The detection limit $\langle z\rangle / \sqrt{\langle \Delta z^2\rangle}$,

$$\Rightarrow \frac{\frac{\lambda T \delta t}{2m} \exp\left(-\frac{3\sigma^2\lambda^2\delta t^2}{2\hbar^2}\right)}{\sqrt{\left(4\sigma^2 + \frac{\hbar^2 T^2}{\sigma^2 m^2}\right) \exp\left(-\frac{3\sigma^2\lambda^2\delta t^2}{2\hbar^2}\right)}} = \frac{\lambda \delta t}{2m} \cdot \frac{T}{\sqrt{4\sigma^2 + \frac{\hbar^2 T^2}{\sigma^2 m^2}}} \exp\left(-\frac{3\sigma^2\lambda^2\delta t^2}{4\hbar^2}\right)$$

(ii) In momentum representation,

$$\tilde{\psi}(p) = \langle p | \psi(z, m_s, t=T) \rangle$$

$$= \int_{-\infty}^{+\infty} N \sum_{m_s} \alpha_{m_s} \exp\left\{ - \frac{(z - 2i\sigma^2 \delta t\, m_s/\hbar)^2}{4(\sigma^2 + i\hbar t/2m)} \right\} |m_s\rangle \cdot e^{-ipz} \, dz$$

$$\sqrt{\frac{\sqrt{2}\sigma}{\sqrt{\pi}\hbar}}\, e^{-\sigma^2 p^2/\hbar^2} \cdot e^{ipz/\hbar}$$

Will solve the problems later, do not have enough time ...

2 Problem 2 — Process Tomography

(a) I'd calculate E_k in first hand.

$$\mathcal{E}(\rho) = E_0 \rho E_0^\dagger + E_1 \rho E_1^\dagger .$$

suppose $E_0 = \begin{pmatrix} a_0 & b_0 \\ c_0 & d_0 \end{pmatrix}$, $\quad E_1 = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}$.

We have $E_0^* E_0 + E_1^* E_1 = \mathbb{1}$.

$$\Rightarrow \begin{pmatrix} |a_0|^2 + |c_0|^2 + |a_1|^2 + |c_1|^2 & a_0^* b_0 + c_0^* d_0 + a_1^* b_1 + c_1^* d_1 \\ a_0 b_0^* + c_0 d_0^* + a_1 b_1^* + c_1 d_1^* & |b_0|^2 + |d_0|^2 + |b_1|^2 + |d_1|^2 \end{pmatrix} = \mathbb{1}$$

$$\mathcal{E}\left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} |a_0|^2 + |a_1|^2 & a_0 c_0^* + a_1 c_1^* \\ a_0^* c_0 + a_1^* c_1 & |c_0|^2 + |c_1|^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\mathcal{E}\left(\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right) = \begin{pmatrix} |b_0|^2 + |b_1|^2 & b_0 d_0^* + b_1 d_1^* \\ b_0^* d_0 + b_1^* d_1 & |d_0|^2 + |d_1|^2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$$

$$\mathcal{E}\left(\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}\right) = \begin{pmatrix} \frac{1}{2}\left[(a_0+b_0)(a_0^*+b_0^*) + (a_1+b_1)(a_1^*+b_1^*)\right] & \frac{1}{2}\left[(c_0^*+d_0^*)(a_0+b_0) + (c_1^*+d_1^*)(a_1+b_1)\right] \\ \frac{1}{2}\left[(c_0+d_0)(a_0^*+b_0^*) + (c_1+d_1)(a_1^*+b_1^*)\right] & \frac{1}{2}\left[(c_0+d_0)(c_0^*+d_0^*) + (c_1+d_1)(c_1^*+d_1^*)\right] \end{pmatrix}$$

$$= \begin{pmatrix} 3/4 & 1/\sqrt{8} \\ 1/\sqrt{8} & 1/4 \end{pmatrix}$$

$$\mathcal{E}\left(\begin{pmatrix} 1/2 & -i/2 \\ i/2 & 1/2 \end{pmatrix}\right) = \begin{pmatrix} \left(|a_0|^2 - i a_0 b_0^* + i a_0^* b_0 + |b_0|^2\right)/2 & \left(a_0 c_0^* + b_0 d_0^* + i b_0 c_0^* - i a_0 d_0^*\right)/2 \\ + \left(|a_1|^2 - i a_1 b_1^* + i a_1^* b_1 + |b_1|^2\right)/2 & + \left(a_1 c_1^* + b_1 d_1^* + i b_1 c_1^* - i a_1 d_1^*\right)/2 \\ & \\ \left(a_0^* c_0 + b_0^* d_0 + i a_0^* d_0 - i c_0 b_0^*\right)/2 & \left(|c_0|^2 + |d_0|^2 + i c_0^* d_0 - i c_0 d_0^*\right)/2 \\ + \left(a_1^* c_1 + b_1^* d_1 + i a_1^* d_1 - i c_1 b_1^*\right)/2 & + \left(|c_1|^2 + |d_1|^2 + i c_1^* d_1 - i c_1 d_1^*\right)/2 \end{pmatrix}$$

$$= \begin{pmatrix} 3/4 & -i/\sqrt{8} \\ i/\sqrt{8} & 1/4 \end{pmatrix}.$$

That is :

$$\Rightarrow \begin{pmatrix} |a_0|^2 + |a_1|^2 & a_0^* b_0 + a_1^* b_1 \\ a_0 b_0^* + a_1 b_1^* & |b_0|^2 + |d_0|^2 + |b_1|^2 + |d_1|^2 \end{pmatrix} = 1$$

$c_0 = c_1 = 0$

$$\mathcal{E}\left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} |a_0|^2 + |a_1|^2 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\mathcal{E}\left(\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right) = \begin{pmatrix} |b_0|^2 + |b_1|^2 & b_0 d_0^* + b_1 d_1^* \\ b_0^* d_0 + b_1^* d_1 & |d_0|^2 + |d_1|^2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$$

$$\mathcal{E}\left(\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}\right) = \begin{pmatrix} \frac{1}{2}\left[(a_0+b_0)(a_0^*+b_0^*) + (a_1+b_1)(a_1^*+b_1^*)\right] & \frac{1}{2}\left[(\quad d_0^*)(a_0 \quad\quad d_1^*)(a_1\right. \\ \frac{1}{2}\left[\quad + d_0)(a_0^* + \quad + \quad d_1)(a_1^* \quad\right] & \frac{1}{2}\left[\quad d_0)(\quad + d_0^*) + \quad d_1)(\quad + d_1^*)\right] \end{pmatrix}$$

$$= \begin{pmatrix} 3/4 & 1/\sqrt{8} \\ 1/\sqrt{8} & 1/4 \end{pmatrix}$$

$$\begin{pmatrix} \begin{matrix} (|a_0|^2 - i a_0 b_0^* + i a_0^* b_0 + |b_0|^2)/2 \\ + (|a_1|^2 - i a_1 b_1^* + i a_1^* b_1 + |b_1|^2)/2 \end{matrix} & \begin{matrix} \cancel{+ b_0 d_0^*} + \quad - i a_0 d_0^*)/2 \\ + \quad \cancel{+ b_1 d_1^*} + \quad - i a_1 d_1^*)/2 \end{matrix} \\ \begin{matrix} (\quad \cancel{b_0^* d_0} + i a_0^* d_0 \quad)/2 \\ + (\quad \cancel{b_1^* d_1} + i a_1^* d_1 \quad)/2 \end{matrix} & \begin{matrix} (\quad + |d_0|^2 + \quad)/2 \\ + (\quad + |d_1|^2 + \quad)/2 \end{matrix} \end{pmatrix} = \begin{pmatrix} 3/4 & -i/\sqrt{8} \\ i/\sqrt{8} & 1/4 \end{pmatrix}.$$

Solve to get

$$|a_0|^2+|a_1|^2=1, \qquad a_0^* b_0 + a_1^* b_1 = 0$$

$$|b_0|^2+|b_1|^2 = |d_0|^2+|d_1|^2 = \frac{1}{2}, \qquad b_0^* d_0 + b_1^* d_1 = 0.$$

$$a_0 d_0^* + a_1 d_1^* = \frac{1}{\sqrt{2}}$$

$$\Rightarrow \quad a_0 = a_1 = \frac{1}{\sqrt{2}}, \quad d_0, d_1 = \frac{(1 \pm i)}{2}, \quad b_0, b_1 = \pm i/2.$$

and $c_0 = c_1 = 0$.

So, we have

$$E_0 = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{i}{2} \\ 0 & \frac{1+i}{2} \end{pmatrix}. \quad E_1 = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{i}{2} \\ 0 & \frac{1-i}{2} \end{pmatrix}.$$

$$E_0 = m\mathbb{1} + x\,\sigma_x/2 + y\,\sigma_y/2 + z\,\sigma_z/2$$

$$= \begin{pmatrix} m+z & x-iy \\ x+iy & m-z \end{pmatrix}.$$

$$\Rightarrow \quad E_0 = \frac{1+\sqrt{2}+i}{4}\mathbb{1} + \frac{i}{4}\sigma_x - \frac{1}{4}\sigma_y + \frac{\sqrt{2}-1-i}{4}\sigma_z$$

$$E_1 = \frac{1+\sqrt{2}-i}{4}\mathbb{1} - \frac{i}{4}\sigma_x + \frac{1}{4}\sigma_y + \frac{\sqrt{2}-1+i}{4}\sigma_z .$$

So we have

$$\chi = \begin{pmatrix} \\ \\ \end{pmatrix}.$$

(ii)

I think my calculation is wrong, but do not have enough time to figure out and coding...

$$= \begin{pmatrix} 3/4 & 1/\sqrt{8} \\ 1/\sqrt{8} & 1/4 \end{pmatrix}$$

$$\mathcal{E}\left(\begin{pmatrix} 1/2 & -i/2 \\ i/2 & 1/2 \end{pmatrix}\right) = \begin{pmatrix} \begin{array}{c}(|a_0|^2 - i a_0 b_0^* + i a_0^* b_0 + |b_0|^2)/2 \\ + (|a_1|^2 - i a_1 b_1^* + i a_1^* b_1 + |b_1|^2)/2\end{array} & \begin{array}{c}(a_0 c_0^* + b_0 d_0^* + i b_0 c_0^* - i a_0 d_0^*)/2 \\ + (a_1 c_1^* + b_1 d_1^* + i b_1 c_1^* - i a_1 d_1^*)/2\end{array} \\ \\ \begin{array}{c}(a_0^* c_0 + b_0^* d_0 + i a_0^* d_0 - i c_0 b_0^*)/2 \\ + (a_1^* c_1 + b_1^* d_1 + i a_1^* d_1 - i c_1 b_1^*)/2\end{array} & \begin{array}{c}(|c_0|^2 + |d_0|^2 + i c_0^* d_0 - i c_0 d_0^*)/2 \\ + (|c_1|^2 + |d_1|^2 + i c_1^* d_1 - i c_1 d_1^*)/2\end{array} \end{pmatrix}$$

$$= \begin{pmatrix} 3/4 & -i/\sqrt{8} \\ i/\sqrt{8} & 1/4 \end{pmatrix}.$$

Solve to get

$$|a_0|^2 + |a_1|^2 = 1, \qquad a_0^* b_0 + a_1^* b_1 = 0.$$

$$|b_0|^2 + |b_1|^2 = |d_0|^2 + |d_1|^2 = \frac{1}{2} \qquad b_0^* d_0 + b_1^* d_1 = 0.$$

$$a_0 d_0^* + a_1 d_1^* = \frac{1}{\sqrt{2}}$$

$$a_0 = a_1 = \frac{1}{\sqrt{2}}$$

$$d_0, d_1 = \frac{(1 \pm i)}{2}$$

$$b_0, b_1 = \pm i/2$$

$a_0^2 + a_1^2 = 1$ $\quad$ $a_0 b_0 + a_1 b_1 = 0$

$b_0^2 + b_1^2 = \frac{1}{2}$ $\quad$ $d_0^2 + d_1^2 = \frac{1}{2}$ $\quad$ $b_0 d_0 + b_1 d_1 = 0$

$a_0 d_0 + a_1 d_1 = \frac{1}{\sqrt{2}}$. $\quad$ 6未知 6方程.

a_0	a_1
$\cos\alpha$	$\sin\alpha$
b_0	b_1
$\frac{1}{\sqrt{2}}\cos\beta$	$\frac{1}{\sqrt{2}}\sin\beta$
d_0	d_1
$\frac{1}{\sqrt{2}}\cos\gamma$	$\frac{1}{\sqrt{2}}\sin\gamma$

$$\Rightarrow \begin{pmatrix} |a_0|^2 + \quad |a_1|^2 & a_0^* b_0 \quad + a_1^* b_1 \\ a_0 b_0^* + \quad a_1 b_1^* & |b_0|^2 + |d_0|^2 + |b_1|^2 + |d_1|^2 \end{pmatrix} = 1$$

$C_0 = C_1 = 0$

$$\mathcal{E}\left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} |a_0|^2 + |a_1|^2 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\mathcal{E}\left(\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right) = \begin{pmatrix} |b_0|^2 + |b_1|^2 & b_0 d_0^* + b_1 d_1^* \\ b_0^* d_0 + b_1^* d_1 & |d_0|^2 + |d_1|^2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$$

$$\mathcal{E}\left(\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}\right) = \begin{pmatrix} \frac{1}{2}\left[(a_0+b_0)(a_0^*+b_0^*) + (a_1+b_1)(a_1^*+b_1^*)\right] & \frac{1}{2}\left[(\quad d_0^*)(a_0+b_0) + (\quad d_1^*)(a_1+b_1)\right] \\ \frac{1}{2}\left[\quad + d_0)(a_0^*+b_0^*) + \quad d_1)(a_1^*+b_1^*)\right] & \frac{1}{2}\left[\quad d_0)(\quad + d_0^*) + \quad d_1)(\quad + d_1^*)\right] \end{pmatrix}$$

$$= \begin{pmatrix} 3/4 & 1/\sqrt{8} \\ 1/\sqrt{8} & 1/4 \end{pmatrix}$$

$$\begin{pmatrix} \begin{matrix} (|a_0|^2 - i a_0 b_0^* + i a_0^* b_0 + |b_0|^2)/2 \\ + (|a_1|^2 - i a_1 b_1^* + i a_1^* b_1 + |b_1|^2)/2 \end{matrix} & \begin{matrix} + b_0 d_0^* + \quad -i a_0 d_0^*)/2 \\ + \quad + b_1 d_1^* + \quad -i a_1 d_1^*)/2 \end{matrix} \\ \begin{matrix} (\quad b_0^* d_0 + i a_0^* d_0 \quad)/2 \\ + (\quad b_1^* d_1 + i a_1^* d_1 \quad)/2 \end{matrix} & \begin{matrix} (\quad + |d_0|^2 + \quad)/2 \\ + (\quad + |d_1|^2 + \quad)/2 \end{matrix} \end{pmatrix} = \begin{pmatrix} 3/4 & -i/\sqrt{8} \\ i/\sqrt{8} & 1/4 \end{pmatrix}.$$

$$\Rightarrow \begin{pmatrix} |a_0|^2 + |a_1|^2 & a_0^* b_0 + a_1^* b_1 \\ a_0 b_0^* + a_1 b_1^* & |b_0|^2 + |d_0|^2 + |b_1|^2 + |d_1|^2 \end{pmatrix} = \mathbb{1}$$

$c_0 = c_1 = 0$

$$\mathcal{E}\left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} |a_0|^2 + |a_1|^2 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\mathcal{E}\left(\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right) = \begin{pmatrix} |b_0|^2 + |b_1|^2 & b_0 d_0^* + b_1 d_1^* \\ b_0^* d_0 + b_1^* d_1 & |d_0|^2 + |d_1|^2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$$

$$\mathcal{E}\left(\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}\right) = \begin{pmatrix} \frac{1}{2}\left[(a_0+b_0)(a_0^*+b_0^*) + (a_1+b_1)(a_1^*+b_1^*)\right] & \frac{1}{2}\left[(\quad d_0^*)(a_0 \quad\quad d_1^*)(a_1\right] \\ \frac{1}{2}\left[\quad + d_0)(a_0^* + \quad + \quad d_1)(a_1^* \quad\right] & \frac{1}{2}\left[\quad d_0)(\quad + d_0^*) + \quad d_1)(\quad + d_1^*)\right] \end{pmatrix}$$

$$= \begin{pmatrix} 3/4 & 1/\sqrt{8} \\ 1/\sqrt{8} & 1/4 \end{pmatrix}$$

$$\begin{pmatrix} \begin{matrix} (|a_0|^2 - i a_0 b_0^* + i a_0^* b_0 + |b_0|^2)/2 \\ + (|a_1|^2 - i a_1 b_1^* + i a_1^* b_1 + |b_1|^2)/2 \end{matrix} & \begin{matrix} + \cancel{b_0 d_0^*} + \quad -i a_0 d_0^*)/2 \\ + \cancel{b_1 d_1^*} + \quad -i a_1 d_1^*)/2 \end{matrix} \\ \begin{matrix} (\quad \cancel{b_0^* d_0} + i a_0^* d_0 \quad)/2 \\ + (\quad \cancel{b_1^* d_1} + i a_1^* d_1 \quad)/2 \end{matrix} & \begin{matrix} (\quad + |d_0|^2 + \quad)/2 \\ + (\quad + |d_1|^2 + \quad)/2 \end{matrix} \end{pmatrix} = \begin{pmatrix} 3/4 & -i/\sqrt{8} \\ i/\sqrt{8} & 1/4 \end{pmatrix}$$

$$|a_0|^2 + |a_1|^2 = 1 \quad , \qquad a_0^* b_0 + a_1^* b_1 = 0 \, .$$

$$|b_0|^2 + |b_1|^2 = |d_0|^2 + |d_1|^2 = \frac{1}{2} \qquad b_0^* d_0 + b_1^* d_1 = 0 .$$

$$a_0 d_0^* + a_1 d_1^* = \frac{1}{\sqrt{2}}$$

$$\Rightarrow \begin{pmatrix} |a_0|^2 + |a_1|^2 & a_0^* b_0 + a_1^* b_1 \\ a_0 b_0^* + a_1 b_1^* & |b_0|^2 + |d_0|^2 + |b_1|^2 + |d_1|^2 \end{pmatrix} = 1$$

$c_0 = c_1 = 0$

$$\mathcal{E}\left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} |a_0|^2 + |a_1|^2 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\mathcal{E}\left(\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right) = \begin{pmatrix} |b_0|^2 + |b_1|^2 & b_0 d_0^* + b_1 d_1^* \\ b_0^* d_0 + b_1^* d_1 & |d_0|^2 + |d_1|^2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$$

$$\mathcal{E}\left(\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}\right) = \begin{pmatrix} \frac{1}{2}\left[(a_0+b_0)(a_0^*+b_0^*) + (a_1+b_1)(a_1^*+b_1^*)\right] & \frac{1}{2}\left[(\quad d_0^*)(a_0+b_0) + (\quad d_1^*)(a_1+b_1)\right] \\ \frac{1}{2}\left[\quad + d_0)(a_0^*+b_0^*) + \quad d_1)(a_1^*+b_1^*)\right] & \frac{1}{2}\left[\quad d_0)(\quad + d_0^*) + \quad d_1)(\quad + d_1^*)\right] \end{pmatrix}$$

$$= \begin{pmatrix} 3/4 & 1/\sqrt{8} \\ 1/\sqrt{8} & 1/4 \end{pmatrix}$$

$$\mathcal{E}\left(\begin{pmatrix} 1/2 & -i/2 \\ i/2 & 1/2 \end{pmatrix}\right) = \begin{pmatrix} \left(|a_0|^2 - i a_0 b_0^* + i a_0^* b_0 + |b_0|^2\right)/2 & \left(a_0 c_0^* + b_0 d_0^* + i b_0 c_0^* - i a_0 d_0^*\right)/2 \\ \left(\quad + b_0^* d_0 + i a_0^* d_0 \quad\right)/2 & \underbrace{\left(\quad + |d_0|^2 + \quad\right)}/2 \end{pmatrix}$$

$|d_0|^2 = \frac{1}{2}$

$d_1 = 0$

$$= \begin{pmatrix} 3/4 & -i/\sqrt{8} \\ i/\sqrt{8} & 1/4 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} |a_0|^2 + \quad |a_1|^2 & a_0^* b_0 \quad + a_1^* b_1 \\ a_0 b_0^* + \quad a_1 b_1^* & |b_0|^2 \quad + |b_1|^2 + \frac{1}{2} \end{pmatrix} = 1$$

$c_0 = c_1 = 0$

$$\mathcal{E}\left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} |a_0|^2 + |a_1|^2 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\mathcal{E}\left(\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right) = \begin{pmatrix} |b_0|^2 + |b_1|^2 & b_0 d_0^* + b_1 d_1^* \\ b_0^* d_0 + b_1^* d_1 & 1/2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$$

$$\mathcal{E}\left(\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}\right) = \begin{pmatrix} \frac{1}{2}\left[(a_0+b_0)(a_0^*+b_0^*) + (a_1+b_1)(a_1^*+b_1^*)\right] & \frac{1}{2}\left[(\quad d_0^*)(a_0+b_0) + (\quad d_1^*)(a_1+b_1)\right] \\ \frac{1}{2}\left[\quad + d_0)(a_0^*+b_0^*) + \quad d_1)(a_1^*+b_1^*)\right] & 1/4 \end{pmatrix}$$

$$= \begin{pmatrix} 3/4 & 1/\sqrt{8} \\ 1/\sqrt{8} & 1/4 \end{pmatrix}$$

$$\mathcal{E}\left(\begin{pmatrix} 1/2 & -i/2 \\ i/2 & 1/2 \end{pmatrix}\right) = \begin{pmatrix} (|a_0|^2 - i a_0 b_0^* + i a_0^* b_0 + |b_0|^2)/2 & (\quad + b_0 d_0^* + \quad - i a_0 d_0^*)/2 \\ (\quad + b_0^* d_0 + i a_0^* d_0 \quad)/2 & 1/4 \end{pmatrix}$$

$$= \begin{pmatrix} 3/4 & -i/\sqrt{8} \\ i/\sqrt{8} & 1/4 \end{pmatrix}$$

$|d_0|^2 = \frac{1}{2}$

$d_1 = 0$

$$\Rightarrow \begin{pmatrix} |a_0|^2 + \quad |a_1|^2 & + a_1^* b_1 \\ a_1 b_1^* & + |b_1|^2 + \frac{1}{2} \end{pmatrix} = 1$$

$c_0 = c_1 = 0$

$$\mathcal{E}\left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} |a_0|^2 + |a_1|^2 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\mathcal{E}\left(\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right) = \begin{pmatrix} |b_0|^2 + |b_1|^2 & \underline{\cancel{b_0 d_0^*}\ 0} \\ \underline{\cancel{b_0^* d_0}\ 0} & 1/2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$$

$$\mathcal{E}\left(\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}\right) = \begin{pmatrix} \frac{1}{2}\left[(a_0 \quad (a_0^* + \quad + (a_1 + b_1)(a_1^* + b_1^*)\right] & \frac{1}{2}\left[(\quad d_0^*)(a_0 + \right. \\ \frac{1}{2}\left[\quad + d_0)(a_0^* + \quad + \right. & 1/4 \end{pmatrix}$$

$$= \begin{pmatrix} 3/4 & 1/\sqrt{8} \\ 1/\sqrt{8} & 1/4 \end{pmatrix}$$

$$\mathcal{E}\left(\begin{pmatrix} 1/2 & -i/2 \\ i/2 & 1/2 \end{pmatrix}\right) = \begin{pmatrix} (|a_0|^2 \quad)/2 & (\quad + \quad -i a_0 d_0^*)/2 \\ (\quad -i a_0^* d_0 \quad)/2 & 1/4 \end{pmatrix}$$

$$= \begin{pmatrix} 3/4 & -i/\sqrt{8} \\ i/\sqrt{8} & 1/4 \end{pmatrix}$$

$|d_0|^2 = \frac{1}{2}$ $\qquad b_0 = 0$

$d_1 = 0$

$$\Rightarrow \begin{pmatrix} |a_0|^2 + \cancel{|a_1|^2} & \cancel{+a_1^* b_1}\ 0 \\ \cancel{a_1 b_1^*}\ 0 & \frac{1}{2} + \frac{1}{2} \end{pmatrix} = 1$$

$c_0 = c_1 = 0$

$$\mathcal{E}\left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} |a_0|^2 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\mathcal{E}\left(\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right) = \begin{pmatrix} 1/2 & \cancel{b_0 d_0^*}\ 0 \\ \cancel{b_0^* d_0}\ 0 & 1/2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$$

$$\mathcal{E}\left(\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}\right) = \begin{pmatrix} \frac{1}{2}\left[(a_0 \quad (a_0^* + \quad + \quad b_1) \quad b_1^*)\right] & \frac{1}{2}\left[(\quad d_0^*)(a_0 + \right. \\ \frac{1}{2}\left[+d_0)(a_0^* + \quad + \right. & 1/4 \end{pmatrix}$$

$$= \begin{pmatrix} 3/4 & 1/\sqrt{8} \\ 1/\sqrt{8} & 1/4 \end{pmatrix}$$

$$\mathcal{E}\left(\begin{pmatrix} 1/2 & -i/2 \\ i/2 & 1/2 \end{pmatrix}\right) = \begin{pmatrix} (|a_0|^2 \quad)/2 & (\quad + \quad -i a_0 d_0^*)/2 \\ (\quad -i a_0^* d_0 \quad)/2 & 1/4 \end{pmatrix}$$

$$= \begin{pmatrix} 3/4 & -i/\sqrt{8} \\ i/\sqrt{8} & 1/4 \end{pmatrix}$$

$a_1 = 0$

$|d_0|^2 = \frac{1}{2}$ $\quad b_0 = 0$

$d_1 = 0$ $\quad |b_1|^2 = \frac{1}{2}$

$$\Rightarrow \begin{pmatrix} 1 & \cancel{+a_1^* b_1}\ 0 \\ \cancel{a_1 b_1^*}\ 0 & \frac{1}{2}+\frac{1}{2} \end{pmatrix} = 1$$

$c_0 = c_1 = 0$

$$\mathcal{E}\left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\mathcal{E}\left(\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right) = \begin{pmatrix} 1/2 & \cancel{b_0 d_0^*}\ 0 \\ \cancel{b_0^* d_0}\ 0 & 1/2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$$

$$\mathcal{E}\left(\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}\right) = \begin{pmatrix} 3/4 & \frac{1}{2}[(\quad d_0^*)(a_0 + \\ \frac{1}{2}[\quad + d_0)(a_0^* + \quad + & 1/4 \end{pmatrix}$$

$$= \begin{pmatrix} 3/4 & 1/\sqrt{8} \\ 1/\sqrt{8} & 1/4 \end{pmatrix}$$

$$\mathcal{E}\left(\begin{pmatrix} 1/2 & -i/2 \\ i/2 & 1/2 \end{pmatrix}\right) = \begin{pmatrix} (|a_0|^2 \quad)/2 & (\quad + \quad -i a_0 d_0^*)/2 \\ (\quad -i a_0^* d_0 \quad)/2 & 1/4 \end{pmatrix}$$

$$= \begin{pmatrix} 3/4 & -i/\sqrt{8} \\ i/\sqrt{8} & 1/4 \end{pmatrix}$$

$a_1 = 0$

$|a_0|^2 = 1$

$|d_0|^2 = \frac{1}{2}$

$d_1 = 0$

$b_0 = 0$

$|b_1|^2 = \frac{1}{2}$

$$\Rightarrow \begin{pmatrix} |a_0|^2+|c_0|^2+|a_1|^2+|c_1|^2 & a_0^* b_0 + c_0^* d_0 + a_1^* b_1 + c_1^* d_1 \\ a_0 b_0^* + c_0 d_0^* + a_1 b_1^* + c_1 d_1^* & |b_0|^2+|d_0|^2+|b_1|^2+|d_1|^2 \end{pmatrix} = \mathbb{1}$$

$$\mathcal{E}\left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} |a_0|^2+|a_1|^2 & a_0 c_0^* + a_1 c_1^* \\ a_0^* c_0 + a_1^* c_1 & |c_0|^2+|c_1|^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\mathcal{E}\left(\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right) = \begin{pmatrix} |b_0|^2+|b_1|^2 & b_0 d_0^* + b_1 d_1^* \\ b_0^* d_0 + b_1^* d_1 & |d_0|^2+|d_1|^2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$$

$$\mathcal{E}\left(\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}\right) = \begin{pmatrix} \frac{1}{2}\left[(a_0+b_0)(a_0^*+b_0^*)+(a_1+b_1)(a_1^*+b_1^*)\right] & \frac{1}{2}\left[(c_0^*+d_0^*)(a_0+b_0)+(c_1^*+d_1^*)(a_1+b_1)\right] \\ \frac{1}{2}\left[(c_0+d_0)(a_0^*+b_0^*)+(c_1+d_1)(a_1^*+b_1^*)\right] & \frac{1}{2}\left[(c_0+d_0)(c_0^*+d_0^*)+(c_1+d_1)(c_1^*+d_1^*)\right] \end{pmatrix}$$

$$= \begin{pmatrix} 3/4 & 1/\sqrt{8} \\ 1/\sqrt{8} & 1/4 \end{pmatrix}$$

$$\mathcal{E}\left(\begin{pmatrix} 1/2 & -i/2 \\ i/2 & 1/2 \end{pmatrix}\right) = \begin{pmatrix} \left(|a_0|^2 - i a_0 b_0^* + i a_0^* b_0 + |b_0|^2\right)/2 & \left(a_0 c_0^* + b_0 d_0^* + i b_0 c_0^* - i a_0 d_0^*\right)/2 \\ \left(a_0^* c_0 + b_0^* d_0 + i a_0^* d_0 - i c_0 b_0^*\right)/2 & \left(|c_0|^2 + |d_0|^2 + i c_0^* d_0 - i c_0 d_0^*\right)/2 \end{pmatrix}$$

$$= \begin{pmatrix} 3/4 & -i/\sqrt{8} \\ i/\sqrt{8} & 1/4 \end{pmatrix}.$$

$$\Rightarrow \begin{pmatrix} |a_0|^2 + |a_1|^2 & \cancel{a_0^* b_0} + a_1^* b_1 \\ \cancel{a_0 b_0^*} + a_1 b_1^* & \cancel{|b_0|^2} + |d_1|^2 \end{pmatrix} = 1$$

$c_0 = c_1 = 0$

$$\mathcal{E}\left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} |a_0|^2 + |a_1|^2 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\mathcal{E}\left(\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right) = \begin{pmatrix} \cancel{|b_0|^2} + |b_1|^2 & 0 \\ 0 & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$$

$$\mathcal{E}\left(\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}\right) = \begin{pmatrix} 3/4 + \frac{1}{2}(a_1 + b_1)(a_1^* + b_1^*)] & \frac{1}{2}[\quad d_0^*)(a_0 \\ \frac{1}{2}[(\quad d_0)(a_0^* & 1/4 \end{pmatrix}$$

$$= \begin{pmatrix} 3/4 & 1/\sqrt{8} \\ 1/\sqrt{8} & 1/4 \end{pmatrix}$$

$$\mathcal{E}\left(\begin{pmatrix} 1/2 & -i/2 \\ i/2 & 1/2 \end{pmatrix}\right) = \begin{pmatrix} 3/4 & (\quad -i a_0 d_0^*)/2 \\ (\quad i a_0^* d_0 \quad)/2 & 1/4 \end{pmatrix}$$

$|a_0|^2 = \frac{3}{2}$

$|a_1|^2 =$

$$= \begin{pmatrix} 3/4 & -i/\sqrt{8} \\ i/\sqrt{8} & 1/4 \end{pmatrix}.$$

$|d_0|^2 = \frac{1}{2}$ $\quad b_0 = 0$

$d_1 = 0.$

$$\Rightarrow \begin{pmatrix} |a_0|^2 + |a_1|^2 & a_1^* b_1 \\ a_1 b_1^* & |b_1|^2 \end{pmatrix} = 1$$

$c_0 = c_1 = 0$

$$\mathcal{E}\left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} |a_0|^2 + |a_1|^2 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$b_0 = 0$

$d_1 = 0$

$$\mathcal{E}\left(\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right) = \begin{pmatrix} |b_1|^2 & 0 \\ 0 & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$$

$$\mathcal{E}\left(\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}\right) = \begin{pmatrix} \frac{1}{2}\left[(a_0 \quad (a_0^* \quad + (a_1 + b_1)(a_1^* + b_1^*)\right] & \frac{1}{2}\left[\quad d_0^*)(a_0 + \right. \\ \frac{1}{2}\left[(\quad d_0)(a_0^* \quad + \quad \right] & \frac{1}{2}\left[(\quad d_0)(\quad d_0^*) + \right. \end{pmatrix}$$

$$= \begin{pmatrix} 3/4 & 1/\sqrt{8} \\ 1/\sqrt{8} & 1/4 \end{pmatrix}$$

$$\mathcal{E}\left(\begin{pmatrix} 1/2 & -i/2 \\ i/2 & 1/2 \end{pmatrix}\right) = \begin{pmatrix} (|a_0|^2 \quad)/2 & (\quad + \quad -i\, a_0 d_0^*)/2 \\ (\quad + i\, a_0^* d_0 \quad)/2 & (\quad |d_0|^2 \quad)/2 \end{pmatrix}$$

$$= \begin{pmatrix} 3/4 & -i/\sqrt{8} \\ i/\sqrt{8} & 1/4 \end{pmatrix}.$$

$|d_0| = \frac{1}{\sqrt{2}}$. $\quad |a_0| = \sqrt{\frac{3}{2}}$

$|b_1| = \frac{1}{\sqrt{2}}$

$$\Rightarrow \begin{pmatrix} |a_0|^2 + & \\ & |b_1|^2 \end{pmatrix} = 1$$

$c_0 = c_1 = 0$

$$\mathcal{E}\left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} |a_0|^2 + & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$b_0 = 0$

$d_1 = 0$

$$\mathcal{E}\left(\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right) = \begin{pmatrix} |b_1|^2 & 0 \\ 0 & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$$

$$\mathcal{E}\left(\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}\right) = \begin{pmatrix} \frac{1}{2}\left[(a_0 \quad (a_0^* \quad + \quad b_1)(\quad + b_1^*)\right] & \frac{1}{2}\left[\quad d_0^*)(a_0 + \right. \\ \frac{1}{2}\left[(\quad d_0)(a_0^* \quad + \quad \right] & \frac{1}{2}\left[(\quad d_0)(\quad d_0^*) + \right. \end{pmatrix}$$

$$= \begin{pmatrix} 3/4 & 1/\sqrt{8} \\ 1/\sqrt{8} & 1/4 \end{pmatrix}$$

$$\mathcal{E}\left(\begin{pmatrix} 1/2 & -i/2 \\ i/2 & 1/2 \end{pmatrix}\right) = \begin{pmatrix} (|a_0|^2 \quad)/2 & (\quad + \quad -i a_0 d_0^*)/2 \\ (\quad + i a_0^* d_0 \quad)/2 & (\quad |d_0|^2 \quad)/2 \end{pmatrix}$$

$$= \begin{pmatrix} 3/4 & -i/\sqrt{8} \\ i/\sqrt{8} & 1/4 \end{pmatrix}.$$

$|d_0| = \frac{1}{\sqrt{2}}$. $|a_0| = \sqrt{\frac{3}{2}}$

$|b_1| = \frac{1}{\sqrt{2}}$ $a_1 = 0$

$$\begin{pmatrix} a_0^* & c_0^* \\ b_0^* & d_0^* \end{pmatrix}\begin{pmatrix} a_0 & b_0 \\ c_0 & d_0 \end{pmatrix} = \begin{matrix} |a_0|^2+|c_0|^2 & a_0^* b_0 + c_0^* d_0 \\ a_0 b_0^* + c_0 d_0^* & |b_0|^2+|d_0|^2 \end{matrix}$$

$$\begin{pmatrix} a_0 & b_0 \\ c_0 & d_0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} a_0^* & c_0^* \\ b_0^* & d_0^* \end{pmatrix} = \begin{pmatrix} a_0^* & c_0^* \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} |a_0|^2 & a_0 c_0^* \\ a_0^* c_0 & |c_0|^2 \end{pmatrix}$$

$$\begin{pmatrix} a_0 & b_0 \\ c_0 & d_0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a_0^* & c_0^* \\ b_0^* & d_0^* \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ b_0^* & d_0^* \end{pmatrix}$$

$$= \begin{pmatrix} |b_0|^2 & b_0 d_0^* \\ b_0^* d_0 & |d_0|^2 \end{pmatrix}$$

$$\begin{pmatrix} a_0 & b_0 \\ c_0 & d_0 \end{pmatrix}\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}\begin{pmatrix} a_0^* & c_0^* \\ b_0^* & d_0^* \end{pmatrix} = \begin{pmatrix} \frac{1}{2}(a_0^*+b_0^*) & \frac{1}{2}(c_0^*+d_0^*) \\ \frac{1}{2}(a_0^*+b_0^*) & \frac{1}{2}(c_0^*+d_0^*) \end{pmatrix}$$

$$= \begin{pmatrix} \dfrac{|a_0|^2 + a_0 b_0^* + a_0^* b_0 + |b_0|^2}{2} & \frac{1}{2}(c_0^*+d_0^*)(a_0+b_0) \\ \frac{1}{2}(c_0+d_0)(a_0^*+b_0^*) & \frac{1}{2}(|c_0|^2+|d_0|^2+c_0^* d_0 + c_0 d_0^*) \end{pmatrix}$$

$$\begin{pmatrix} a_0 & b_0 \\ c_0 & d_0 \end{pmatrix}\begin{pmatrix} 1/2 & -i/2 \\ i/2 & 1/2 \end{pmatrix}\begin{pmatrix} a_0^* & c_0^* \\ b_0^* & d_0^* \end{pmatrix} = \begin{pmatrix} \frac{1}{2}(a_0^* - i b_0^*) & \frac{1}{2}(c_0^* - i d_0^*) \\ \frac{1}{2}(i a_0^* + b_0^*) & \frac{1}{2}(i c_0^* + d_0^*) \end{pmatrix}$$

$$= \begin{pmatrix} (|a_0|^2 - i a_0 b_0^* + i a_0^* b_0 + |b_0|^2)/2 & (a_0 c_0^* - i a_0 d_0^* + i b_0 c_0^* + b_0 d_0^*)/2 \\ (a_0^* c_0 - i c_0 b_0^* + i a_0^* d_0 + b_0^* d_0)/2 & (|c_0|^2 - i c_0 d_0^* + i c_0^* d_0 + |d_0|^2)/2 \end{pmatrix}$$

For $|\psi(z, m_s=\frac{1}{2}, T)\rangle$,

$$\langle z(T)\rangle = \int_{-\infty}^{+\infty} \langle\psi| z |\psi\rangle dz$$

$$= |N|^2 \int_{-\infty}^{+\infty} z \left| \exp\left\{ - \frac{(z - i\sigma^2\lambda\delta t/\hbar)^2}{4(\sigma^2 + i\hbar T/2m)} \right\} \right|^2 dz \cdot |\alpha_{\frac{1}{2}}|^2$$

$$= |N|^2 |\alpha_{\frac{1}{2}}|^2 \int_{-\infty}^{+\infty} z \, Re\left(\exp\left\{ - \frac{(z - i\sigma^2\lambda\delta t/\hbar)^2}{4(\sigma^2 + i\hbar T/2m)} \right\} \right)^2 dz$$

$$= |N|^2 |\alpha_{\frac{1}{2}}|^2 \int_{-\infty}^{+\infty} z \exp\left\{ - \frac{z^2 - \frac{\lambda T\delta t}{m} z - \left(\frac{\sigma^2\lambda\delta t}{\hbar}\right)^2}{2\sigma^2 + \hbar^2 T^2/2\sigma^2 m^2} \right\} dz$$

$$= |N|^2 |\alpha_{\frac{1}{2}}|^2 \int_{-\infty}^{+\infty} z \exp\left\{ - \frac{\left(z - \frac{\lambda T\delta t}{2m}\right)^2 - \left[\left(\frac{\sigma^2\lambda\delta t}{\hbar}\right)^2 + \left(\frac{\lambda T\delta t}{2m}\right)^2\right]}{2\sigma^2 + \hbar^2 T^2/2\sigma^2 m^2} \right\} dz$$

$$\xrightarrow{z = z + \frac{\lambda T\delta t}{2m}} \quad = |N|^2 |\alpha_{\frac{1}{2}}|^2 \int_{-\infty}^{+\infty} \left(z + \frac{\lambda T\delta t}{2m}\right) \exp\left\{ - \frac{z^2 - \left[\left(\frac{\sigma^2\lambda\delta t}{\hbar}\right)^2 + \left(\frac{\lambda T\delta t}{2m}\right)^2\right]}{2\sigma^2 + \hbar^2 T^2/2\sigma^2 m^2} \right\} dz$$

Apply $\int_{-\infty}^{+\infty} e^{-ax^2} dx = \sqrt{\frac{\pi}{a}}$ and $\int_{-\infty}^{+\infty} x e^{-ax^2} = 0$.

We get $$\langle z(T)\rangle = |N|^2 |\alpha_{\frac{1}{2}}|^2 \frac{\lambda T\delta t}{2m} \exp\left[\frac{\left(\frac{\sigma^2\lambda\delta t}{\hbar}\right)^2 + \left(\frac{\lambda T\delta t}{2m}\right)^2}{2\sigma^2 + \hbar^2 T^2/2\sigma^2 m^2} \right] \sqrt{2\pi\sigma^2 + \frac{\pi\hbar^2 T^2}{2\sigma^2 m^2}}$$

$$= \frac{|\alpha_\uparrow|^2}{\sqrt{1 + \frac{T^2\hbar^2}{4m^2\sigma^4}}} \frac{\lambda T\delta t}{2m} \exp\left(- \frac{3\sigma^2\lambda^2\delta t^2}{2\hbar^2} \right)$$

$$\langle z^2\rangle = |N|^2|\alpha_{+\frac{1}{2}}|^2\int_{-\infty}^{+\infty} z^2 \exp\left\{-\frac{\left(z-\frac{\lambda T\delta t}{2m}\right)^2-\left[\left(\frac{\sigma^2\lambda\delta t}{\hbar}\right)^2+\left(\frac{\lambda T\delta t}{2m}\right)^2\right]}{2\sigma^2+\hbar^2T^2/2\sigma^2m^2}\right\}dz$$

$$= |N|^2|\alpha_{+\frac{1}{2}}|^2\int_{-\infty}^{+\infty}\left[z^2-\frac{\lambda T\delta t}{m}z+\left(\frac{\lambda T\delta t}{2m}\right)^2\right]\exp\left\{-\frac{z^2-\left[\left(\frac{\sigma^2\lambda\delta t}{\hbar}\right)^2+\left(\frac{\lambda T\delta t}{2m}\right)^2\right]}{2\sigma^2+\hbar^2T^2/2\sigma^2m^2}\right\}dz$$

Apply $\int_{-\infty}^{+\infty} x^2 e^{-ax^2}dx = \sqrt{\frac{\pi}{a}}\cdot\frac{1}{2a}$. we get

$$\langle z^2\rangle = |N|^2|\alpha_{+\frac{1}{2}}|^2\left\{\sqrt{2\pi\sigma^2+\frac{\pi\hbar^2T^2}{2\sigma^2m^2}}\cdot\left(4\sigma^2+\frac{\hbar^2T^2}{\sigma^2m^2}\right)\exp\left(\frac{\sigma^2\lambda^2\delta t^2}{2\hbar^2}\right)\right.$$

$$\left.+\frac{\lambda^2T^2\delta t^2}{4m^2}\exp\left(\frac{\sigma^2\lambda^2\delta t^2}{2\hbar^2}\right)\cdot\left(2\pi\sigma^2+\frac{\pi\hbar^2T^2}{2\sigma^2m^2}\right)\right\}$$

So, $\langle \Delta z^2\rangle = \langle z^2\rangle - \langle z\rangle^2$

$=$

$$|N|^2 = \frac{1}{\sqrt{2\pi}\sigma}\cdot\frac{1}{1+\frac{T^2\hbar^2}{4m^2\sigma^4}}\cdot\exp\left(-\frac{2\sigma^2\lambda^2\delta t^2}{\hbar^2}\right)$$

$$\langle z\rangle^2 = |\alpha_{+\frac{1}{2}}|^4\cdot\sqrt{2\pi}\sigma|N|^2\,\frac{\lambda^2T^2\delta t^2}{4m^2}\exp\left(-\frac{\sigma^2\lambda^2\delta t^2}{\hbar^2}\right)$$

$$-\frac{iT}{2m\hbar}p^2 \qquad -\frac{\sigma^2}{\hbar^2}\left[\left(p^2-\frac{\cancel{4}\,2\,i\lambda m_s\delta t}{\hbar}p\right)-\frac{\cancel{4}\lambda^2\cancel{m_s^2}\delta t^2}{\hbar^2}\right]$$

$$e^{2x} \qquad -\left(\frac{iT}{2m\hbar}+\frac{\sigma^2}{\hbar^2}\right)p^2+\frac{\cancel{4}\,2\,i\lambda\sigma^2 m_s\delta t}{\hbar^3}+\frac{\sigma^2\lambda^2\delta t^2}{4\hbar^4}$$

$$\frac{\sigma^2\lambda\delta t}{\cancel{\hbar}}\cdot\frac{\cancel{\hbar}T}{2m}$$

$$e^{x+iy}=e^x(\cos y+i\sin y)$$

$$\frac{z-iA}{2(\sigma^2+iB)}=\frac{(z-iA)(\sigma^2-iB)}{2(\sigma^4+B^2)}$$

$$\frac{z\sigma^2-AB}{2(\sigma^4+B^2)}$$

$$z\Rightarrow z-\frac{\lambda\delta t}{2m}$$

$$\int_{-\infty}^{+\infty}\left(z+\frac{\lambda\delta t}{2m}\right)\exp\left\{-\frac{\sigma^2 z}{\sigma^4+\left(\frac{\hbar T}{2m}\right)^2}\right\}dz$$

$$\exp\left\{-\frac{(z-i\sigma^2\lambda\delta t/\hbar)^2}{4(\sigma^2+i\hbar T/2m)}\right\}$$

$$A$$

$$\frac{z^2-\left(\frac{\sigma^2\lambda\delta t}{\hbar}\right)^2-\frac{2i\sigma^2\lambda\delta t}{\hbar}z}{4\sigma^2+2i\hbar T/m}$$

$$\frac{A-iB}{C+iD}\Rightarrow\frac{(A-iB)(C-iD)}{C^2+D^2} \qquad \frac{AC-BD}{C^2+D^2}$$

$$\frac{\left[z^2-\left(\frac{\sigma^2\lambda\delta t}{\hbar}\right)^2\right]4\sigma^2-\frac{2\sigma^2\lambda\delta t}{\cancel{\hbar}}z\cdot\frac{2\cancel{\hbar}T}{m}}{16\sigma^4+4\hbar^2T^2/m^2}$$

$$=\left\{\left[z^2-\left(\frac{\sigma^2\lambda\delta t}{\hbar}\right)^2\right]-\frac{\lambda\delta t T}{m}z\right\}\Big/\left(4\sigma^2+\hbar^2T^2/\sigma^2m^2\right)$$

$$\Rightarrow \sum_n i\hbar \psi_n \dot{C}_n(t) e^{-iE_nt/\hbar} = H'\psi_1.$$

$$C_n^{(1)} = \frac{1}{-i\hbar}\int_0^t \langle n|\lambda\sigma_z \otimes \hat{z}|1\rangle e^{iE_nt'/\hbar}\,dt'$$

of which $|n\rangle = \frac{1}{(2\pi\hbar)^{\frac{3}{2}}} e^{i(p_x+p_y+p_z)/\hbar} \chi_{m_s}(S_z)$

$$\Rightarrow |1\rangle = \frac{1}{(2\pi\hbar)^{\frac{3}{2}}} e^{i(p_x+p_y+p_z)/\hbar} \chi_{-\frac{1}{2}}. \quad E_1 = \frac{p_{x_0}^2}{2m} - \frac{\hbar}{2} g\mu_B B_z.$$

$$|2\rangle = \frac{1}{(2\pi\hbar)^{\frac{3}{2}}} e^{i(p_x+p_y+p_z)/\hbar} \chi_{+\frac{1}{2}}. \quad E_2 = \frac{p_{x_0}^2}{2m} + \frac{\hbar}{2} g\mu_B B_z.$$

$$\left[\frac{\left(\frac{\sigma^2\lambda\delta t}{\hbar}\right)^2 + \left(\frac{\lambda T\delta t}{2m}\right)^2}{2\sigma^2 + \hbar^2 T^2/2\sigma^2 m^2}\right]$$

$$\frac{4\left(\frac{\sigma^4 m^2}{\hbar^2} + \frac{T^2\hbar^2}{4m^2}\right)(\lambda\delta t)^2 \cdot 2\sigma^2 m^2 / 4\hbar^2}{4\sigma^4 m^2 + \hbar^2 T^2}$$

$$(\lambda\delta t)^2 \cdot \frac{\sigma^2}{2\hbar^2}$$

$$|N|^2 = \frac{1}{\sqrt{2\pi}\sigma} \cdot \frac{1}{1+\frac{T^2\hbar^2}{4m^2\sigma^4}} \cdot \exp\left(-\frac{2\sigma^2\lambda^2\delta t^2}{\hbar^2}\right)$$

$$\langle z\rangle^2 = \cdot\sqrt{2\pi}\sigma|N|^2 \frac{\lambda^2 T^2\delta t^2}{4m^2} \exp\left(-\frac{\sigma^2\lambda^2\delta t^2}{\hbar^2}\right)$$

$$\psi_0 = \frac{1}{(2\pi\sigma_0^2)^{3/4}} \exp\left(\frac{-\vec{r}^2}{4\sigma_0^2} + i\vec{k}\cdot\vec{r}\right)$$

$$U = e^{\left[\frac{-it}{2m\hbar}(p_x^2+p_y^2+p_z^2) - \frac{it}{\hbar}\mu_c(B_0+bz)\sigma_z\right]}$$

$$\hat{U} = e^{-\frac{1}{6}k}\, e^{-\frac{it\mu_c}{2m\hbar}(p_x^2+p_y^2)}\, e^{-\frac{it\mu_c}{\hbar}(B_0+bz)\sigma_z}\, e^{\frac{it^2\mu_c b}{2m\hbar}p_z\sigma_z}\, e^{\frac{-it}{2m\hbar}p_z^2}$$

$$\hat{U}\psi_0(\alpha|\uparrow\rangle + \beta|\downarrow\rangle)$$

$$E_0 = m_0 \widetilde{E_0} + m_1 \widetilde{E_1} + m_2 \widetilde{E_2} + m_3 \widetilde{E_3}$$

4x4

$$E_1 = n_0 \widetilde{E_0} + n_1 \widetilde{E_1} + n_2 \widetilde{E_2} + n_3 \widetilde{E_3}$$

Homework 1 yuxiang@uchicago.edu

1 Stern-Gerlach Measurement

(a) (i) $H = \frac{\vec{p}^2}{2m} + H' \quad (0 < t \leq \delta t)$

$$= \frac{\vec{p}^2}{2m} + \lambda \sigma_z \otimes \hat{z}$$

So, $H = \begin{cases} \frac{\vec{p}^2}{2m} + \lambda \sigma_z \otimes \hat{z} & (0 < t \leq \delta t) \\ \frac{\vec{p}^2}{2m} & (\delta t < t \leq T + \delta T) \end{cases}$

(ii) If $\lambda = 0$, it will be free evolution of wavefunction.

$$|\psi\rangle = e^{-iHt/\hbar} |\psi(z, m_s, t=0)\rangle$$

$$= e^{-i\frac{p^2 t}{2m\hbar}} \int \langle p|\psi(z, m_s, t=0)\rangle |p\rangle dp$$

$$= \int e^{-\frac{ip^2 t}{2m\hbar}} \tilde{\phi}(p) e^{ipz/\hbar} dp \cdot (\alpha_\downarrow |\downarrow\rangle + \alpha_\uparrow |\uparrow\rangle)$$

of which $\tilde{\phi}(p) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{+\infty} \frac{1}{\sqrt{\sqrt{2\pi}\sigma}} e^{-\frac{z^2}{4\sigma^2}} e^{-ipz/\hbar} dz$

$$= \frac{\sqrt{2}\sigma}{\sqrt{\hbar}} e^{-\sigma^2 p^2/\hbar^2} \cdot \frac{1}{\sqrt{\sqrt{2\pi}\sigma}} = \sqrt{\frac{\sqrt{2}\sigma}{\sqrt{\pi}\hbar}} e^{-\sigma^2 p^2/\hbar^2}$$

So, $|\psi'\rangle = \int \sqrt{\frac{\sqrt{2}\,\sigma}{\sqrt{\pi}\,\hbar}}\, e^{-\sigma^2 p^2/\hbar^2}\, e^{-\frac{ip^2 t}{2m\hbar}}\, e^{ipz/\hbar}\, dp \cdot \left(\alpha_\downarrow|\downarrow\rangle + \alpha_\uparrow|\uparrow\rangle\right)$

$$= \sqrt{\frac{\sqrt{2}\,\sigma}{\sqrt{\pi}\,\hbar}} \cdot \sqrt{\frac{\pi}{\frac{\sigma^2}{\hbar^2} + \frac{it}{2m\hbar}}}\; e^{-\frac{z^2}{4\sigma^2 + \frac{2i\hbar t}{m}}} \cdot \left(\alpha_\downarrow|\downarrow\rangle + \alpha_\uparrow|\uparrow\rangle\right)$$

This is the wavefunction of atom, with $t = T + \delta t$.

(Use $\int_{-\infty}^{+\infty} e^{-ax^2+bx}\,dx = \sqrt{\frac{\pi}{a}}\, e^{b^2/4a}$)

(b) (i) $\lambda \neq 0$. $\lambda \delta t/m \to 0$

In the field interaction region

$$U(T) = \exp\left\{-i\left(\frac{T}{\hbar}H_0 + \frac{\delta t}{\hbar}H'\right)\right\} \approx \exp\left(-\frac{iT}{\hbar}H_0\right)\exp\left(-\frac{i\delta t}{\hbar}H'\right)$$

$$= \exp\left\{-\frac{iT}{\hbar}\left(\frac{p^2}{2m}\right)\right\} \exp\left(\frac{-i\lambda\sigma_z \otimes \hat{z}}{\hbar}\,\delta t\right)$$

(ii) Apply $U(T)$ to $|\psi(z, m_s, t=0)\rangle$ we get

$$U(T)\,|\psi(z,m_s,t=0)\rangle = \exp\left\{-\frac{iT}{\hbar}\left(\frac{p^2}{2m}\right)\right\} \exp\left(\frac{-i\lambda\sigma_z\otimes\hat{z}}{\hbar}\,\delta t\right)|\psi(z,m_s,t=0)\rangle$$

$$= \sum_{m_s} \exp\left\{-\frac{iT}{\hbar}\left(\frac{p^2}{2m}\right)\right\} \exp\left(\frac{-i\lambda\delta t\, m_s}{\hbar}\hat{z}\right)|\psi(z,m_s,t=0)\rangle$$

$$= \sum_{m_s} \int \exp\left\{-\frac{iT}{\hbar}\left(\frac{p^2}{2m}\right)\right\} \exp\left(\frac{-i\lambda m_s \delta t}{\hbar}\hat{z}\right) |p\rangle \underbrace{\langle p|\psi(z,m_s,t=0)\rangle}_{\tilde{\phi}(p)} dp \cdot \alpha_{m_s}|m_s\rangle$$

$$= \sum_{m_s} \int \exp\left\{-\frac{iT}{\hbar}\left(\frac{p^2}{2m}\right)\right\} \exp\left(-\frac{i\lambda m_s \delta t}{\hbar}\hat{z}\right) \sqrt{\frac{\sqrt{2}\,\sigma}{\sqrt{\pi}\,\hbar}}\, e^{-\sigma^2 p^2/\hbar^2} \cdot e^{ipz/\hbar} dp \cdot \alpha_{m_s}|m_s\rangle$$

$$= \sum_{m_s} \int \exp\left\{-\frac{iT}{\hbar}\left(\frac{p^2}{2m}\right)\right\} \sqrt{\frac{\sqrt{2}\,\sigma}{\sqrt{\pi}\,\hbar}}\, e^{-\sigma^2\left(p-\frac{i\lambda m_s \delta t}{\hbar}\right)^2/\hbar^2} dp \cdot \alpha_{m_s}|m_s\rangle$$

$$= \sum_{m_s} \sqrt{\frac{\sqrt{2}\,\sigma}{\sqrt{\pi}\,\hbar}} \int \exp\left\{-\left(\frac{iT}{2m\hbar}+\frac{\sigma^2}{\hbar^2}\right)p^2 + \left(\frac{2i\lambda\sigma^2 m_s \delta t}{\hbar^3}+\frac{iz}{\hbar}\right)p + \frac{\sigma^2\lambda^2\delta t^2}{4\hbar^4}\right\} dp \; \alpha_{m_s}|m_s\rangle$$

Use $\int_{-\infty}^{+\infty} e^{-ax^2+bx}dx = \sqrt{\frac{\pi}{a}}\, e^{b^2/4a}$, we get

$$\Rightarrow \quad = \sqrt{\frac{\sqrt{2}\,\sigma}{\sqrt{\pi}\,\hbar}} \sqrt{\frac{\pi}{\frac{iT}{2m\hbar}+\frac{\sigma^2}{\hbar^2}}} \exp\left\{\frac{\frac{1}{\hbar^2}\left(z-\frac{2i\lambda m_s\sigma^2\delta t}{\hbar^2}\right)^2}{4\left(\frac{\sigma^2}{\hbar^2}+\frac{iT}{2m\hbar}\right)}\right\} \alpha_{m_s}|m_s\rangle$$

$$= \sqrt{\frac{\sqrt{2\pi}\,\sigma}{1+\frac{iT\hbar}{2m\sigma^2}}} \cdot \frac{\hbar}{\sigma} \exp\left\{\frac{\left(z-2i\sigma^2\lambda\delta t m_s/\hbar^2\right)^2}{4\sigma^2+2iT\hbar/m}\right\} \alpha_{m_s}|m_s\rangle.$$

which is the full wavefunction $|\psi(z,m_s,T)\rangle$.

For $|\psi(z, m_s=\frac{1}{2}, T)\rangle$,

$$\langle z(T)\rangle = \int_{-\infty}^{+\infty} \langle\psi| z |\psi\rangle dz$$

$$= |N|^2 \int_{-\infty}^{+\infty} z \left| \exp\left\{-\frac{(z - i\sigma^2\lambda\delta t/\hbar)^2}{4(\sigma^2 + i\hbar T/2m)}\right\}\right|^2 dz \cdot$$

$$= |N|^2 \int_{-\infty}^{+\infty} z \, \mathrm{Re}\left(\exp\left\{-\frac{(z - i\sigma^2\lambda\delta t/\hbar)^2}{4(\sigma^2 + i\hbar T/2m)}\right\}\right)^2 dz$$

$$= |N|^2 \int_{-\infty}^{+\infty} z \exp\left\{-\frac{z^2 - \frac{\lambda T\delta t}{m} z - \left(\frac{\sigma^2\lambda\delta t}{\hbar}\right)^2}{2\sigma^2 + \hbar^2 T^2/2\sigma^2 m^2}\right\} dz$$

$$= |N|^2 \int_{-\infty}^{+\infty} z \exp\left\{-\frac{\left(z - \frac{\lambda T\delta t}{2m}\right)^2 - \left[\left(\frac{\sigma^2\lambda\delta t}{\hbar}\right)^2 + \left(\frac{\lambda T\delta t}{2m}\right)^2\right]}{2\sigma^2 + \hbar^2 T^2/2\sigma^2 m^2}\right\} dz$$

$$\xrightarrow{z = z + \frac{\lambda T\delta t}{2m}} = |N|^2 \int_{-\infty}^{+\infty} \left(z + \frac{\lambda T\delta t}{2m}\right) \exp\left\{-\frac{z^2 - \left[\left(\frac{\sigma^2\lambda\delta t}{\hbar}\right)^2 + \left(\frac{\lambda T\delta t}{2m}\right)^2\right]}{2\sigma^2 + \hbar^2 T^2/2\sigma^2 m^2}\right\} dz$$

Apply $\int_{-\infty}^{+\infty} e^{-ax^2} dx = \sqrt{\frac{\pi}{a}}$ and $\int_{-\infty}^{+\infty} x e^{-ax^2} = 0$.

We get $$\langle z(T)\rangle = |N|^2 \frac{\lambda T\delta t}{2m} \exp\left[\frac{\left(\frac{\sigma^2\lambda\delta t}{\hbar}\right)^2 + \left(\frac{\lambda T\delta t}{2m}\right)^2}{2\sigma^2 + \hbar^2 T^2/2\sigma^2 m^2}\right] \sqrt{2\pi\sigma^2 + \frac{\pi\hbar^2 T^2}{2\sigma^2 m^2}}$$

$$= \frac{\lambda T\delta t}{2m} \exp\left(-\frac{3\sigma^2\lambda^2\delta t^2}{2\hbar^2}\right)$$

$$\langle z^2\rangle = |N|^2\int_{-\infty}^{+\infty} z^2\exp\left\{-\frac{\left(z-\frac{\lambda T\delta t}{2m}\right)^2-\left[\left(\frac{\sigma^2\lambda\delta t}{\hbar}\right)^2+\left(\frac{\lambda T\delta t}{2m}\right)^2\right]}{2\sigma^2+\hbar^2T^2/2\sigma^2m^2}\right\}dz$$

$$= |N|^2\int_{-\infty}^{+\infty}\left[z^2-\frac{\lambda T\delta t}{m}z+\left(\frac{\lambda T\delta t}{2m}\right)^2\right]\exp\left\{-\frac{z^2-\left[\left(\frac{\sigma^2\lambda\delta t}{\hbar}\right)^2+\left(\frac{\lambda T\delta t}{2m}\right)^2\right]}{2\sigma^2+\hbar^2T^2/2\sigma^2m^2}\right\}dz$$

Apply $\int_{-\infty}^{+\infty} x^2 e^{-ax^2}dx = \sqrt{\frac{\pi}{a}}\cdot\frac{1}{2a}$. we get

$$\langle z^2\rangle = |N|^2\left\{\sqrt{2\pi\sigma^2+\frac{\pi\hbar^2T^2}{2\sigma^2m^2}}\cdot\left(4\sigma^2+\frac{\hbar^2T^2}{\sigma^2m^2}\right)\exp\left(\frac{\sigma^2\lambda^2\delta t^2}{2\hbar^2}\right)\right.$$

$$\left.+\frac{\lambda^2T^2\delta t^2}{4m^2}\exp\left(\frac{\sigma^2\lambda^2\delta t^2}{2\hbar^2}\right)\cdot\sqrt{2\pi\sigma^2+\frac{\pi\hbar^2T^2}{2\sigma^2m^2}}\right\}$$

$$= \left(4\sigma^2+\frac{\hbar^2T^2}{\sigma^2m^2}+\frac{\lambda^2T^2\delta t^2}{4m^2}\right)\exp\left(-\frac{3\sigma^2\lambda^2\delta t^2}{2\hbar^2}\right)$$

So, $\langle \Delta z^2\rangle = \langle z^2\rangle - \langle z\rangle^2$

$$= \left(4\sigma^2+\frac{\hbar^2T^2}{\sigma^2m^2}\right)\exp\left(-\frac{3\sigma^2\lambda^2\delta t^2}{2\hbar^2}\right)$$

items do not cancell.

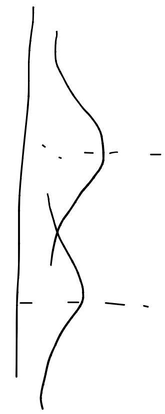
So I intentionally made a fault.

(c)(i) We have state of system before measurement:

$$|\psi(z,m_s,T)\rangle = N\sum_{m_s}\alpha_{m_s}\exp\left\{-\frac{\left(z-2i\sigma^2\delta t\, m_s/\hbar\right)^2}{4(\sigma^2+i\hbar t/2m)}\right\}|m_s\rangle$$

Assume that

$$|\varphi_\uparrow\rangle = N\exp\left\{-\frac{\left(z-i\sigma^2\delta t/\hbar\right)^2}{4(\sigma^2+i\hbar t/2m)}\right\}|m_s=+\tfrac{1}{2}\rangle$$



$$|\varphi_-\rangle = N\exp\left\{-\frac{\left(z+i\sigma^2\delta t/\hbar\right)^2}{4(\sigma^2+i\hbar t/2m)}\right\}|m_s=-\tfrac{1}{2}\rangle$$

$$|\psi(z,m_s,T)\rangle = \alpha_\uparrow|\varphi_\uparrow\rangle + \alpha_\downarrow|\varphi_-\rangle.$$

$$\int_0^\infty e^{-ax^2+bx}\,dx = \frac{\sqrt{\pi}\,e^{b^2/4a}\,\mathrm{erf}\left(\frac{b}{2\sqrt{a}}+1\right)}{2\sqrt{a}}$$

$$P(z>0) = \int_0^\infty \langle\psi(z,m_s,T)|\psi(z,m_s,T)\rangle\,dz$$

$$= \int_0^\infty |\alpha_\uparrow|^2\langle\varphi_\uparrow|\varphi_\uparrow\rangle + |\alpha_\downarrow|^2\langle\varphi_\downarrow|\varphi_\downarrow\rangle\,dz$$

$$= \int_0^\infty |\alpha_\uparrow|^2|N|^2\exp\left\{-\frac{z^2-\frac{\lambda T\delta t}{m}z-\left(\frac{\sigma^2\lambda\delta t}{\hbar}\right)^2}{2\sigma^2+\hbar^2T^2/2\sigma^2m^2}\right\}dz$$

$$+\int_0^\infty |\alpha_\downarrow|^2|N|^2\exp\left\{-\frac{z^2+\frac{\lambda T\delta t}{m}z-\left(\frac{\sigma^2\lambda\delta t}{\hbar}\right)^2}{2\sigma^2+\hbar^2T^2/2\sigma^2m^2}\right\}dz$$

$\frac{\lambda \delta t}{m} \to 0$

$$\Longrightarrow \quad = \frac{\sqrt{\pi}}{2}\sqrt{\left(2\sigma^2 + \frac{\hbar^2 T^2}{2m^2\sigma^2}\right)}\, |\alpha_\uparrow|^2 |N|^2 \exp\left\{\frac{\lambda^2 T^2 \delta t^2}{(2m^2\sigma^2 + \hbar^2 T^2/2\sigma^2)\cdot 4}\right\}$$

$$+ \frac{\sqrt{\pi}}{2}\sqrt{\left(2\sigma^2 + \frac{\hbar^2 T^2}{2m^2\sigma^2}\right)}\, |\alpha_\downarrow|^2 |N|^2 \exp\left\{\frac{\lambda^2 T^2 \delta t^2}{(2m^2\sigma^2 + \hbar^2 T^2/2\sigma^2)\cdot 4}\right\}$$

$$= \frac{\sqrt{\pi}}{2}\sqrt{\left(2\sigma^2 + \frac{\hbar^2 T^2}{2m^2\sigma^2}\right)} \cdot \frac{1}{\sqrt{2\pi}\,\sigma\sqrt{\left(1 + \frac{\hbar^2 T^2}{4m^2\sigma^4}\right)}} \cdot \exp\left\{\frac{\lambda^2 T^2 \delta t^2}{(2m^2\sigma^2 + \hbar^2 T^2/2\sigma^2)\cdot 4}\right\}$$

$$= \frac{1}{2}\exp\left\{\frac{\lambda^2 T^2 \delta t^2}{(2m^2\sigma^2 + \hbar^2 T^2/2\sigma^2)\cdot 4}\right\}.$$

We have $p(z>0) = \langle \psi(z, m_s, T) | M_+^\dagger M_+ | \psi(z, m_s, T) \rangle$

$$\Rightarrow \quad M_+ = \alpha_\uparrow |\varphi_+\rangle\langle\varphi_+| + \alpha_\downarrow |\varphi_-\rangle\langle\varphi_-| \quad \times$$

(ii) $\mathcal{E} = \frac{1}{2}\mathcal{Y}_+ + \frac{1}{2}\mathcal{Y}_-$

$$\Rightarrow \quad \rho = \frac{1}{2}\left(|\uparrow\rangle\langle\uparrow| + |\downarrow\rangle\langle\downarrow|\right).$$

$$\rho' = M^{+} \rho M \quad , \qquad \rho^{Q\,\prime} = \mathrm{Tr}_{M}(M^{+}\rho M).$$

Should have $M^{+} = \langle \mu | U | \varphi \rangle$

$$= \langle + | U |$$

(d)(i) To maximize distinguishability,

we should have $2m^2\sigma^2 = \dfrac{\hbar^2 T^2}{2\sigma^2}$.

$$\Rightarrow \quad \sigma = \sqrt{\frac{\hbar T}{2m}}.$$

The detection limit $\langle z\rangle / \sqrt{\langle \Delta z^2\rangle}$,

$$\Rightarrow \quad \frac{\dfrac{\lambda T\delta t}{2m}\exp\left(-\dfrac{3\sigma^2\lambda^2\delta t^2}{2\hbar^2}\right)}{\sqrt{\left(4\sigma^2+\dfrac{\hbar^2T^2}{\sigma^2 m^2}\right)\exp\left(-\dfrac{3\sigma^2\lambda^2\delta t^2}{2\hbar^2}\right)}} = \frac{\lambda\delta t}{2m}\cdot\frac{T}{\sqrt{4\sigma^2+\dfrac{\hbar^2T^2}{\sigma^2m^2}}}\exp\left(-\frac{3\sigma^2\lambda^2\delta t^2}{4\hbar^2}\right)$$

$$= \frac{\lambda\delta t}{8\hbar}\exp\left(-\frac{3T\lambda^2\delta t^2}{8m\hbar}\right).$$

(ii) In momentum representation,

$$\tilde{\psi}(p) = \langle p | \psi(z, m_s, t = T) \rangle$$

$$= \int_{-\infty}^{+\infty} N \sum_{m_s} \alpha_{m_s} \exp\left\{ - \frac{(z - 2i\sigma^2 \delta t\, m_s/\hbar)^2}{4(\sigma^2 + i\hbar t/2m)} \right\} |m_s\rangle \cdot e^{-ipz} dz$$

$$\sqrt{\frac{\sqrt{2}\,\sigma}{\sqrt{\pi}\,\hbar}}\, e^{-\sigma^2 p^2/\hbar^2} \cdot e^{ipz/\hbar}$$

Then calculate

$$\int p |\tilde{\psi}(p)|^2 dp \quad \text{and} \quad \int p^2 |\tilde{\psi}(p)|^2 dp \ \ldots$$

Will solve the problems later, do not have enough time ...

2 Problem 2 – Process Tomography

(a) I'd calculate E_k in first hand.

$$\mathcal{E}(\rho) = E_0 \rho E_0^\dagger + E_1 \rho E_1^\dagger .$$

suppose $E_0 = \begin{pmatrix} a_0 & b_0 \\ c_0 & d_0 \end{pmatrix}$, $\quad E_1 = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}$.

We have $E_0^* E_0 + E_1^* E_1 = \mathbb{1}$.

$$\Rightarrow \begin{pmatrix} |a_0|^2 + |c_0|^2 + |a_1|^2 + |c_1|^2 & a_0^* b_0 + c_0^* d_0 + a_1^* b_1 + c_1^* d_1 \\ a_0 b_0^* + c_0 d_0^* + a_1 b_1^* + c_1 d_1^* & |b_0|^2 + |d_0|^2 + |b_1|^2 + |d_1|^2 \end{pmatrix} = \mathbb{1}$$

$$\mathcal{E}\left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} |a_0|^2 + |a_1|^2 & a_0 c_0^* + a_1 c_1^* \\ a_0^* c_0 + a_1^* c_1 & |c_0|^2 + |c_1|^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\mathcal{E}\left(\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right) = \begin{pmatrix} |b_0|^2 + |b_1|^2 & b_0 d_0^* + b_1 d_1^* \\ b_0^* d_0 + b_1^* d_1 & |d_0|^2 + |d_1|^2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$$

$$\mathcal{E}\left(\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}\right) = \begin{pmatrix} \frac{1}{2}\left[(a_0+b_0)(a_0^*+b_0^*) + (a_1+b_1)(a_1^*+b_1^*)\right] & \frac{1}{2}\left[(c_0^*+d_0^*)(a_0+b_0) + (c_1^*+d_1^*)(a_1+b_1)\right] \\ \frac{1}{2}\left[(c_0+d_0)(a_0^*+b_0^*) + (c_1+d_1)(a_1^*+b_1^*)\right] & \frac{1}{2}\left[(c_0+d_0)(c_0^*+d_0^*) + (c_1+d_1)(c_1^*+d_1^*)\right] \end{pmatrix}$$

$$= \begin{pmatrix} 3/4 & 1/\sqrt{8} \\ 1/\sqrt{8} & 1/4 \end{pmatrix}$$

$$\mathcal{E}\left(\begin{pmatrix} 1/2 & -i/2 \\ i/2 & 1/2 \end{pmatrix}\right) = \begin{pmatrix} \begin{matrix} \left(|a_0|^2 - i a_0 b_0^* + i a_0^* b_0 + |b_0|^2\right)/2 \\ + \left(|a_1|^2 - i a_1 b_1^* + i a_1^* b_1 + |b_1|^2\right)/2 \end{matrix} & \begin{matrix} \left(a_0 c_0^* + b_0 d_0^* + i b_0 c_0^* - i a_0 d_0^*\right)/2 \\ + \left(a_1 c_1^* + b_1 d_1^* + i b_1 c_1^* - i a_1 d_1^*\right)/2 \end{matrix} \\ \\ \begin{matrix} \left(a_0^* c_0 + b_0^* d_0 + i a_0^* d_0 - i c_0 b_0^*\right)/2 \\ + \left(a_1^* c_1 + b_1^* d_1 + i a_1^* d_1 - i c_1 b_1^*\right)/2 \end{matrix} & \begin{matrix} \left(|c_0|^2 + |d_0|^2 + i c_0^* d_0 - i c_0 d_0^*\right)/2 \\ + \left(|c_1|^2 + |d_1|^2 + i c_1^* d_1 - i c_1 d_1^*\right)/2 \end{matrix} \end{pmatrix}$$

$$= \begin{pmatrix} 3/4 & -i/\sqrt{8} \\ i/\sqrt{8} & 1/4 \end{pmatrix}.$$

That is :

$$\Rightarrow \begin{pmatrix} |a_0|^2 + |a_1|^2 & a_0^* b_0 + a_1^* b_1 \\ a_0 b_0^* + a_1 b_1^* & |b_0|^2 + |d_0|^2 + |b_1|^2 + |d_1|^2 \end{pmatrix} = \mathbb{1}$$

$c_0 = c_1 = 0$

$$\mathcal{E}\left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} |a_0|^2 + |a_1|^2 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\mathcal{E}\left(\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right) = \begin{pmatrix} |b_0|^2 + |b_1|^2 & b_0 d_0^* + b_1 d_1^* \\ b_0^* d_0 + b_1^* d_1 & |d_0|^2 + |d_1|^2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$$

$$\mathcal{E}\left(\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}\right) = \begin{pmatrix} \frac{1}{2}\left[(a_0+b_0)(a_0^*+b_0^*) + (a_1+b_1)(a_1^*+b_1^*)\right] & \frac{1}{2}\left[(\quad d_0^*)(a_0 \quad\quad d_1^*)(a_1\right. \\ \frac{1}{2}\left[\quad + d_0)(a_0^* + \quad + \quad d_1)(a_1^* \quad \right] & \frac{1}{2}\left[\quad d_0)(\quad + d_0^*) + \quad d_1)(\quad + d_1^*)\right] \end{pmatrix}$$

$$= \begin{pmatrix} 3/4 & 1/\sqrt{8} \\ 1/\sqrt{8} & 1/4 \end{pmatrix}$$

$$\begin{pmatrix} \begin{array}{l} (|a_0|^2 - i a_0 b_0^* + i a_0^* b_0 + |b_0|^2)/2 \\ + (|a_1|^2 - i a_1 b_1^* + i a_1^* b_1 + |b_1|^2)/2 \end{array} & + \begin{array}{l} \cancel{+ b_0 d_0^*} + \quad -i a_0 d_0^*)/2 \\ \cancel{+ b_1 d_1^*} + \quad -i a_1 d_1^*)/2 \end{array} \\ \begin{array}{l} (\quad \cancel{b_0^* d_0} + i a_0^* d_0 \quad)/2 \\ + (\quad \cancel{b_1^* d_1} + i a_1^* d_1 \quad)/2 \end{array} & \begin{array}{l} (\quad + |d_0|^2 + \quad)/2 \\ + (\quad + |d_1|^2 + \quad)/2 \end{array} \end{pmatrix} = \begin{pmatrix} 3/4 & -i/\sqrt{8} \\ i/\sqrt{8} & 1/4 \end{pmatrix}$$

Solve to get

$$|a_0|^2 + |a_1|^2 = 1, \qquad a_0^* b_0 + a_1^* b_1 = 0$$

$$|b_0|^2 + |b_1|^2 = |d_0|^2 + |d_1|^2 = \frac{1}{2}, \qquad b_0^* d_0 + b_1^* d_1 = 0.$$

$$a_0 d_0^* + a_1 d_1^* = \frac{1}{\sqrt{2}}$$

$$\Rightarrow \quad a_0 = a_1 = \frac{1}{\sqrt{2}}, \quad d_0, d_1 = \frac{(1 \pm i)}{2}, \quad b_0, b_1 = \pm i/2.$$

and $\quad c_0 = c_1 = 0.$

So, we have

$$E_0 = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{i}{2} \\ 0 & \frac{1+i}{2} \end{pmatrix}. \quad E_1 = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{i}{2} \\ 0 & \frac{1-i}{2} \end{pmatrix}.$$

$$E_0 = m\mathbb{1} + x\,\sigma_x/2 + y\,\sigma_y/2 + z\,\sigma_z/2$$

$$= \begin{pmatrix} m+z & x-iy \\ x+iy & m-z \end{pmatrix}.$$

$$\Rightarrow \quad E_0 = \frac{1+\sqrt{2}+i}{4}\mathbb{1} + \frac{i}{4}\sigma_x - \frac{1}{4}\sigma_y + \frac{\sqrt{2}-1-i}{4}\sigma_z$$

$$E_1 = \frac{1+\sqrt{2}-i}{4}\mathbb{1} - \frac{i}{4}\sigma_x + \frac{1}{4}\sigma_y + \frac{\sqrt{2}-1+i}{4}\sigma_z .$$

So we have

$$\chi = \begin{pmatrix} & \\ & \end{pmatrix} .$$

(ii)

I think my calculation is wrong, but do not have enough time to figure out and coding...

为何 Rabi 振荡的条件里面包含 H 非简并?

初始不是 H 的本征态. 无 Rabi 振荡

2 能级系统最普遍的形式为

$$H = \begin{pmatrix} a_0+a_3 & a_1-ia_2 \\ a_1+ia_2 & a_0-a_3 \end{pmatrix} \longrightarrow H = a_0\sigma_0 + a_1\sigma_1 + a_2\sigma_2 + a_3\sigma_3$$

为什么是这样的形式?

表示什么含义?

HW 2 yuxiang@uchicago.edu

1 Problem 1 — Different approaches to spectroscopy

Part (a) - Rabi Spectroscopy

$$|\psi(t)\rangle = e^{-iHt}|0\rangle$$

$$H = \hbar\frac{\delta\omega}{2}\sigma_z + \hbar\frac{\Omega}{2}\sigma_x = \begin{pmatrix} \frac{\hbar\delta\omega}{2} & \frac{\hbar\Omega}{2} \\ \frac{\hbar\Omega}{2} & -\frac{\hbar\delta\omega}{2} \end{pmatrix}$$

is a time-independant Hamiltonian with eigenvalues:

$$|\lambda I - H| = \begin{pmatrix} \lambda - \frac{\hbar\delta\omega}{2} & -\frac{\hbar\Omega}{2} \\ -\frac{\hbar\Omega}{2} & \lambda + \frac{\hbar\delta\omega}{2} \end{pmatrix}$$

$$\lambda_1 = \frac{\hbar}{2}\sqrt{\Omega^2 + (\delta\omega)^2},$$

$$\text{eigenstate } |+\rangle = \frac{1}{\sqrt{\Omega^2 + \left(\delta\omega + \sqrt{\Omega^2+(\delta\omega)^2}\right)^2}}\left(\Omega,\ \delta\omega + \sqrt{\Omega^2+(\delta\omega)^2}\right)$$

$$\lambda_2 = -\frac{\hbar}{2}\sqrt{\Omega^2+(\delta\omega)^2},$$

$$\text{eigenstate } |-\rangle = \frac{1}{\sqrt{\Omega^2+\left(-\delta\omega+\sqrt{\Omega^2+(\delta\omega)^2}\right)^2}}\left(\Omega, -\delta\omega + \sqrt{\Omega^2+(\delta\omega)^2}\right)$$

Since that

$$H = \hbar \frac{\delta\omega}{2}\sigma_z + \hbar\frac{\Omega}{2}\sigma_x = \vec{r}\cdot\vec{\sigma} = |\vec{r}|\,\hat{r}\cdot\vec{\sigma}$$

$$|\vec{r}| = \frac{\hbar}{2}\sqrt{\Omega^2+(\delta\omega)^2}, \quad \hat{r} = \left(\frac{\Omega}{\sqrt{\Omega^2+(\delta\omega)^2}},\ 0,\ \frac{\delta\omega}{\sqrt{\Omega^2+(\delta\omega)^2}}\right)$$

$$e^{-iHt/\hbar} = e^{-i\vec{r}\cdot\vec{\sigma}t/\hbar} = 1\cos\frac{|\vec{r}|t}{\hbar} - i\sigma_r \sin\frac{|\vec{r}|t}{\hbar}$$

So, $|\psi(t)\rangle = e^{-iHt}|0\rangle$ missing a factor $e^{i\delta\omega t/2}$?

$$= \cos\frac{|\vec{r}|t}{\hbar}|0\rangle - i\left(\frac{\Omega}{\sqrt{\Omega^2+(\delta\omega)^2}}|1\rangle + \frac{\delta\omega}{\sqrt{\Omega^2+(\delta\omega)^2}}|0\rangle\right)\sin\frac{|\vec{r}|t}{\hbar}$$

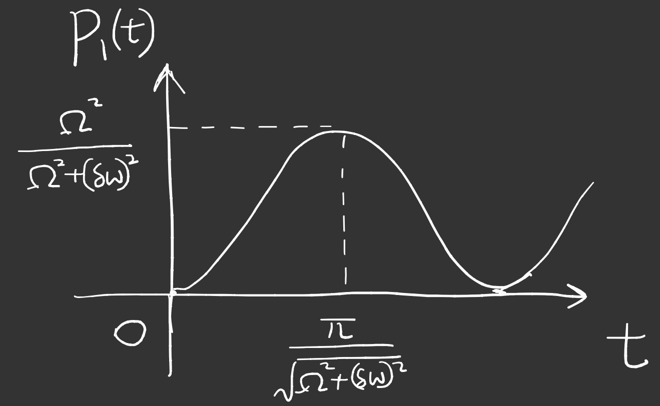
$$= \left(\cos\frac{|\vec{r}|t}{\hbar} - i\frac{\delta\omega}{\sqrt{\Omega^2+(\delta\omega)^2}}\sin\frac{|\vec{r}|t}{\hbar}\right)|0\rangle - i\frac{\Omega}{\sqrt{\Omega^2+(\delta\omega)^2}}\sin\frac{|\vec{r}|t}{\hbar}|1\rangle$$

With probability

$$P(|\psi\rangle = |1\rangle) = |\langle 1|\psi(t)\rangle|^2 = \frac{\Omega^2}{\Omega^2+(\delta\omega)^2}\sin^2\frac{|\vec{r}|t}{\hbar}$$

$$= \frac{\Omega^2}{\Omega^2+(\delta\omega)^2}\sin^2\left(\frac{\sqrt{\Omega^2+(\delta\omega)^2}}{2}t\right).$$

Could be sketched as :



Linewidth : $$\delta P = \frac{dP}{d\Omega}\,\delta\Omega + \frac{dP}{d(\delta\omega)}\,\delta(\delta\omega)$$

$$= \frac{2\Omega(\delta\omega)^2}{(\Omega^2+(\delta\omega)^2)^2}\sin^2\left(\frac{\sqrt{\Omega^2+(\delta\omega)^2}}{2}t\right)\delta\Omega$$

$$+ \frac{\Omega^2}{\Omega^2+(\delta\omega)^2}\sin\left(\frac{\sqrt{\Omega^2+(\delta\omega)^2}}{2}t\right)\cos\left(\frac{\sqrt{\Omega^2+(\delta\omega)^2}}{2}t\right)\cdot\frac{\Omega t}{\sqrt{\Omega^2+(\delta\omega)^2}}\,\delta\Omega$$

$$- \frac{2\Omega^2\delta\omega}{(\Omega^2+(\delta\omega)^2)^2}\sin^2\left(\frac{\sqrt{\Omega^2+(\delta\omega)^2}}{2}t\right)\delta(\delta\omega)$$

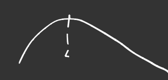
$$+ \frac{\Omega^2}{\Omega^2+(\delta\omega)^2}\sin\left(\frac{\sqrt{\Omega^2+(\delta\omega)^2}}{2}t\right)\cos\left(\frac{\sqrt{\Omega^2+(\delta\omega)^2}}{2}t\right)\cdot\frac{\delta\omega t}{\sqrt{\Omega^2+(\delta\omega)^2}}\,\delta(\delta\omega)$$

For a fixed Ω, we have $\delta\Omega = 0$.

$$\delta P = \frac{-2x\delta x}{(1+x^2)^2}\sin^2\left(\sqrt{1+x^2}\,\frac{\Omega t}{2}\right) + \frac{\delta x}{1+x^2}\sin\left(\sqrt{1+x^2}\,\frac{\Omega t}{2}\right)\cos\left(\sqrt{1+x^2}\,\frac{\Omega t}{2}\right)$$

δP around $t = \dfrac{\pi}{\sqrt{\Omega^2+(\delta\omega)^2}}$ is

$$\delta P\left(t=\frac{\pi}{\sqrt{\Omega^2+(\delta\omega)^2}}\right) = \frac{-2x\,\delta x}{(1+x^2)^2} . \qquad x = \frac{\delta\omega}{\Omega} .$$

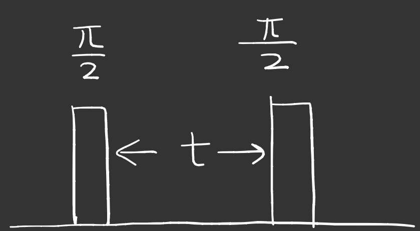


Should be maximized so that

$$x = \frac{1}{\sqrt{3}} \Rightarrow \Omega = \sqrt{3}\,\delta\omega .$$

(Does not make sense to me)

Part (b) — Ramsey Spectroscopy



$\tau = \frac{\pi}{2}/\Omega .$ $t .$

As is shown above,

$$|\psi(t)\rangle = e^{-iHt}\,|0\rangle$$

$$= \left(\cos\frac{|\vec{r}|t}{\hbar} - i\frac{\delta\omega}{\sqrt{\Omega^2+(\delta\omega)^2}}\sin\frac{|\vec{r}|t}{\hbar}\right)|0\rangle - i\frac{\Omega}{\sqrt{\Omega^2+(\delta\omega)^2}}\sin\frac{|\vec{r}|t}{\hbar}\,|1\rangle$$

$e^{i\delta\omega t/2}$ $\qquad$ $e^{-i\delta\omega t/2 - i\phi}$

After $t = \frac{\pi}{2\Omega}$, $\qquad \tilde{\Omega} = \sqrt{\Omega^2 + (\delta\omega)^2}$

$$\left|\psi\left(t=\frac{\pi}{2\Omega}\right)\right\rangle = \left[\cos\left(\frac{\tilde{\Omega}\pi}{4\Omega}\right) - i\frac{\delta\omega}{\tilde{\Omega}}\sin\left(\frac{\tilde{\Omega}\pi}{4\Omega}\right)\right]|0\rangle \cdot e^{i\frac{\pi\delta\omega}{4\Omega}}$$

$$\parallel$$

$$\psi_\tau \qquad\qquad - i\frac{\Omega}{\tilde{\Omega}}\sin\left(\frac{\tilde{\Omega}\pi}{4\Omega}\right)|1\rangle \cdot e^{-i\frac{\pi\delta\omega}{4\Omega} - i\phi}$$

it will free evolute for time duration t,

$$|\psi(t+\tau)\rangle = e^{-iHt/\hbar}|\psi_\tau\rangle$$

$e^{-iHt/\hbar}$ could be written in a matrix form

$$\begin{pmatrix} e^{i\delta\omega t/2}\cos(\tilde{\Omega}t/2) - i\frac{\delta\omega}{\tilde{\Omega}}\sin(\tilde{\Omega}t/2) & -ie^{i\delta\omega t/2 + i\phi}\frac{\Omega}{\tilde{\Omega}}\sin(\tilde{\Omega}t/2) \\ -ie^{-i\delta\omega t/2 - i\phi}\frac{\Omega}{\tilde{\Omega}}\sin(\tilde{\Omega}t/2) & e^{-i\delta\omega t/2}\cos(\tilde{\Omega}t/2) + i\frac{\delta\omega}{\tilde{\Omega}}\sin(\tilde{\Omega}t/2) \end{pmatrix}$$

Let us do an approximation for ψ_τ. assume $\delta\omega = 0$.

$$\Rightarrow \psi_\tau = \frac{1}{\sqrt{2}}e^{i\frac{\pi\delta\omega}{4\Omega}}|0\rangle - i\frac{1}{\sqrt{2}}e^{-i\frac{\pi\delta\omega}{4\Omega} - i\phi}|1\rangle$$

$$= \underbrace{\frac{1}{\sqrt{2}} e^{i\frac{\pi\delta\omega}{4\Omega}}}_{\text{neglected}} \left(|0\rangle - i e^{-i\frac{\pi\delta\omega}{2\Omega} - i\phi} |1\rangle \right)$$

The $\frac{\pi}{2}$ pulse operation is like

$$\Pi_2 = \begin{pmatrix} \frac{1}{\sqrt{2}} e^{i\frac{\pi\delta\omega}{4\Omega}} & -i\frac{1}{\sqrt{2}} e^{i\frac{\pi\delta\omega}{4\Omega}} \\ -i\frac{1}{\sqrt{2}} e^{-i\frac{\pi\delta\omega}{4\Omega} - i\phi} & \frac{1}{\sqrt{2}} e^{-i\frac{\pi\delta\omega}{4\Omega}} \end{pmatrix}$$

So after evolution of time duration t,

$$|\psi(t+\tau)\rangle = e^{-iHt/\hbar} |\psi_\tau\rangle$$

$$= e^{-iHt/\hbar} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} e^{-i\frac{\pi\delta\omega}{2\Omega} - i\phi} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{\sqrt{2}} e^{i\delta\omega t/2} \cos(\tilde{\Omega} t/2) - \frac{i}{\sqrt{2}} \frac{\delta\omega}{\tilde{\Omega}} \sin(\tilde{\Omega} t/2) \\ \qquad + \frac{1}{\sqrt{2}} e^{i\delta\omega t/2 + i\phi} \frac{\Omega}{\tilde{\Omega}} \sin(\tilde{\Omega} t/2)\, e^{-i\frac{\pi\delta\omega}{2\Omega} - i\phi} \\ \\ \frac{-1}{\sqrt{2}} e^{-i\frac{\pi\delta\omega}{2\Omega} - i\phi} \frac{\delta\omega}{\tilde{\Omega}} \sin(\tilde{\Omega} t/2) \\ \\ -\frac{i}{\sqrt{2}} e^{-i\delta\omega t/2 - i\phi} \frac{\Omega}{\tilde{\Omega}} \sin(\tilde{\Omega} t/2) + \frac{i}{\sqrt{2}} e^{-i\frac{\pi\delta\omega}{2\Omega} - i\phi - i\delta\omega t/2} \cos(\tilde{\Omega} t/2) \end{pmatrix}$$

$$
\simeq \begin{pmatrix} \frac{1}{\sqrt{2}} e^{i\delta\omega t/2} \cos(\tilde{\Omega} t/2) + \frac{1}{2} e^{i\delta\omega t/2 + i\phi} \frac{\Omega}{\tilde{\Omega}} \sin(\tilde{\Omega} t/2)\, e^{-i\frac{\pi\delta\omega}{2\Omega} - i\phi} \\ \\ -\frac{i}{\sqrt{2}} e^{-i\delta\omega t/2 - i\phi} \frac{\Omega}{\tilde{\Omega}} \sin(\tilde{\Omega} t/2) + \frac{i}{\sqrt{2}} e^{-i\frac{\pi\delta\omega}{2\Omega} - i\phi - i\delta\omega t/2} \cos(\tilde{\Omega} t/2) \end{pmatrix}
$$

After the second $\frac{\pi}{2}$ pulse,

$$
|\psi(t+2\tau)\rangle = \Pi_2 \, |\psi(t+\tau)\rangle .
$$

Probability (excited state) $= |c(1)|^2$.

with $c(1) =$

$$
-i\frac{1}{\sqrt{2}} e^{-i\frac{\pi\delta\omega}{4\Omega} - i\phi} \left(\frac{1}{\sqrt{2}} e^{i\delta\omega t/2} \cos(\tilde{\Omega} t/2) + \frac{1}{\sqrt{2}} e^{i\delta\omega t/2 + i\phi} \frac{\Omega}{\tilde{\Omega}} \sin(\tilde{\Omega} t/2)\, e^{-i\frac{\pi\delta\omega}{2\Omega} - i\phi} \right)
$$

$$
+ \frac{1}{\sqrt{2}} e^{-i\frac{\pi\delta\omega}{4\Omega}} \left(-\frac{i}{\sqrt{2}} e^{-i\delta\omega t/2 - i\phi} \frac{\Omega}{\tilde{\Omega}} \sin(\tilde{\Omega} t/2) + \frac{i}{\sqrt{2}} e^{-i\frac{\pi\delta\omega}{2\Omega} - i\phi - i\delta\omega t/2} \cos(\tilde{\Omega} t/2) \right)
$$

$$= \frac{i}{2} e^{-i\frac{\pi\delta\omega}{4\Omega} - i\phi} \cos(\tilde{\Omega} t/2) \left[e^{i\delta\omega t/2} + e^{-i\frac{\pi\delta\omega}{2\Omega} - i\delta\omega t/2} \right]$$

$$- \frac{i}{2} e^{-i\frac{\pi\delta\omega}{4\Omega} - i\phi} \sin(\tilde{\Omega} t/2) \left[e^{-i\delta\omega t/2} + e^{i\delta\omega t/2 - i\frac{\pi\delta\omega}{2\Omega}} \right] \frac{\Omega}{\tilde{\Omega}}$$

$$|c(1)| = \frac{1}{2} \left| \cos(\tilde{\Omega} t/2) \left(e^{i\delta\omega t/2} + e^{-i\frac{\pi\delta\omega}{2\Omega} - i\delta\omega t/2} \right) \right.$$

$$\left. - \frac{\Omega}{\tilde{\Omega}} \sin(\tilde{\Omega} t/2) \left(e^{-i\delta\omega t/2} + e^{i\delta\omega t/2 - i\frac{\pi\delta\omega}{2\Omega}} \right) \right|$$

$$= \frac{1}{2} \left\{ (a^2+b^2) \left[\cos^2\left(\frac{\tilde{\Omega} t}{2}\right) + \frac{\Omega^2}{\tilde{\Omega}^2} \sin^2\left(\frac{\tilde{\Omega} t}{2}\right) \right] \right.$$

$$\left. + \frac{\Omega}{\tilde{\Omega}} \sin(\tilde{\Omega} t)(b-a) \right\}^{\frac{1}{2}}$$

with $$a = \mathrm{Re}\left(e^{-i\delta\omega t/2} + e^{i\delta\omega t/2 - i\frac{\pi\delta\omega}{2\Omega}} \right)$$

$$= \cos\left(\frac{\delta\omega t}{2}\right) + \cos\left(\frac{\delta\omega t}{2} - \frac{\pi\delta\omega}{2\Omega}\right)$$

$$= \left[1 + \cos\left(\frac{\pi\delta\omega}{2\Omega}\right) \right] \cos\frac{\delta\omega t}{2} + \sin\frac{\pi\delta\omega}{2\Omega} \sin\frac{\delta\omega t}{2}$$

$$= 2\cos^2\left(\frac{\pi\delta\omega}{4\Omega}\right) \cos\left(\frac{\delta\omega t}{2}\right) + 2\sin\left(\frac{\pi\delta\omega}{4\Omega}\right) \cos\left(\frac{\pi\delta\omega}{4\Omega}\right) \sin\frac{\delta\omega t}{2}$$

$$b = \mathrm{Im}\left(e^{-i\delta\omega t/2} + e^{i\delta\omega t/2 - i\frac{\pi\delta\omega}{2\Omega}}\right)$$

$$= \sin\left(\delta\omega t/2 - \frac{\pi\delta\omega}{2\Omega}\right) - \sin\left(\frac{\delta\omega t}{2}\right)$$

$$= \sin\frac{\delta\omega t}{2}\cos\frac{\pi\delta\omega}{2\Omega} - \cos\frac{\delta\omega t}{2}\sin\frac{\pi\delta\omega}{2\Omega} - \sin\frac{\delta\omega t}{2}$$

?

$$= -2\sin\frac{\delta\omega t}{2}\sin^2\frac{\pi\delta\omega}{4\Omega} - 2\cos\frac{\delta\omega t}{2}\sin\frac{\pi\delta\omega}{4\Omega}\cos\frac{\pi\delta\omega}{4\Omega}$$

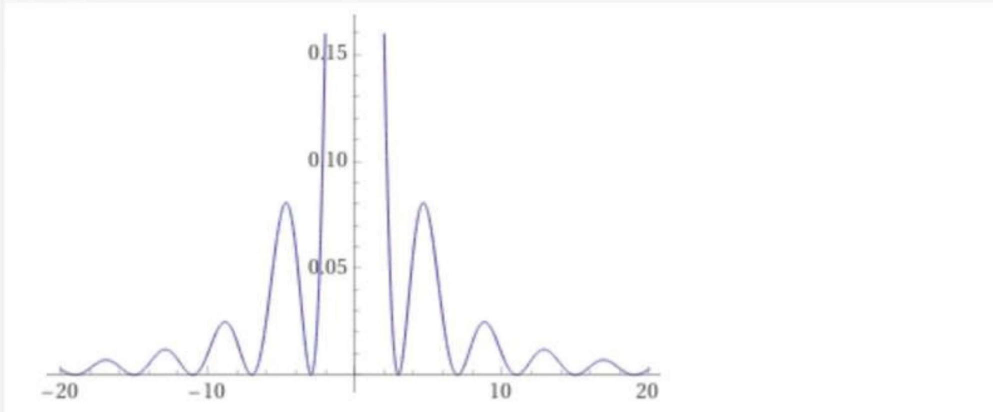
$\Rightarrow$

$$P = \frac{\Omega^2}{\tilde{\Omega}^2}\left(\cos\frac{\delta\omega t}{2} + \frac{\delta\omega}{\tilde{\Omega}}\sin\frac{\delta\omega t}{2}\right)^2$$

Input interpretation

plot	$\frac{1}{1+a^2}\left(\cos\left(\frac{\pi}{2}\times\frac{a}{2}\right) + \frac{a}{\sqrt{1+a^2}}\sin\left(\frac{\pi}{2}\times\frac{a}{2}\right)\right)^2$	$a = -20$ to 20

Plot



Linewidth could be found by

Part (c) — Optical Pumping

$$\begin{cases} \frac{d}{dt}\rho_{00} = \Gamma\rho_{11} + \frac{i}{2}\left(\Omega^*\bar{\rho}_{10} - \Omega\bar{\rho}_{01}\right) \\ \frac{d}{dt}\rho_{11} = -\Gamma\rho_{11} + \frac{i}{2}\left(\Omega\bar{\rho}_{01} - \Omega^*\bar{\rho}_{10}\right) \\ \frac{d}{dt}\bar{\rho}_{01} = -\left(\frac{\Gamma}{2} + i\delta\omega\right)\bar{\rho}_{01} + \frac{i}{2}\Omega^*\left(\rho_{11} - \rho_{00}\right) \\ \frac{d}{dt}\bar{\rho}_{10} = -\left(\frac{\Gamma}{2} - i\delta\omega\right)\bar{\rho}_{10} + \frac{i}{2}\Omega^*\left(\rho_{00} - \rho_{11}\right) \end{cases}$$

$$\bar{\rho}_{01} = \rho_{01}e^{-i\delta\omega t}$$

$$\bar{\rho}_{10} = \rho_{10}e^{i\delta\omega t}$$

$\Rightarrow$

$$\begin{cases} \frac{d}{dt}\rho_{00} = \Gamma\rho_{11} + \frac{i}{2}\left(\Omega^*\rho_{10}e^{i\delta\omega t} - \Omega\rho_{01}e^{-i\delta\omega t}\right) \\ \frac{d}{dt}\rho_{11} = -\Gamma\rho_{11} + \frac{i}{2}\left(\Omega\rho_{01}e^{-i\delta\omega t} - \Omega^*\rho_{10}e^{i\delta\omega t}\right) \\ e^{-i\delta\omega t}\frac{d\rho_{01}}{dt} - \cancel{i\delta\omega e^{-i\delta\omega t}\rho_{01}} = -\left(\frac{\Gamma}{2} + \cancel{i\delta\omega}\right)\rho_{01}e^{-i\delta\omega t} + \frac{i}{2}\Omega^*\left(\rho_{11} - \rho_{00}\right) \\ e^{i\delta\omega t}\frac{d\rho_{10}}{dt} + \cancel{i\delta\omega e^{i\delta\omega t}\rho_{10}} = -\left(\frac{\Gamma}{2} - \cancel{i\delta\omega}\right)\rho_{10}e^{i\delta\omega t} + \frac{i}{2}\Omega^*\left(\rho_{00} - \rho_{11}\right) \end{cases}$$

$$\Rightarrow \begin{cases} \frac{d}{dt}\rho_{00} = \Gamma\rho_{11} + \frac{i}{2}\left(\Omega^*\rho_{10}e^{i\delta\omega t} - \Omega\rho_{01}e^{-i\delta\omega t}\right) \\ \frac{d}{dt}\rho_{11} = -\Gamma\rho_{11} + \frac{i}{2}\left(\Omega\rho_{01}e^{-i\delta\omega t} - \Omega^*\rho_{10}e^{i\delta\omega t}\right) \\ \frac{d}{dt}\rho_{01} = -\frac{\Gamma}{2}\rho_{01} + \frac{i}{2}\Omega^*e^{i\delta\omega t}\left(\rho_{11} - \rho_{00}\right) \\ \frac{d}{dt}\rho_{10} = -\frac{\Gamma}{2}\rho_{10} + \frac{i}{2}\Omega^*e^{-i\delta\omega t}\left(\rho_{00} - \rho_{11}\right) \end{cases}$$

We have matrix :

$$O = \begin{bmatrix} 0 & -\frac{i}{2}\Omega e^{-i\delta\omega t} & \frac{i}{2}\Omega^* e^{i\delta\omega t} & \Gamma \\ \frac{i}{2}\Omega^* e^{i\delta\omega t} & -\frac{\Gamma}{2} & 0 & \frac{i}{2}\Omega^* e^{i\delta\omega t} \\ \frac{i}{2}\Omega^* e^{-i\delta\omega t} & 0 & -\frac{\Gamma}{2} & -\frac{i}{2}\Omega^* e^{-i\delta\omega t} \\ 0 & \frac{i}{2}\Omega e^{-i\delta\omega t} & -\frac{i}{2}\Omega^* e^{i\delta\omega t} & -\Gamma \end{bmatrix}$$

with eigenvalues :

$$|I\lambda - O| = \begin{bmatrix} \lambda & \frac{i}{2}\Omega e^{-i\delta\omega t} & -\frac{i}{2}\Omega^* e^{i\delta\omega t} & -\Gamma \\ -\frac{i}{2}\Omega^* e^{i\delta\omega t} & \lambda + \frac{\Gamma}{2} & 0 & -\frac{i}{2}\Omega^* e^{i\delta\omega t} \\ -\frac{i}{2}\Omega^* e^{-i\delta\omega t} & 0 & \lambda + \frac{\Gamma}{2} & \frac{i}{2}\Omega^* e^{-i\delta\omega t} \\ 0 & -\frac{i}{2}\Omega e^{-i\delta\omega t} & \frac{i}{2}\Omega^* e^{i\delta\omega t} & \lambda + \Gamma \end{bmatrix}$$

$$= \lambda\left\{\left(\lambda+\frac{\Gamma}{2}\right)\left[\left(\lambda+\frac{\Gamma}{2}\right)(\lambda+\Gamma)+\frac{\Omega^{*2}}{4}\right]\right.$$

$$\left. -\frac{i}{2}\Omega e^{-i\delta\omega t}\cdot\frac{i}{2}\Omega^{*}e^{i\delta\omega t}\cdot\left(\lambda+\frac{\Gamma}{2}\right)\right\}$$

$$+\frac{i}{2}\Omega^{*}e^{i\delta\omega t}\left\{\frac{i}{2}\Omega e^{-i\delta\omega t}\left[\left(\lambda+\frac{\Gamma}{2}\right)(\lambda+\Gamma)+\frac{\Omega^{*2}}{4}\right]\right.$$

$$\left. -\frac{i}{2}\Omega e^{-i\delta\omega t}\left[\frac{\Omega^{*2}}{4}+\Gamma\left(\lambda+\frac{\Gamma}{2}\right)\right]\right\}$$

$$-\frac{i}{2}\Omega^{*}e^{i\delta\omega t}\left\{i\Omega e^{-i\delta\omega t}\cdot\left(-\frac{\Omega^{*2}}{4}\right)+\left(\lambda+\frac{\Gamma}{2}\right)\cdot\frac{i}{2}\Omega^{*}e^{i\delta\omega t}\lambda\right.$$

$$\left. +\frac{i}{2}\Omega e^{-i\delta\omega t}\cdot\frac{\Omega^{*2}}{4}e^{2i\delta\omega t}\right\}$$

$$= \lambda\left(\lambda+\frac{\Gamma}{2}\right)\left[\left(\lambda+\frac{\Gamma}{2}\right)(\lambda+\Gamma)+\frac{\Omega^{*2}}{2}\right]-\frac{|\Omega|^2}{4}\lambda\left(\lambda+\frac{\Gamma}{2}\right)$$

Should not do in this way.

Let's just find the steady state.

with $d\rho_{11}/dt = 0$

Suppose at steady state $\rho_{01} = A$, $\rho_{10} = B$.

$$\bar{\rho}_{01} = A e^{-i\delta\omega t}, \quad \bar{\rho}_{10} = B e^{i\delta\omega t}.$$

In this case

$$\begin{cases} \frac{d}{dt}\rho_{00} = \Gamma \rho_{11} + \frac{i}{2}\left(\Omega^* B e^{i\delta\omega t} - \Omega A e^{-i\delta\omega t}\right) = 0 \\ \frac{d}{dt}\rho_{11} = -\Gamma \rho_{11} + \frac{i}{2}\left(\Omega A e^{-i\delta\omega t} - \Omega^* B e^{i\delta\omega t}\right) = 0 \\ \frac{dA}{dt} e^{-i\delta\omega t} - i\delta\omega A e^{-i\delta\omega t} = -\left(\frac{\Gamma}{2} + i\delta\omega\right) A e^{-i\delta\omega t} + \frac{i}{2}\Omega^*(\rho_{11} - \rho_{00}) \\ \frac{dB}{dt} e^{i\delta\omega t} + i\delta\omega B e^{i\delta\omega t} = -\left(\frac{\Gamma}{2} - i\delta\omega\right) B e^{i\delta\omega t} + \frac{i}{2}\Omega^*(\rho_{00} - \rho_{11}) \end{cases}$$

$$\Rightarrow \quad \Gamma \rho_{11} = \frac{i}{2}\left(\Omega A e^{-i\delta\omega t} - \Omega^* B e^{i\delta\omega t}\right)$$

$$\rho_{11} - \rho_{00} = \frac{\Gamma A e^{-i\delta\omega t} + \frac{dA}{dt}}{i\Omega^*} = \frac{-\Gamma B e^{i\delta\omega t} - \frac{dB}{dt}}{i\Omega^*}$$

const

Suppose $A =$

2 Problem 2 – Laser cooling and trapping

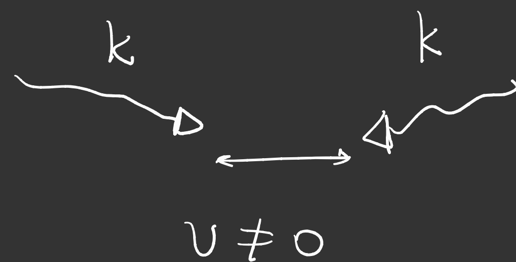
Part (a) — Radiation Pressure

$$\rho_{11} = \frac{|\Omega|^2}{2\Omega^2 + \Gamma^2 + 4\delta\omega^2}$$

$$F_{scatter} = \frac{\Delta \vec{p}}{\Delta t} = \frac{\hbar \vec{k}}{\tau / f_e} = \frac{\hbar \vec{k}}{1/\Gamma f_e} \quad \rho_{11}$$

$$= \hbar \vec{k} \, \frac{|\Omega|^2 \Gamma}{2\Omega^2 + \Gamma^2 + 4\delta\omega^2} .$$

Part (b) — Doppler Cooling



$$F_{total} = F_+ - F_-$$

For population of moving atom,

$$\omega + kv \longrightarrow \rho \uparrow$$

$$\omega - kv \longrightarrow \rho \downarrow$$

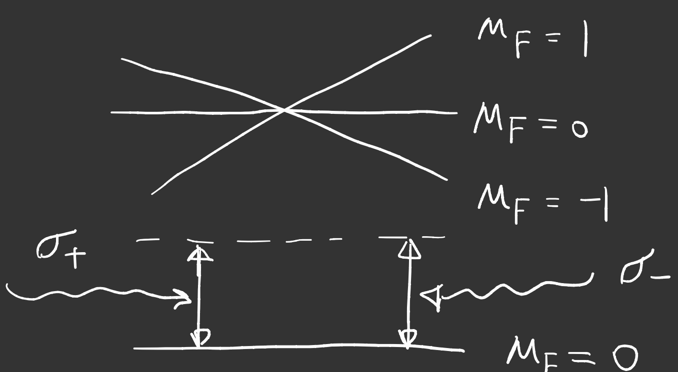
$$F_{total} = \hbar k \frac{|\Omega + kv|^2 \Gamma}{2(\Omega+kv)^2 + \Gamma^2 + 4\delta\omega^2} - \hbar k \frac{|\Omega - kv|^2 \Gamma}{2(\Omega-kv)^2 + \Gamma^2 + 4\delta\omega^2}$$

$$= \hbar k \Gamma \frac{4\Omega k v (\Gamma^2 + 4\delta\omega^2)}{(\Gamma^2 + 4\delta\omega^2 + 2\Omega^2 + 2k^2v^2)^2 - (4\Omega k v)^2}$$

$$= 4\hbar k^2 \frac{\Omega (\Gamma^2 + 4\delta\omega^2)}{(\Gamma^2 + 4\delta\omega^2 + 2\Omega^2 + 2k^2v^2)^2 - (4\Omega k v)^2} v$$

$$= 4\hbar k^2 \frac{\Omega / (\Gamma^2 + 4\delta\omega^2)}{\left(1 + \frac{2\Omega^2}{\Gamma^2 + 4\delta\omega^2}\right)^2 + 4k^2v^2\left(1 + \frac{k^2v^2}{\Gamma^2 + 4\delta\omega^2}\right)} v$$

Part (c) — Magneto-optical Trap (MOT)



$$F\left(\omega + kv - \omega_0 - \frac{g\mu_B}{\hbar}\frac{\partial B}{\partial z} z\right)$$

$$F_{MOT} = -\eta v - \frac{\Omega}{k}\frac{g\mu_B}{\hbar}\frac{\partial B}{\partial z} z.$$

$x - x_0$

$$F = F_+ + F_-$$

$$= \frac{1}{2}\hbar k \Gamma \frac{S_0}{1+S_0+(2\Delta_+/\Gamma)^2} + \frac{1}{2}\hbar k \Gamma \frac{S_0}{1+S_0+(2\Delta_-/\Gamma)^2}$$

with $\Delta_+ = \delta\omega - kv + \mu B/\hbar$.

$\Delta_- = \delta\omega + kv - \mu B/\hbar$.

$$S_0 = \frac{I}{I_0}.$$

Let $\gamma_m = g_J \mu_B/\hbar$.

$$F_\pm(x-x_0) = \hbar \vec{k} \frac{|\Omega|^2 \Gamma}{2\Omega^2 + \Gamma^2 + 4\underbrace{\delta\omega^2}_{\delta\omega \pm \gamma_m \frac{\partial B}{\partial z} z}}.$$

For small $\vec{v}$,

$$F = -\eta \vec{v} - \frac{\Omega}{k} \frac{g\mu_B}{\hbar} \frac{\partial B}{\partial z} z.$$

$$F_{total} = \hbar k \frac{|\Omega + kv|^2 \Gamma}{2(\Omega+kv)^2 + \Gamma^2 + 4\delta\omega^2}$$

$$- \hbar k \frac{|\Omega - kv|^2 \Gamma}{2(\Omega-kv)^2 + \Gamma^2 + 4\delta\omega^2}$$

$$= \hbar k \Gamma$$

$$\frac{a+b}{c+d} - \frac{a-b}{c-d}$$

$$\frac{\overbrace{(\Omega^2 + k^2v^2 + 2\Omega k v)}^{a+b}}{\underbrace{[\Gamma^2 + 4\delta\omega^2 + 2\Omega^2 + 2k^2v^2 + 4\Omega k v]}_{c+d}} - \frac{(\Omega^2 + k^2v^2 - 2\Omega k v)}{\Gamma^2 + 4\delta\omega^2 + 2\Omega^2 + 2k^2v^2 - 4\Omega k v}$$

$$= \frac{2(bc-ad)}{c^2-d^2}$$

$$ac + bc - ad - bd$$

$$-(ac + ad - bc - bd)$$

$$= 2(bc - ad)$$

$$= \frac{2 \cdot [2\Omega k v \cdot (\Gamma^2 + 4\delta\omega^2 + 2\Omega^2 + 2k^2v^2) - (\Omega^2 + k^2v^2) 4\Omega k v]}{(\Gamma^2 + 4\delta\omega^2 + 2\Omega^2 + 2k^2v^2)^2 - (4\Omega k v)^2}$$

$$=$$

HW 2 yuxiang@uchicago.edu

1 Problem 1 — Different approaches to spectroscopy

Part (a) - Rabi Spectroscopy

$$|\psi(t)\rangle = e^{-iHt}|0\rangle$$

$$H = \hbar\frac{\delta\omega}{2}\sigma_z + \hbar\frac{\Omega}{2}\sigma_x = \begin{pmatrix} \frac{\hbar\delta\omega}{2} & \frac{\hbar\Omega}{2} \\ \frac{\hbar\Omega}{2} & -\frac{\hbar\delta\omega}{2} \end{pmatrix}$$

is a time-independant Hamiltonian with eigenvalues:

$$|\lambda I - H| = \begin{pmatrix} \lambda-\frac{\hbar\delta\omega}{2} & -\frac{\hbar\Omega}{2} \\ -\frac{\hbar\Omega}{2} & \lambda+\frac{\hbar\delta\omega}{2} \end{pmatrix}$$

$$\lambda_1 = \frac{\hbar}{2}\sqrt{\Omega^2+(\delta\omega)^2},$$

eigenstate $|+\rangle = \dfrac{1}{\sqrt{\Omega^2+\left(\delta\omega+\sqrt{\Omega^2+(\delta\omega)^2}\right)^2}}\left(\Omega,\ \delta\omega+\sqrt{\Omega^2+(\delta\omega)^2}\right)$

$$\lambda_2 = -\frac{\hbar}{2}\sqrt{\Omega^2+(\delta\omega)^2},$$

eigenstate $|-\rangle = \dfrac{1}{\sqrt{\Omega^2+\left(-\delta\omega+\sqrt{\Omega^2+(\delta\omega)^2}\right)^2}}\left(\Omega,\ -\delta\omega+\sqrt{\Omega^2+(\delta\omega)^2}\right)$

Since that

$$H = \hbar \frac{\delta\omega}{2}\sigma_z + \hbar\frac{\Omega}{2}\sigma_x = \vec{r}\cdot\vec{\sigma} = |\vec{r}|\,\hat{r}\cdot\vec{\sigma}$$

$$|\vec{r}| = \frac{\hbar}{2}\sqrt{\Omega^2+(\delta\omega)^2}, \quad \hat{r} = \left(\frac{\Omega}{\sqrt{\Omega^2+(\delta\omega)^2}},\ 0,\ \frac{\delta\omega}{\sqrt{\Omega^2+(\delta\omega)^2}}\right)$$

$$e^{-iHt/\hbar} = e^{-i\vec{r}\cdot\vec{\sigma}t/\hbar} = 1\cos\frac{|\vec{r}|t}{\hbar} - i\sigma_r \sin\frac{|\vec{r}|t}{\hbar}$$

So, $|\psi(t)\rangle = e^{-iHt}|0\rangle$ missing a factor $e^{i\delta\omega t/2}$?

$$= \cos\frac{|\vec{r}|t}{\hbar}|0\rangle - i\left(\frac{\Omega}{\sqrt{\Omega^2+(\delta\omega)^2}}|1\rangle + \frac{\delta\omega}{\sqrt{\Omega^2+(\delta\omega)^2}}|0\rangle\right)\sin\frac{|\vec{r}|t}{\hbar}$$

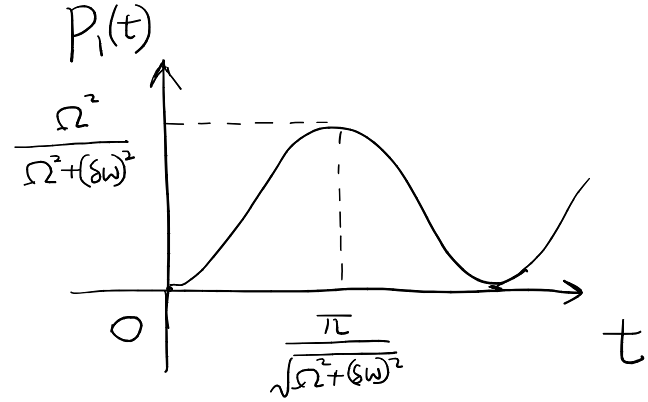
$$= \left(\cos\frac{|\vec{r}|t}{\hbar} - i\frac{\delta\omega}{\sqrt{\Omega^2+(\delta\omega)^2}}\sin\frac{|\vec{r}|t}{\hbar}\right)|0\rangle - i\frac{\Omega}{\sqrt{\Omega^2+(\delta\omega)^2}}\sin\frac{|\vec{r}|t}{\hbar}|1\rangle$$

With probability

$$P(|\psi\rangle = |1\rangle) = |\langle 1|\psi(t)\rangle|^2 = \frac{\Omega^2}{\Omega^2+(\delta\omega)^2}\sin^2\frac{|\vec{r}|t}{\hbar}$$

$$= \frac{\Omega^2}{\Omega^2+(\delta\omega)^2}\sin^2\left(\frac{\sqrt{\Omega^2+(\delta\omega)^2}}{2}t\right).$$

Could be sketched as:



Linewidth :
$$\delta P = \frac{dP}{d\Omega}\delta\Omega + \frac{dP}{d(\delta\omega)}\delta(\delta\omega)$$

$$= \frac{2\Omega(\delta\omega)^2}{(\Omega^2+(\delta\omega)^2)^2}\sin^2\left(\frac{\sqrt{\Omega^2+(\delta\omega)^2}}{2}t\right)\delta\Omega$$

$$+\frac{\Omega^2}{\Omega^2+(\delta\omega)^2}\sin\left(\frac{\sqrt{\Omega^2+(\delta\omega)^2}}{2}t\right)\cos\left(\frac{\sqrt{\Omega^2+(\delta\omega)^2}}{2}t\right)\cdot\frac{\Omega t}{\sqrt{\Omega^2+(\delta\omega)^2}}\delta\Omega$$

$$-\frac{2\Omega^2\delta\omega}{(\Omega^2+(\delta\omega)^2)^2}\sin^2\left(\frac{\sqrt{\Omega^2+(\delta\omega)^2}}{2}t\right)\delta(\delta\omega)$$

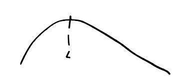
$$+\frac{\Omega^2}{\Omega^2+(\delta\omega)^2}\sin\left(\frac{\sqrt{\Omega^2+(\delta\omega)^2}}{2}t\right)\cos\left(\frac{\sqrt{\Omega^2+(\delta\omega)^2}}{2}t\right)\cdot\frac{\delta\omega t}{\sqrt{\Omega^2+(\delta\omega)^2}}\delta(\delta\omega)$$

For a fixed Ω, we have $\delta\Omega=0$.

$$\delta P = \frac{-2x\delta x}{(1+x^2)^2}\sin^2\left(\sqrt{1+x^2}\,\frac{\Omega t}{2}\right) + \frac{\delta x}{1+x^2}\sin\left(\sqrt{1+x^2}\,\frac{\Omega t}{2}\right)\cos\left(\sqrt{1+x^2}\,\frac{\Omega t}{2}\right)$$

δP around $t = \dfrac{\pi}{\sqrt{\Omega^2+(\delta\omega)^2}}$ is

$$\delta P\left(t=\frac{\pi}{\sqrt{\Omega^2+(\delta\omega)^2}}\right) = \frac{-2x\,\delta x}{(1+x^2)^2}. \qquad x = \frac{\delta\omega}{\Omega}.$$

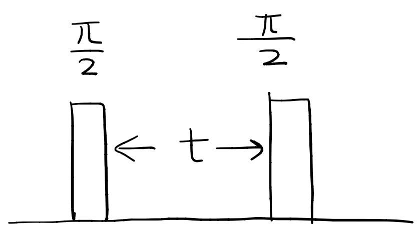


Should be maximized so that

$$x = \frac{1}{\sqrt{3}} \Rightarrow \Omega = \sqrt{3}\,\delta\omega.$$

(Does not make sense to me)

Part (b) – Ramsey Spectroscopy



$\tau = \frac{\pi}{2}/\Omega$. t.

As is shown above,

$$|\psi(t)\rangle = e^{-iHt}|0\rangle$$

$$= \left(\cos\frac{|\vec{r}|t}{\hbar} - i\frac{\delta\omega}{\sqrt{\Omega^2+(\delta\omega)^2}}\sin\frac{|\vec{r}|t}{\hbar}\right)|0\rangle - i\frac{\Omega}{\sqrt{\Omega^2+(\delta\omega)^2}}\sin\frac{|\vec{r}|t}{\hbar}|1\rangle$$

$e^{i\delta\omega t/2}$ $\qquad\qquad$ $e^{-i\delta\omega t/2 - i\phi}$

After $t = \frac{\pi}{2\Omega}$, $\qquad \tilde{\Omega} = \sqrt{\Omega^2 + (\delta\omega)^2}$

$$|\psi(t = \tfrac{\pi}{2\Omega})\rangle = \left[\cos\left(\frac{\tilde{\Omega}\pi}{4\Omega}\right) - i\frac{\delta\omega}{\tilde{\Omega}}\sin\left(\frac{\tilde{\Omega}\pi}{4\Omega}\right)\right]|0\rangle \cdot e^{i\frac{\pi\delta\omega}{4\Omega}}$$

$$\parallel$$

$$\psi_\tau \qquad\qquad - i\frac{\Omega}{\tilde{\Omega}}\sin\left(\frac{\tilde{\Omega}\pi}{4\Omega}\right)|1\rangle \cdot e^{-i\frac{\pi\delta\omega}{4\Omega} - i\phi}$$

it will free evolute for time duration t,

$$|\psi(t+\tau)\rangle = e^{-iHt/\hbar}|\psi_\tau\rangle$$

$e^{-iHt/\hbar}$ could be written in a matrix form

$$\begin{pmatrix} e^{i\delta\omega t/2}\cos(\tilde{\Omega}t/2) - i\frac{\delta\omega}{\tilde{\Omega}}\sin(\tilde{\Omega}t/2) & -ie^{i\delta\omega t/2 + i\phi}\frac{\Omega}{\tilde{\Omega}}\sin(\tilde{\Omega}t/2) \\ -ie^{-i\delta\omega t/2 - i\phi}\frac{\Omega}{\tilde{\Omega}}\sin(\tilde{\Omega}t/2) & e^{-i\delta\omega t/2}\cos(\tilde{\Omega}t/2) + i\frac{\delta\omega}{\tilde{\Omega}}\sin(\tilde{\Omega}t/2) \end{pmatrix}$$

Let us do an approximation for ψ_τ. assume $\delta\omega = 0$.

$$\Rightarrow \psi_\tau = \frac{1}{\sqrt{2}}e^{i\frac{\pi\delta\omega}{4\Omega}}|0\rangle - i\frac{1}{\sqrt{2}}e^{-i\frac{\pi\delta\omega}{4\Omega} - i\phi}|1\rangle$$

$$= \frac{1}{\sqrt{2}} \underbrace{e^{i\frac{\pi\delta\omega}{4\Omega}}}_{\text{neglected}} \left(|0\rangle - i e^{-i\frac{\pi\delta\omega}{2\Omega} - i\phi} |1\rangle \right)$$

The $\frac{\pi}{2}$ pulse operation is like

$$\Pi_2 = \begin{pmatrix} \frac{1}{\sqrt{2}} e^{i\frac{\pi\delta\omega}{4\Omega}} & -i\frac{1}{\sqrt{2}} e^{i\frac{\pi\delta\omega}{4\Omega}} \\ -i\frac{1}{\sqrt{2}} e^{-i\frac{\pi\delta\omega}{4\Omega} - i\phi} & \frac{1}{\sqrt{2}} e^{-i\frac{\pi\delta\omega}{4\Omega}} \end{pmatrix}$$

So after evolution of time duration t,

$$|\psi(t+\tau)\rangle = e^{-iHt/\hbar} |\psi_\tau\rangle$$

$$= e^{-iHt/\hbar} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} e^{-i\frac{\pi\delta\omega}{2\Omega} - i\phi} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{\sqrt{2}} e^{i\delta\omega t/2} \cos(\tilde{\Omega}t/2) - \frac{i}{\sqrt{2}} \frac{\delta\omega}{\tilde{\Omega}} \sin(\tilde{\Omega}t/2) + \frac{1}{\sqrt{2}} e^{i\delta\omega t/2 + i\phi} \frac{\Omega}{\tilde{\Omega}} \sin(\tilde{\Omega}t/2)\, e^{-i\frac{\pi\delta\omega}{2\Omega} - i\phi} \\ \\ \frac{-1}{\sqrt{2}} e^{-i\frac{\pi\delta\omega}{2\Omega} - i\phi} \frac{\delta\omega}{\tilde{\Omega}} \sin(\tilde{\Omega}t/2) \\ \\ -\frac{i}{\sqrt{2}} e^{-i\delta\omega t/2 - i\phi} \frac{\Omega}{\tilde{\Omega}} \sin(\tilde{\Omega}t/2) + \frac{i}{\sqrt{2}} e^{-i\frac{\pi\delta\omega}{2\Omega} - i\phi - i\delta\omega t/2} \cos(\tilde{\Omega}t/2) \end{pmatrix}$$

$$
\simeq \begin{pmatrix} \frac{1}{\sqrt{2}} e^{i\delta\omega t/2} \cos(\tilde{\Omega} t/2) + \frac{1}{2} e^{i\delta\omega t/2 + i\phi} \frac{\Omega}{\tilde{\Omega}} \sin(\tilde{\Omega} t/2)\, e^{-i\frac{\pi\delta\omega}{2\Omega} - i\phi} \\ \\ -\frac{i}{\sqrt{2}} e^{-i\delta\omega t/2 - i\phi} \frac{\Omega}{\tilde{\Omega}} \sin(\tilde{\Omega} t/2) + \frac{i}{\sqrt{2}} e^{-i\frac{\pi\delta\omega}{2\Omega} - i\phi - i\delta\omega t/2} \cos(\tilde{\Omega} t/2) \end{pmatrix}
$$

After the second $\frac{\pi}{2}$ pulse,

$$
|\psi(t+2\tau)\rangle = \Pi_2 \, |\psi(t+\tau)\rangle .
$$

$$
\text{Probability (excited state)} = |c(1)|^2 .
$$

with $c(1) =$

$$
-i\frac{1}{\sqrt{2}} e^{-i\frac{\pi\delta\omega}{4\Omega} - i\phi} \left(\frac{1}{\sqrt{2}} e^{i\delta\omega t/2} \cos(\tilde{\Omega} t/2) + \frac{1}{\sqrt{2}} e^{i\delta\omega t/2 + i\phi} \frac{\Omega}{\tilde{\Omega}} \sin(\tilde{\Omega} t/2)\, e^{-i\frac{\pi\delta\omega}{2\Omega} - i\phi} \right)
$$

$$
+ \frac{1}{\sqrt{2}} e^{-i\frac{\pi\delta\omega}{4\Omega}} \left(-\frac{i}{\sqrt{2}} e^{-i\delta\omega t/2 - i\phi} \frac{\Omega}{\tilde{\Omega}} \sin(\tilde{\Omega} t/2) + \frac{i}{\sqrt{2}} e^{-i\frac{\pi\delta\omega}{2\Omega} - i\phi - i\delta\omega t/2} \cos(\tilde{\Omega} t/2) \right)
$$

$$= \frac{i}{2} e^{-\frac{i\pi\delta\omega}{4\Omega} - i\phi} \cos(\tilde{\Omega} t/2) \left[e^{i\delta\omega t/2} + e^{-i\frac{\pi\delta\omega}{2\Omega} - i\delta\omega t/2} \right]$$

$$- \frac{i}{2} e^{-\frac{i\pi\delta\omega}{4\Omega} - i\phi} \sin(\tilde{\Omega} t/2) \left[e^{-i\delta\omega t/2} + e^{i\delta\omega t/2 - i\frac{\pi\delta\omega}{2\Omega}} \right] \frac{\Omega}{\tilde{\Omega}}$$

$$|c(1)| = \frac{1}{2} \Big| \cos(\tilde{\Omega} t/2) \left(e^{i\delta\omega t/2} + e^{-i\frac{\pi\delta\omega}{2\Omega} - i\delta\omega t/2} \right)$$

$$- \frac{\Omega}{\tilde{\Omega}} \sin(\tilde{\Omega} t/2) \left(e^{-i\delta\omega t/2} + e^{i\delta\omega t/2 - i\frac{\pi\delta\omega}{2\Omega}} \right) \Big|$$

$$= \frac{1}{2} \left\{ (a^2+b^2) \left[\cos^2\left(\frac{\tilde{\Omega} t}{2}\right) + \frac{\Omega^2}{\tilde{\Omega}^2} \sin^2\left(\frac{\tilde{\Omega} t}{2}\right) \right] \right.$$

$$\left. + \frac{\Omega}{\tilde{\Omega}} \sin(\tilde{\Omega} t)(b-a) \right\}^{\frac{1}{2}}$$

with $$a = \mathrm{Re}\left(e^{-i\delta\omega t/2} + e^{i\delta\omega t/2 - i\frac{\pi\delta\omega}{2\Omega}} \right)$$

$$= \cos\left(\frac{\delta\omega t}{2}\right) + \cos\left(\frac{\delta\omega t}{2} - \frac{\pi\delta\omega}{2\Omega}\right)$$

$$= \left[1 + \cos\left(\frac{\pi\delta\omega}{2\Omega}\right) \right] \cos\frac{\delta\omega t}{2} + \sin\frac{\pi\delta\omega}{2\Omega} \sin\frac{\delta\omega t}{2}$$

$$= 2\cos^2\left(\frac{\pi\delta\omega}{4\Omega}\right) \cos\left(\frac{\delta\omega t}{2}\right) + 2\sin\left(\frac{\pi\delta\omega}{4\Omega}\right) \cos\left(\frac{\pi\delta\omega}{4\Omega}\right) \sin\frac{\delta\omega t}{2}$$

$$b = \mathrm{Im}\left(e^{-i\delta\omega t/2} + e^{i\delta\omega t/2 - i\frac{\pi\delta\omega}{2\Omega}}\right)$$

$$= \sin\left(\delta\omega t/2 - \frac{\pi\delta\omega}{2\Omega}\right) - \sin\left(\frac{\delta\omega t}{2}\right)$$

$$= \sin\frac{\delta\omega t}{2}\cos\frac{\pi\delta\omega}{2\Omega} - \cos\frac{\delta\omega t}{2}\sin\frac{\pi\delta\omega}{2\Omega} - \sin\frac{\delta\omega t}{2}$$

?

$$= -2\sin\frac{\delta\omega t}{2}\sin^2\frac{\pi\delta\omega}{4\Omega} - 2\cos\frac{\delta\omega t}{2}\sin\frac{\pi\delta\omega}{4\Omega}\cos\frac{\pi\delta\omega}{4\Omega}$$

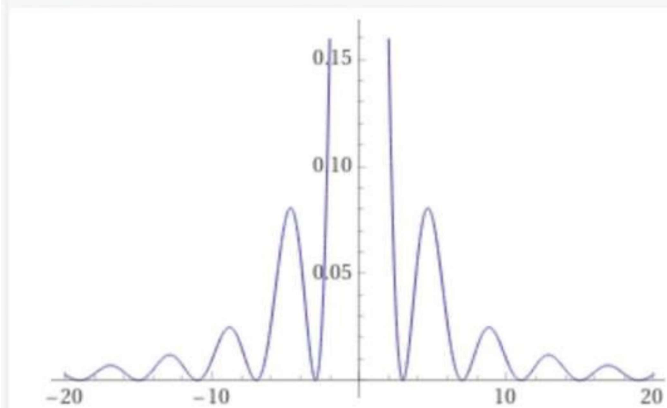
$\Rightarrow$

$$P = \frac{\Omega^2}{\tilde{\Omega}^2}\left(\cos\frac{\delta\omega t}{2} + \frac{\delta\omega}{\tilde{\Omega}}\sin\frac{\delta\omega t}{2}\right)^2$$

Input interpretation

plot	$\frac{1}{1+a^2}\left(\cos\left(\frac{\pi}{2}\times\frac{a}{2}\right) + \frac{a}{\sqrt{1+a^2}}\sin\left(\frac{\pi}{2}\times\frac{a}{2}\right)\right)^2$	$a = -20$ to 20

Plot



Linewidth could be found by

Part (c) — Optical Pumping

$$\begin{cases} \frac{d}{dt}\rho_{00} = \Gamma\rho_{11} + \frac{i}{2}\left(\Omega^*\bar{\rho}_{10} - \Omega\bar{\rho}_{01}\right) \\ \frac{d}{dt}\rho_{11} = -\Gamma\rho_{11} + \frac{i}{2}\left(\Omega\bar{\rho}_{01} - \Omega^*\bar{\rho}_{10}\right) \\ \frac{d}{dt}\bar{\rho}_{01} = -\left(\frac{\Gamma}{2} + i\delta\omega\right)\bar{\rho}_{01} + \frac{i}{2}\Omega^*\left(\rho_{11} - \rho_{00}\right) \\ \frac{d}{dt}\bar{\rho}_{10} = -\left(\frac{\Gamma}{2} - i\delta\omega\right)\bar{\rho}_{10} + \frac{i}{2}\Omega^*\left(\rho_{00} - \rho_{11}\right) \end{cases}$$

$$\bar{\rho}_{01} = \rho_{01}e^{-i\delta\omega t}$$

$$\bar{\rho}_{10} = \rho_{10}e^{i\delta\omega t}$$

$\Rightarrow$

$$\begin{cases} \frac{d}{dt}\rho_{00} = \Gamma\rho_{11} + \frac{i}{2}\left(\Omega^*\rho_{10}e^{i\delta\omega t} - \Omega\rho_{01}e^{-i\delta\omega t}\right) \\ \frac{d}{dt}\rho_{11} = -\Gamma\rho_{11} + \frac{i}{2}\left(\Omega\rho_{01}e^{-i\delta\omega t} - \Omega^*\rho_{10}e^{i\delta\omega t}\right) \\ e^{-i\delta\omega t}\frac{d\rho_{01}}{dt} \cancel{- i\delta\omega e^{-i\delta\omega t}\rho_{01}} = -\left(\frac{\Gamma}{2} \cancel{+ i\delta\omega}\right)\rho_{01}e^{-i\delta\omega t} + \frac{i}{2}\Omega^*\left(\rho_{11} - \rho_{00}\right) \\ e^{i\delta\omega t}\frac{d\rho_{10}}{dt} \cancel{+ i\delta\omega e^{i\delta\omega t}\rho_{10}} = -\left(\frac{\Gamma}{2} \cancel{- i\delta\omega}\right)\rho_{10}e^{i\delta\omega t} + \frac{i}{2}\Omega^*\left(\rho_{00} - \rho_{11}\right) \end{cases}$$

$\Rightarrow$

$$\begin{cases} \frac{d}{dt}\rho_{00} = \Gamma\rho_{11} + \frac{i}{2}\left(\Omega^*\rho_{10}e^{i\delta\omega t} - \Omega\rho_{01}e^{-i\delta\omega t}\right) \\ \frac{d}{dt}\rho_{11} = -\Gamma\rho_{11} + \frac{i}{2}\left(\Omega\rho_{01}e^{-i\delta\omega t} - \Omega^*\rho_{10}e^{i\delta\omega t}\right) \\ \frac{d}{dt}\rho_{01} = -\frac{\Gamma}{2}\rho_{01} + \frac{i}{2}\Omega^* e^{i\delta\omega t}\left(\rho_{11} - \rho_{00}\right) \\ \frac{d}{dt}\rho_{10} = -\frac{\Gamma}{2}\rho_{10} + \frac{i}{2}\Omega^* e^{-i\delta\omega t}\left(\rho_{00} - \rho_{11}\right) \end{cases}$$

We have matrix:

$$O = \begin{bmatrix} 0 & -\frac{i}{2}\Omega e^{-i\delta\omega t} & \frac{i}{2}\Omega^* e^{i\delta\omega t} & \Gamma \\ \frac{i}{2}\Omega^* e^{i\delta\omega t} & -\frac{\Gamma}{2} & 0 & \frac{i}{2}\Omega^* e^{i\delta\omega t} \\ \frac{i}{2}\Omega^* e^{-i\delta\omega t} & 0 & -\frac{\Gamma}{2} & -\frac{i}{2}\Omega^* e^{-i\delta\omega t} \\ 0 & \frac{i}{2}\Omega e^{-i\delta\omega t} & -\frac{i}{2}\Omega^* e^{i\delta\omega t} & -\Gamma \end{bmatrix}$$

with eigenvalues:

$$|I\lambda - O| = \begin{bmatrix} \lambda & \frac{i}{2}\Omega e^{-i\delta\omega t} & -\frac{i}{2}\Omega^* e^{i\delta\omega t} & -\Gamma \\ -\frac{i}{2}\Omega^* e^{i\delta\omega t} & \lambda+\frac{\Gamma}{2} & 0 & -\frac{i}{2}\Omega^* e^{i\delta\omega t} \\ -\frac{i}{2}\Omega^* e^{-i\delta\omega t} & 0 & \lambda+\frac{\Gamma}{2} & \frac{i}{2}\Omega^* e^{-i\delta\omega t} \\ 0 & -\frac{i}{2}\Omega e^{-i\delta\omega t} & \frac{i}{2}\Omega^* e^{i\delta\omega t} & \lambda+\Gamma \end{bmatrix}$$

$$= \lambda \left\{ \left(\lambda + \frac{\Gamma}{2}\right) \left[\left(\lambda + \frac{\Gamma}{2}\right)(\lambda + \Gamma) + \frac{\Omega^{*2}}{4} \right] \right.$$

$$\left. - \frac{i}{2}\Omega e^{-i\delta\omega t} \cdot \frac{i}{2}\Omega^* e^{i\delta\omega t} \cdot \left(\lambda + \frac{\Gamma}{2}\right) \right\}$$

$$+ \frac{i}{2}\Omega^* e^{i\delta\omega t} \left\{ \frac{i}{2}\Omega e^{-i\delta\omega t} \left[\left(\lambda + \frac{\Gamma}{2}\right)(\lambda + \Gamma) + \frac{\Omega^{*2}}{4} \right] \right.$$

$$\left. - \frac{i}{2}\Omega e^{-i\delta\omega t} \left[\frac{\Omega^{*2}}{4} + \Gamma\left(\lambda + \frac{\Gamma}{2}\right) \right] \right\}$$

$$- \frac{i}{2}\Omega^* e^{i\delta\omega t} \left\{ i\Omega e^{-i\delta\omega t} \cdot \left(- \frac{\Omega^{*2}}{4} \right) + \left(\lambda + \frac{\Gamma}{2}\right) \cdot \frac{i}{2}\Omega^* e^{i\delta\omega t} \lambda \right.$$

$$\left. + \frac{i}{2}\Omega e^{-i\delta\omega t} \cdot \frac{\Omega^{*2}}{4} e^{2i\delta\omega t} \right\}$$

$$= \lambda \left(\lambda + \frac{\Gamma}{2}\right) \left[\left(\lambda + \frac{\Gamma}{2}\right)(\lambda + \Gamma) + \frac{\Omega^{*2}}{2} \right] - \frac{|\Omega|^2}{4} \lambda \left(\lambda + \frac{\Gamma}{2}\right)$$

Should not do in this way.

Let's just find the steady state.

with $d\rho_{11}/dt = 0$

Suppose at steady state $\rho_{01} = A$, $\rho_{10} = B$.

$$\bar{\rho}_{01} = A e^{-i\delta\omega t}, \quad \bar{\rho}_{10} = B e^{i\delta\omega t}$$

In this case

$$\left\{\begin{aligned}
&\frac{d}{dt}\rho_{00} = \Gamma\rho_{11} + \frac{i}{2}\left(\Omega^* B e^{i\delta\omega t} - \Omega A e^{-i\delta\omega t}\right) = 0 \\
&\frac{d}{dt}\rho_{11} = -\Gamma\rho_{11} + \frac{i}{2}\left(\Omega A e^{-i\delta\omega t} - \Omega^* B e^{i\delta\omega t}\right) = 0 \\
&\frac{dA}{dt} e^{-i\delta\omega t} - i\delta\omega A e^{-i\delta\omega t} = -\left(\frac{\Gamma}{2} + i\delta\omega\right) A e^{-i\delta\omega t} + \frac{i}{2}\Omega^*(\rho_{11} - \rho_{00}) \\
&\frac{dB}{dt} e^{i\delta\omega t} + i\delta\omega B e^{i\delta\omega t} = -\left(\frac{\Gamma}{2} - i\delta\omega\right) B e^{i\delta\omega t} + \frac{i}{2}\Omega^*(\rho_{00} - \rho_{11})
\end{aligned}\right.$$

$$\Rightarrow \quad \Gamma\rho_{11} = \frac{i}{2}\left(\Omega A e^{-i\delta\omega t} - \Omega^* B e^{i\delta\omega t}\right)$$

$$\rho_{11} - \rho_{00} = \frac{\Gamma A e^{-i\delta\omega t} + \frac{dA}{dt}}{i\Omega^*} = \frac{-\Gamma B e^{i\delta\omega t} - \frac{dB}{dt}}{i\Omega^*}$$

const

Suppose $A =$

$$\frac{dA}{dt} = -\Gamma A e^{-i\delta\omega t} + C$$

$$-\Gamma e^{-\Gamma t}\left(a_1 e^{i\delta\omega t} + a_2 e^{-i\delta\omega t}\right)$$

$$+ e^{-\Gamma t}\left(i\delta\omega\, a_1 e^{i\delta\omega t} - i\delta\omega\, a_2 e^{-i\delta\omega t}\right)$$

$$A = e^{-\Gamma t}\left(a_1 e^{i\delta\omega t} + a_2 e^{-i\delta\omega t}\right)$$

2 Problem 2 — Laser cooling and trapping

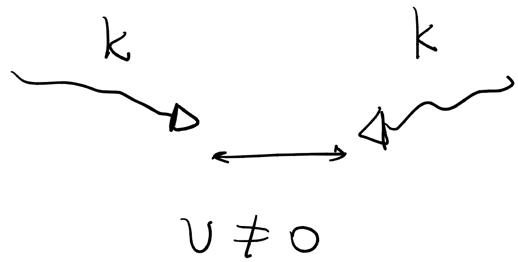
Part (a) — Radiation Pressure

$$\rho_{11} = \frac{|\Omega|^2}{2\Omega^2 + \Gamma^2 + 4\delta\omega^2}$$

$$F_{scatter} = \frac{\Delta \vec{p}}{\Delta t} = \frac{\hbar \vec{k}}{\tau / f_e} = \frac{\hbar \vec{k}}{1/\Gamma f_e} \quad \longleftarrow \rho_{11}$$

$$= \hbar \vec{k} \frac{|\Omega|^2 \Gamma}{2\Omega^2 + \Gamma^2 + 4\delta\omega^2}.$$

Part (b) — Doppler Cooling



$$F_{total} = F_+ - F_-$$

For population of moving atom,

$$\omega + kv \longrightarrow \rho \uparrow$$

$$\omega - kv \longrightarrow \rho \downarrow$$

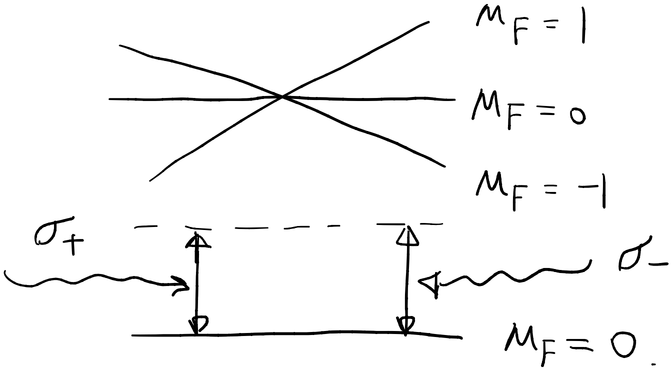
$$F_{total} = \hbar k \frac{|\Omega+kv|^2 \Gamma}{2(\Omega+kv)^2+\Gamma^2+4\delta\omega^2} - \hbar k \frac{|\Omega-kv|^2\Gamma}{2(\Omega-kv)^2+\Gamma^2+4\delta\omega^2}$$

$$= \hbar k \Gamma \frac{4\Omega k v\left(\Gamma^2+4\delta\omega^2\right)}{\left(\Gamma^2+4\delta\omega^2+2\Omega^2+2k^2v^2\right)^2-(4\Omega k v)^2}$$

$$= 4\hbar k^2 \frac{\Omega\left(\Gamma^2+4\delta\omega^2\right)}{\left(\Gamma^2+4\delta\omega^2+2\Omega^2+2k^2v^2\right)^2-(4\Omega k v)^2}\, v$$

$$= 4\hbar k^2 \frac{\Omega/\left(\Gamma^2+4\delta\omega^2\right)}{\left(1+\frac{2\Omega^2}{\Gamma^2+4\delta\omega^2}\right)^2+4k^2v^2\left(1+\frac{k^2v^2}{\Gamma^2+4\delta\omega^2}\right)}\, v$$

Part (c) — Magneto-optical Trap (MOT)



$$F\left(\omega + kv - \omega_0 - \frac{g\mu_B}{\hbar}\frac{\partial B}{\partial z} z\right)$$

$$F_{MOT} = -\eta v - \frac{\Omega}{k}\frac{g\mu_B}{\hbar}\frac{\partial B}{\partial z} z.$$

$x - x_0$

$$F = F_+ + F_-$$

$$= \frac{1}{2}\hbar k\Gamma \frac{s_0}{1+s_0+(2\Delta_+/\Gamma)^2} + \frac{1}{2}\hbar k\Gamma \frac{s_0}{1+s_0+(2\Delta_-/\Gamma)^2}$$

with $\Delta_+ = \delta\omega - kv + \mu B/\hbar$.

$$S_0 = \frac{I}{I_0}.$$

$\Delta_- = \delta\omega + kv - \mu B/\hbar$.

Let $\gamma_m = g_J \mu_B/\hbar$.

$$F_\pm(x-x_0) = \hbar\vec{k}\frac{|\Omega|^2\Gamma}{2\Omega^2+\Gamma^2+4\underbrace{\delta\omega^2}_{\delta\omega \pm \gamma_m \frac{\partial B}{\partial z} z}}.$$

For small $\vec{v}$,

$$F = -\eta\vec{v} - \frac{\Omega}{k}\frac{g\mu_B}{\hbar}\frac{\partial B}{\partial z}z.$$

$$F_{total} = \hbar k \frac{|\Omega + kv|^2 \Gamma}{2(\Omega + kv)^2 + \Gamma^2 + 4\delta\omega^2}$$

$$- \hbar k \frac{|\Omega - kv|^2 \Gamma}{2(\Omega - kv)^2 + \Gamma^2 + 4\delta\omega^2}$$

$$= \hbar k \Gamma$$

$$\frac{a+b}{c+d} - \frac{a-b}{c-d}$$

$$= \frac{2(bc - ad)}{c^2 - d^2}$$

$$ac + bc - ad - bd$$

$$-(ac + ad - bc - bd)$$

$$= 2(bc - ad)$$

$$\frac{\overbrace{(\Omega^2 + k^2v^2 + 2\Omega kv)}^{a+b}}{\underbrace{[\Gamma^2 + 4\delta\omega^2 + 2\Omega^2 + 2k^2v^2 + 4\Omega kv]}_{c+d}} - \frac{(\Omega^2 + k^2v^2 - 2\Omega kv)}{\Gamma^2 + 4\delta\omega^2 + 2\Omega^2 + 2k^2v^2 - 4\Omega kv}$$

$$= \frac{2 \cdot \left[2\Omega kv \cdot (\Gamma^2 + 4\delta\omega^2 + 2\Omega^2 + 2k^2v^2) - (\Omega^2 + k^2v^2) 4\Omega kv\right]}{(\Gamma^2 + 4\delta\omega^2 + 2\Omega^2 + 2k^2v^2)^2 - (4\Omega kv)^2}$$

$$=$$

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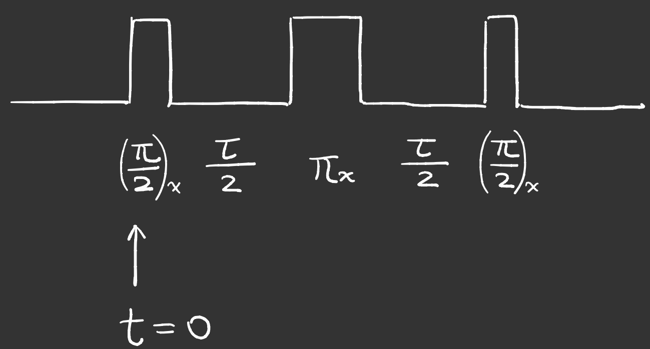
1 Problem 1 — Spin echo and quantum measurement

Part (a) – Sensing of AC field



initial state $|\phi_i\rangle = |m_s = 0\rangle$

AC field : $\vec{B}(t) = B\sin(2\omega t + \phi_0)\,\hat{z}$.

Interaction of AC field could be described by :

$$H_{MW} = g\mu_B\, \vec{B}\cdot\vec{S_z} = g\mu_B B_{AC}\sin(2\omega t+\phi_0) S_z .$$

Transforming to microwave frame $\tilde{H} = U' H U$. $U = e^{\pm i 2\omega t S_z}$

$$= \cos(2\omega t) \pm i S_z \sin(2\omega t)$$

$$\tilde{H} = \begin{pmatrix} \cos(\omega_{mw}t) - i\sin(\omega_{mw}t) & 0 & 0 \\ 0 & \cos(\omega_{mw}t) & 0 \\ 0 & 0 & \cos(\omega_{mw}t) - i\sin(\omega_{mw}t) \end{pmatrix} g\mu_B B \begin{pmatrix} \sin(2\omega t+\phi_0) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\sin(2\omega t+\phi_0) \end{pmatrix} \begin{pmatrix} \cos(\omega_{mw}t) + i\sin(\omega_{mw}t) & 0 & 0 \\ 0 & \cos(\omega_{mw}t) & 0 \\ 0 & 0 & \cos(\omega_{mw}t) + i\sin(\omega_{mw}t) \end{pmatrix}$$

$$= \begin{pmatrix} \cos(\omega_{mw}t) - i\sin(\omega_{mw}t) & 0 & 0 \\ 0 & \cos(\omega_{mw}t) & 0 \\ 0 & 0 & \cos(\omega_{mw}t) - i\sin(\omega_{mw}t) \end{pmatrix} g\mu_B B \begin{pmatrix} \cos(\omega t)\sin(2\omega t+\phi_0) + i\sin(\omega t)\sin(2\omega t+\phi_0) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\sin(2\omega t+\phi_0)\cos\omega t + i\sin(2\omega t+\phi_0)\sin(\omega t) \end{pmatrix}$$

$$= \begin{pmatrix} g\mu_B B \sin(2\omega t+\phi_0) \pm \hbar\omega_{mw} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & g\mu_B B \sin(2\omega t+\phi_0) \pm \hbar\omega_{mw} \end{pmatrix}$$

(should be wrong, will check later)

After $\left(\frac{\pi}{2}\right)_x$ pulse $\quad |\phi_0\rangle = |m_s=0\rangle \longrightarrow |\phi_1\rangle = \frac{|0\rangle+|1\rangle}{\sqrt{2}}$.

After $\frac{\tau}{2}$ evolution $\quad |\phi_2\rangle = e^{-i\tilde{H}t/2\hbar}|\phi_1\rangle$

$$= \frac{1}{\sqrt{2}}\left(|0\rangle + e^{-\int_0^{\frac{\tau}{2}} i(g\mu_B \tilde{B}_z - \hbar\omega_{mw})\frac{d\tau}{\hbar}}|1\rangle\right)$$

$$= \frac{1}{\sqrt{2}}\left(|0\rangle + e^{-i\varphi_0}|1\rangle\right)$$

After π_x pulse $\quad |\phi_3\rangle = \frac{1}{\sqrt{2}}\left(e^{-i\varphi_0}|0\rangle + |1\rangle\right)$

After $\frac{\tau}{2}$ evolution $\quad |\phi_4\rangle = \frac{1}{\sqrt{2}}\left(e^{-i\varphi_0}|0\rangle + e^{-\int_{\frac{\tau}{2}}^{\tau} i(g\mu_B \tilde{B}_z - \hbar\omega_{mw})\frac{d\tau}{\hbar}}|1\rangle\right)$

$$= \frac{1}{\sqrt{2}}\left(e^{-i\varphi_0}|0\rangle + e^{-i\varphi_1}|1\rangle\right)$$

After $\left(\frac{\pi}{2}\right)_x$ pulse: $\quad |\phi_5\rangle = \left(e^{-i\varphi_0} + e^{-i\varphi_1}\right)|0\rangle + \left(e^{-i\varphi_0} - e^{-i\varphi_1}\right)|1\rangle$

So, $\quad |\langle\phi_5|\phi_0\rangle|^2 = \left|e^{-i\varphi_0} + e^{-i\varphi_1}\right|^2$.

$$= (\cos\varphi_0 + \cos\varphi_1)^2 + (\sin\varphi_0 + \sin\varphi_1)^2$$

$$= 2 + 2\cos\varphi_0\cos\varphi_1 + 2\sin\varphi_0\sin\varphi_1$$

$$= 2 + 2\cos(\varphi_0 - \varphi_1) = 4\cos^2\left(\frac{\varphi_0-\varphi_1}{2}\right).$$

With $\varphi_0 = \int_0^{\frac{\tau}{2}} (g\mu_B\tilde{B}_z - \hbar\omega_{mw})\frac{d\tau}{\hbar}$, $\quad \varphi_1 = \int_{\frac{\tau}{2}}^{\tau} (g\mu_B\tilde{B}_z - \hbar\omega_{mw})\frac{d\tau}{\hbar}$.

$$\frac{\varphi_0-\varphi_1}{2} = \left[\int_0^{\frac{\tau}{2}} g\mu_B B\sin(2\omega t+\phi_0)\,dt - \int_{\frac{\tau}{2}}^{\tau} g\mu_B B\sin(2\omega t+\phi_0)\,dt\right]/\hbar$$

$$= \frac{g\mu_B B}{2\omega\hbar}\left[\cos(\phi_0) - \cos(\omega\tau+\phi_0)\right] - \frac{g\mu_B B}{2\omega\hbar}\left(\cos(2\omega\tau+\phi_0) - (\cos\omega\tau+\phi_0)\right)$$

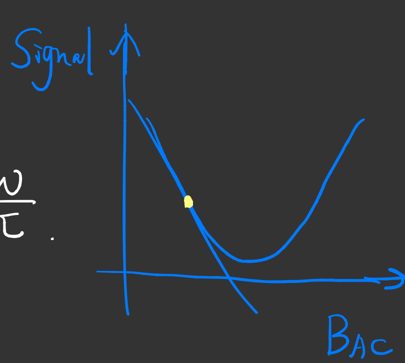
$$= \frac{g\mu_B B}{2\omega\hbar}\left[\cos\phi_0 - \cos(2\omega\tau+\phi_0)\right]$$

$$= \frac{g\mu_B B}{\omega\hbar}\sin(\omega\tau)\sin(\omega\tau+\phi_0)$$

Different from answer $\quad \delta\phi = \frac{4g\mu_B B_{AC}}{2\pi\nu}\sin^2\left(\frac{\pi\nu\tau}{2}\right)\cos(\pi\nu\tau + \phi_{AC})$.

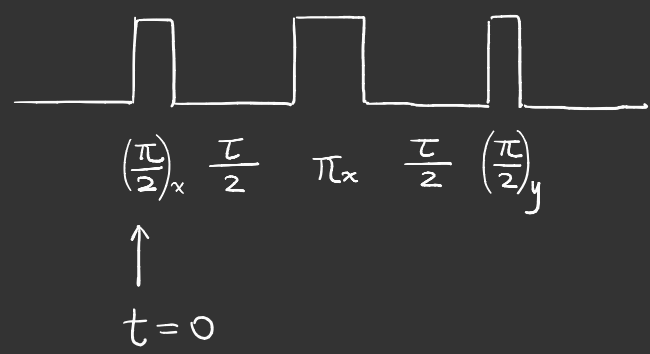
Sensitivity $\quad \delta B_{min} \propto \sqrt{\nu}/F(1/\nu)$.

maximized for $\tau = \frac{1}{\nu}$ with $\nu = \frac{\omega}{\pi}$.



Is at point with maximum slope.

Part (b) – Optimized sensitivity



$$|\phi_4\rangle = \frac{1}{\sqrt{2}}\left(e^{-i\varphi_0}|0\rangle + e^{-i\varphi_1}|1\rangle\right)$$

$$|\phi_5'\rangle = \frac{1}{2}\left[\left(e^{-i\varphi_0} + i e^{-i\varphi_1}\right)|0\rangle + \left(i e^{-i\varphi_0} - e^{-i\varphi_1}\right)|1\rangle\right]$$

$$|\langle\phi_5'|\phi_0\rangle|^2 = \left[(\cos\varphi_0 + \sin\varphi_1)^2 + (\cos\varphi_1 - \sin\varphi_0)^2\right]$$

$$= 2 + 2\sin\varphi_1\cos\varphi_0 - 2\sin\varphi_0\cos\varphi_1 = 2\left[1 + \sin(\varphi_1 - \varphi_0)\right].$$

$$\delta\phi = \frac{4g\mu_B B}{2\pi\nu}\sin^2\left(\frac{\pi\nu\tau}{2}\right)\cos(\pi\nu\tau + \phi_{AC}).$$

with $\tau = \frac{1}{\nu}$ and $\phi_{AC} = 0$. $\quad \delta\phi = -\frac{4g\mu_B B}{2\pi\nu}$.

Sensitivity: $\delta B_{min} = \sigma_S^N / dS_B$

$= \sigma_S\sqrt{\frac{\tau}{T}} / dS_B$ maximized at $\tau = \frac{1}{\nu}$

$= \frac{2\pi}{\omega}$

Part (c) — Sensing in non-Markovian environment

$$\vec{B}_{nuc} = B_{nuc} \sin(\omega\tau + \phi_0)\,\hat{z}.$$

$$\delta\phi = \frac{4 g \mu_B B_{AC}}{2\pi\nu} \sin^2\left(\frac{\pi\nu\tau}{2}\right) \cos(\pi\nu\tau + \phi_{AC}) \qquad \nu = \frac{\omega}{2\pi}$$

$$= \frac{4 g \mu_B B_{nuc}}{\omega} \sin^2\left(\frac{\omega\tau}{4}\right) \cos\left(\frac{\omega\tau}{2} + \phi_0\right)$$

$-\pi \sim \pi$

$$P = \cos^2\left(\frac{\delta\phi}{2}\right) = \cos^2\left[\underbrace{\frac{2 g \mu_B B_{nuc}}{\omega} \sin^2\left(\frac{\omega\tau}{4}\right) \cos\left(\frac{\omega\tau}{2} + \phi_0\right)}\right].$$

this term be in range $\left[-\frac{2 g \mu_B B_{nuc}}{\omega} \sin^2\left(\frac{\omega\tau}{4}\right), \frac{2 g \mu_B B_{nuc}}{\omega} \sin^2\left(\frac{\omega\tau}{4}\right)\right].$

should exceed $\frac{\pi}{2}$, so $\quad \frac{2 g \mu_B B_{nuc}}{\omega} \sin^2\left(\frac{\omega\tau}{4}\right) \geq \frac{\pi}{2}$

$$\omega\tau \geq 4 \cdot \sin^{-1}\left(\sqrt{\frac{\omega\pi}{4 g \mu_B B_{nuc}}}\right).$$

2 Problem 2 — Quantum logic enhanced sensing

Part (a) — Sensing—logical qubit entanglement

original state =

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|0_A\rangle + |1_A\rangle\right) \otimes |\psi_B\rangle.$$

evolution =

$$e^{-iHt/\hbar}|\psi\rangle, \quad H = g\,\sigma_{z,A}\,\sigma_{z,B}.$$

$$|\psi(t)\rangle = e^{-ig\sigma_{z,A}t/\hbar}\frac{1}{\sqrt{2}}\left(|0_A\rangle + |1_A\rangle\right) \otimes e^{-ig\sigma_{z,B}t/\hbar}|\psi_B\rangle$$

$$= \left[\cos\left(\frac{gt}{\hbar}\right) - i\sin\left(\frac{gt}{\hbar}\right)\sigma_{z,A}\right]\frac{|0_A\rangle + |1_A\rangle}{\sqrt{2}} \otimes \left[\cos\left(\frac{gt}{2}\right) - i\sin\left(\frac{gt}{2}\right)\sigma_{z,B}\right]|\psi_B\rangle$$

$$= \left\{\frac{1}{\sqrt{2}}\left[\cos\left(\frac{gt}{\hbar}\right) - i\sin\left(\frac{gt}{\hbar}\right)\right]|0\rangle_A + \frac{1}{\sqrt{2}}\left[\cos\left(\frac{gt}{\hbar}\right) + i\sin\left(\frac{gt}{\hbar}\right)\right]|1\rangle_A\right\}$$

$$\otimes \left[\cos\left(\frac{gt}{2}\right) - i\sin\left(\frac{gt}{2}\right)\sigma_{z,B}\right]|\psi_B\rangle$$

After the $\frac{\pi}{2}$ pulse,

$$|\psi(t=T^{+})\rangle = \left\{\frac{1}{\sqrt{2}}\left[\cos\left(\frac{gT}{\hbar}\right) - i\sin\left(\frac{gT}{\hbar}\right)\right]|0\rangle_A + \frac{1}{\sqrt{2}}e^{i\phi}\left[\cos\left(\frac{gT}{\hbar}\right) + i\sin\left(\frac{gT}{\hbar}\right)\right]|1\rangle_A\right\}$$

$$\otimes \left[\cos\left(\frac{gT}{2}\right) - i\sin\left(\frac{gT}{2}\right)\sigma_{z,B}\right]|\psi_B\rangle$$

$$\mathbb{1}\cos\left(\frac{gT}{2}\right) - i\sin\left(\frac{gT}{2}\right)\sigma_{z,B} = \begin{pmatrix} \cos\left(\frac{gT}{2}\right) - i\sin\left(\frac{gT}{2}\right) & 0 \\ 0 & \cos\left(\frac{gT}{2}\right) + i\sin\left(\frac{gT}{2}\right) \end{pmatrix}$$

Under which circumstance is it a projective measurement for $|\psi_B\rangle$? $gT =$

Should trace out the subsystem A.

Calculate density matrix and evolution?

Part (b) – Readout Fidelity

$$F = 1 - \mathrm{Max}\left[P_{err}^{0}, P_{err}^{1}\right].$$

$P_{err}^{0} = P(N \text{ rep}, |0_B\rangle, 1 \text{ or more photons})$

$P_{err}^{1} = P(N \text{ rep}, |1_B\rangle, 0 \text{ photons})$.

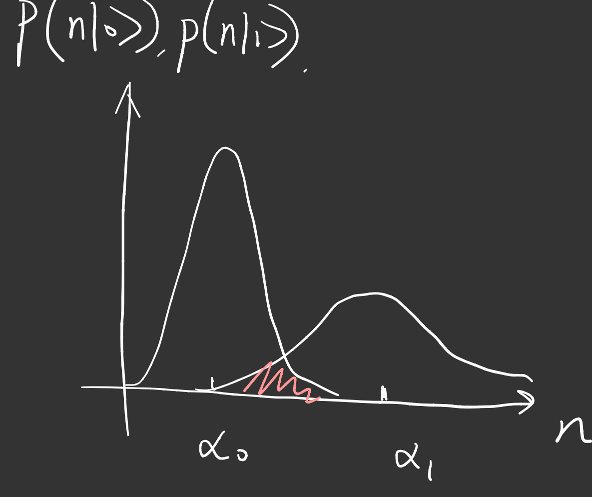
with proper parameter setting of gT.

$$\eta_0 = \frac{1 - P_{err}^0}{1 - P_{err}^0 + 1 - P_{err}^1}.$$

$$= \frac{1 - P_{err}^0}{2 - P_{err}^0 - P_{err}^1}. \qquad \eta_1 = \cdots$$

So, $F = 1 - \max\left\{ \frac{1 - P_{err}^0}{2 - P_{err}^0 - P_{err}^1}, \frac{1 - P_{err}^1}{2 - P_{err}^0 - P_{err}^1} \right\}$,

(c) Extra Credit



For P_{err} dependance of N,

$$P_{err}(N) = \int_0^N \alpha(n)\, dn.$$

with $\alpha(n) = P_0 \alpha_0 + P_1 \alpha_1$.

Should calculate the overlapping area - - -

$$= \begin{pmatrix} g\mu_B B\sin(2\omega t+\phi_0) \pm \hbar\omega_{mw} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & g\mu_B B\sin(2\omega t+\phi_0) \pm \hbar\omega_{mw} \end{pmatrix}$$

(should be wrong, will check later)

After $\left(\frac{\pi}{2}\right)_x$ pulse $\quad |\phi_0\rangle = |m_s=0\rangle \longrightarrow |\phi_1\rangle = \frac{|0\rangle+|1\rangle}{\sqrt{2}}$

After $\frac{\tau}{2}$ evolution $\quad |\phi_2\rangle = e^{-i\tilde{H}t/2\hbar}|\phi_1\rangle$

$$= \frac{1}{\sqrt{2}}\left(|0\rangle + e^{-i(g\mu_B\tilde{B_z} - \hbar\omega_{mw})\frac{\tau}{2\hbar}}|1\rangle\right)$$

$$= \frac{1}{\sqrt{2}}\left(|0\rangle + e^{-i\left(\frac{g\mu_B B_z}{\hbar}\sin(\omega\tau+\phi_0) - \omega_{mw}\right)\frac{\tau}{2}}|1\rangle\right)$$

After π_x pulse $\quad |\phi_3\rangle = \frac{1}{\sqrt{2}}\left(e^{-i\left(\frac{g\mu_B B_z}{\hbar}\sin(\omega\tau+\phi_0) - \omega_{mw}\right)\frac{\tau}{2}}|0\rangle + |1\rangle\right)$

After $\frac{\tau}{2}$ evolution $\quad |\phi_4\rangle = \frac{1}{\sqrt{2}}\left(e^{-i\left(\frac{g\mu_B B_z}{\hbar}\sin(\omega\tau+\phi_0) - \omega_{mw}\right)\frac{\tau}{2}}|0\rangle\right.$

$$\left. + e^{-i\left(\frac{g\mu_B B_z}{\hbar}\sin(2\omega\tau+\phi_0) - \omega_{mw}\right)\frac{\tau}{2}}|1\rangle\right)$$

After $\left(\frac{\pi}{2}\right)_x$ pulse: $\quad |\phi_5\rangle = \frac{1}{\sqrt{2}}\left(e^{-i\left(\frac{g\mu_B B_z}{\hbar}\sin(2\omega\tau+\phi_0) - \omega_{mw}\right)\frac{\tau}{2}}\cdot|0\rangle\right.$

$$\left. + e^{-i\left(\frac{g\mu_B B_z}{\hbar}\sin(\omega\tau+\phi_0) - \omega_{mw}\right)\frac{\tau}{2}}\cdot|1\rangle\right)$$

$$= \frac{1}{\sqrt{2}}\left(e^{-i\phi_0}|0\rangle + e^{-i\phi_0}|1\rangle\right).$$

$$|\langle\phi_S|\phi_0\rangle|^2 = \sin^2(\delta\phi/2).$$

$$\text{with } \delta\phi = 2\pi g\mu_B B_z\left[\sin(\omega\tau+\phi_0) - \sin(\omega\tau+\phi_0)\right]$$

$$= 2\pi g\mu_B B_z \cdot 2\cos\left(\frac{\omega}{2}+\phi_0\right)\sin\left(\frac{\omega\tau}{2}\right)$$

$$\begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -i \end{pmatrix}$$

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

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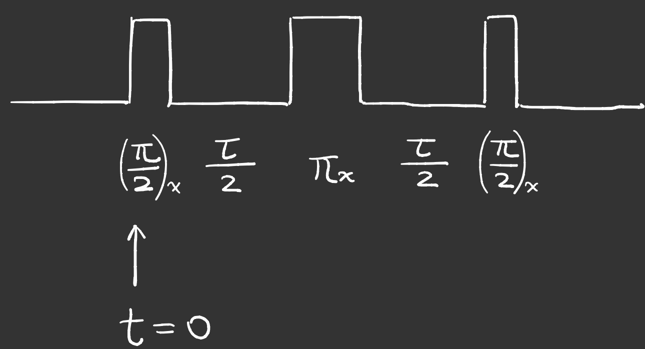
1 Problem 1 — Spin echo and quantum measurement

Part (a) - Sensing of AC field



initial state $|\phi_i\rangle = |m_s=0\rangle$

AC field : $\vec{B}(t) = B\sin(2\omega t+\phi_0)\,\hat{z}$.

Interaction of microwave field could be described by :

$$H_{MW} = g\mu_B \vec{B}\cdot\vec{S_z} = g\mu_B B_{AC}\sin(2\omega t+\phi_0)S_z.$$

After $\left(\frac{\pi}{2}\right)_x$ pulse $|\phi_0\rangle = |m_s=0\rangle \longrightarrow |\phi_1\rangle = \frac{|0\rangle+|1\rangle}{\sqrt{2}}$.

After evolution

$$|\phi_2\rangle = e^{iH\tau/\hbar}\,|\phi_1\rangle ,$$

$$= \left(\cos\frac{\omega\tau}{2} + i S_z \sin\frac{\omega\tau}{2}\right)|\phi_1\rangle$$

$$= \frac{1}{\sqrt{2}}\left(\cos\frac{\omega\tau}{2} + i\sin\frac{\omega\tau}{2}\right)|0\rangle + \frac{1}{\sqrt{2}}\cos\left(\frac{\omega\tau}{2}\right)|1\rangle.$$

After π_x pulse

$$|\phi_3\rangle = \frac{1}{\sqrt{2}}\cos\left(\frac{\omega\tau}{2}\right)|0\rangle + \frac{1}{\sqrt{2}}\left(\cos\left(\frac{\omega\tau}{2}\right) + i\sin\left(\frac{\omega\tau}{2}\right)\right)|1\rangle$$

After evolution $\frac{\tau}{2}$:

$$|\phi_4\rangle =$$

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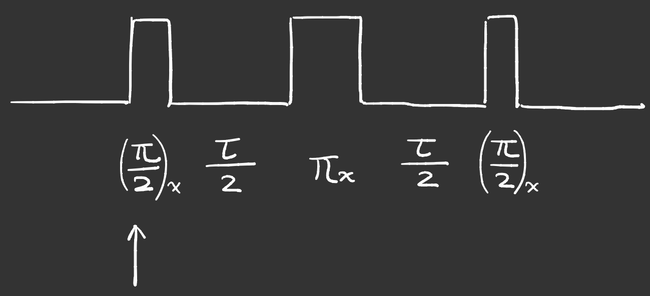
1 Problem 1 — Spin echo and quantum measurement

Part (a) - Sensing of AC field



initial state $|\phi_i\rangle = |m_s = 0\rangle$

AC field : $\vec{B}(t) = B\sin(2\omega t + \phi_0)\,\hat{z}$.

Interaction of microwave field could be described by :

$$H_{MW} = \frac{g\mu_B}{\hbar}\vec{B}\cdot\vec{\sigma_z} =$$

In x direction, $S \longrightarrow S_x$.

$$H_{MW} = \frac{g_e\mu_B B_{MW}}{\sqrt{2}}\begin{pmatrix} 0 & \cos(\omega_{MW}t) & 0 \\ \cos(\omega_{MW}t) & 0 & \cos(\omega_{MW}t) \\ 0 & \cos(\omega_{MW}t) & 0 \end{pmatrix}.$$

Transforming to microwave frame by

$$\tilde{H} = U' H U. \qquad U = e^{\pm i\omega_{MW}tS_z}.$$

$$= \cos(\omega_{MW}t) + i\sin(\omega_{MW}t)\hat{S}_z.$$

We get

$$\begin{pmatrix} \cos(\omega_{mw}t)+i\sin(\omega_{mw}t) & 0 & 0 \\ 0 & \cos(\omega_{mw}t) & 0 \\ 0 & 0 & \cos(\omega_{mw}t)+i\sin(\omega_{mw}t) \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{e^{i\cos\omega t}}{\sqrt{2}} \\ \frac{\cos\omega t}{\sqrt{2}} \\ 0 \end{pmatrix}$$

$$e^{iH\tau/\hbar} = e^{i g \mu_B B_{AC} \sin(2\omega t+\phi_0) S_z \tau/\hbar}$$

$$\tilde{H} = \begin{pmatrix} \cos(\omega_{mw}t) - i\sin(\omega_{mw}t) & 0 & 0 \\ 0 & \cos(\omega_{mw}t) & 0 \\ 0 & 0 & \cos(\omega_{mw}t) - i\sin(\omega_{mw}t) \end{pmatrix} \frac{g_e \mu_B B_{mw}}{\sqrt{2}} \begin{pmatrix} 0 & \cos(\omega_{mw}t) & 0 \\ \cos(\omega_{mw}t) & 0 & \cos(\omega_{mw}t) \\ 0 & \cos(\omega_{mw}t) & 0 \end{pmatrix} \begin{pmatrix} \cos(\omega_{mw}t) + i\sin(\omega_{mw}t) & 0 & 0 \\ 0 & \cos(\omega_{mw}t) & 0 \\ 0 & 0 & \cos(\omega_{mw}t) + i\sin(\omega_{mw}t) \end{pmatrix}$$

$$= \begin{pmatrix} \cos(\omega_{mw}t) - i\sin(\omega_{mw}t) & 0 & 0 \\ 0 & \cos(\omega_{mw}t) & 0 \\ 0 & 0 & \cos(\omega_{mw}t) - i\sin(\omega_{mw}t) \end{pmatrix} \frac{g \mu_B B_{mw}}{\sqrt{2}} \begin{pmatrix} 0 & \cos^2(\omega_{mw}t) & 0 \\ \cos^2(\omega_{mw}t) + i\sin(\omega_{mw}t)\cos(\omega_{mw}t) & 0 & \cos^2(\omega_{mw}t) + i\sin(\omega_{mw}t)\cos(\omega_{mw}t) \\ 0 & \cos^2(\omega_{mw}t) & 0 \end{pmatrix}$$

$$= \frac{g \mu_B B_{mw}}{\sqrt{2}} \begin{pmatrix} 0 & \cos^3(\omega_{mw}t) - i\sin(\omega_{mw}t)\cos^2(\omega_{mw}t) & 0 \\ \cos^3(\omega_{mw}t) + i\sin(\omega_{mw}t)\cos^2(\omega_{mw}t) & 0 & \cos^3(\omega_{mw}t) + i\sin(\omega_{mw}t)\cos^2(\omega_{mw}t) \\ 0 & \cos^3(\omega_{mw}t) + i\sin(\omega_{mw}t)\cos^2(\omega_{mw}t) & 0 \end{pmatrix}$$

$$\xrightarrow{\text{Rotating wave approximation}} \quad = \frac{g \mu_B B_{mw}}{\sqrt{2}} \Bigg($$

$$\cos^2(\omega t) =$$

Suppose resonance condition, so the first $\left(\frac{\pi}{2}\right)_x$ pulse sends $|\phi_i\rangle$ to:

$$|\phi_0\rangle = |m_s = 0\rangle \longrightarrow |\phi_1\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}.$$

$$\tilde{H} = \begin{pmatrix} \cos(\omega_{mw}t) & i\sin(\omega_{mw}t) & 0 \\ i\sin(\omega_{mw}t) & \cos(\omega_{mw}t) & i\sin(\omega_{mw}t) \\ 0 & i\sin(\omega_{mw}t) & \cos(\omega_{mw}t) \end{pmatrix} \frac{g_e\mu_B B_{MW}}{\sqrt{2}} \begin{pmatrix} 0 & \cos(\omega_{mw}t) & 0 \\ \cos(\omega_{mw}t) & 0 & \cos(\omega_{mw}t) \\ 0 & \cos(\omega_{mw}t) & 0 \end{pmatrix} \begin{pmatrix} \cos(\omega_{mw}t) & -i\sin(\omega_{mw}t) & 0 \\ -i\sin(\omega_{mw}t) & \cos(\omega_{mw}t) & -i\sin(\omega_{mw}t) \\ 0 & -i\sin(\omega_{mw}t) & \cos(\omega_{mw}t) \end{pmatrix}$$

$$= \begin{pmatrix} \cos(\omega_{mw}t) & i\sin(\omega_{mw}t) & 0 \\ i\sin(\omega_{mw}t) & \cos(\omega_{mw}t) & i\sin(\omega_{mw}t) \\ 0 & i\sin(\omega_{mw}t) & \cos(\omega_{mw}t) \end{pmatrix} \frac{g_e\mu_B B_{MW}}{\sqrt{2}} \begin{pmatrix} -i\sin(\omega_{mw}t)\cos(\omega_{mw}t) & \cos^2(\omega_{mw}t) & -i\sin(\omega_{mw}t)\cos(\omega_{mw}t) \\ \cos^2(\omega_{mw}t) & -2i\sin(\omega_{mw}t)\cos(\omega_{mw}t) & \cos^2(\omega_{mw}t) \\ -i\sin(\omega_{mw}t)\cos(\omega_{mw}t) & \cos^2(\omega_{mw}t) & -i\sin(\omega_{mw}t)\cos(\omega_{mw}t) \end{pmatrix}$$

$$= \frac{g_e\mu_B B_{MW}}{\sqrt{2}} \begin{pmatrix} 0 \\ \cos(\omega_{mw}t) \\ 0 \end{pmatrix}$$

Suppose resonance condition, so the first $\left(\frac{\pi}{2}\right)_x$ pulse sends $|\phi_i\rangle$

to : $\quad |\phi_0\rangle = |m_s = 0\rangle \longrightarrow |\phi_1\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}$.