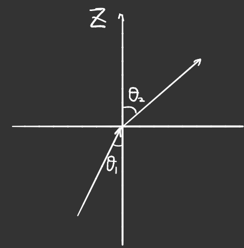
HW 1 Adv Classical Mechanics

Yuxiang Pei yuxiang@uchicago.edu

(1) We have energy conservation equation:



$$\frac{1}{2}m\vec{v}_1^2 + V_1 = \frac{1}{2}m\vec{v}_2^2 + V_2$$

$$\Rightarrow |\vec{v}_2| = \sqrt{\frac{2(V_1-V_2)}{m} + \vec{v}_1^2}$$

There's only changes in $\frac{\partial V}{\partial z}$, So we have $\vec{p}$ conservation in x, y direction $|\vec{v}_1|\sin\theta = |\vec{v}_2|\sin\theta_2$

$$\Rightarrow \sin\theta_2 = \frac{|\vec{v}_1|\sin\theta}{|\vec{v}_2|} = \frac{|\vec{v}_1|\sin\theta}{\sqrt{\frac{2(V_1-V_2)}{m}+\vec{v}_1^2}}.$$

So, the value of $\vec{v}_2$ is $\sqrt{\frac{2(V_1-V_2)}{m}+\vec{v}_1^2}$,

and the angle with z axis is $\sin^{-1}\left(\frac{|\vec{v}_1|\sin\theta}{\sqrt{\frac{2(V_1-V_2)}{m}+\vec{v}_1^2}}\right)$

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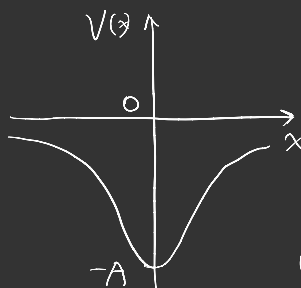
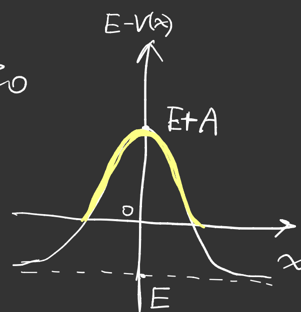
(2) (a) $V(x) = -A/\cosh^2\beta x$.



$$E = \frac{1}{2}mv^2 + V(x)$$

$$\Rightarrow \frac{1}{2}mv^2 = E - V(x) \geq 0$$



① For the case of $-A < E < 0$ as shown on the right,

the particle will be trapped in a certain range of x.

② For $E < -A$, then $\frac{1}{2}mv^2 < 0$.

So the particle will not move or cannot present in this potential.

③ For the case of $E \geq 0$,

since $E - V(x) = \frac{1}{2}mv^2 \geq 0$ is true for any point on the x axis, the particle could move on the axis without being trapped.

To be more detailed about case ①, let's look into its zero point. when $E - V(x) = 0 \Rightarrow E = -A/\cosh^2 \beta x$

$$\Rightarrow E + \frac{4A}{(e^{-\beta x} + e^{\beta x})^2} = 0 \Rightarrow e^{-\beta x} + e^{\beta x} = \sqrt{\frac{-4A}{E}}$$

$$\Rightarrow e^{\beta x} = \frac{1}{2}\left(\sqrt{\frac{-4A}{E}} + \sqrt{\frac{-4A}{E} - 4}\right)$$ (discard the minus root)

So, $$x = \pm \ln\left(\sqrt{\frac{-A}{E}} + \sqrt{\frac{-A-E}{E}}\right).$$

That is, in the case of $-A < E < 0$,

the particle will be trapped in the range

$$\left(-\ln\left(\sqrt{\frac{-A}{E}} + \sqrt{\frac{-A-E}{E}}\right), \ln\left(\sqrt{\frac{-A}{E}} + \sqrt{\frac{-A-E}{E}}\right)\right).$$

(b) $F = -\frac{\partial U}{\partial x}$.

The bound point is $x=0$.

Around $x=0$, we have

$$-\frac{\partial U}{\partial x} = -2A \tanh x \operatorname{sech}^2 x \approx -2Ax.$$

T=2\pi\sqrt{m/k}

So, around $x=0$,

$$m\ddot{x} + 2Ax = 0. \quad \Rightarrow \quad T = 2\pi\sqrt{\frac{2A}{m}}.$$

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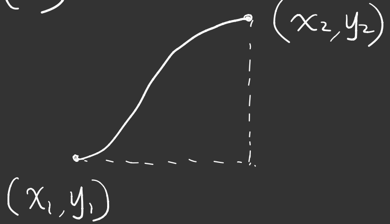
Also, it is possible to get accurate result without doing approximation

(3)



Suppose (x_1, y_1) is the origin point.

then $\frac{1}{2}mv_0^2 = \frac{1}{2}mv^2 - mgy$

$$\Rightarrow v = \sqrt{v_0^2 - 2gy}.$$

$\frac{dl}{dt} = v$, then $dt = \frac{dl}{v} = \frac{\sqrt{dx^2+dy^2}}{v} = \frac{\sqrt{1+y'^2}}{\sqrt{v_0^2-2gy}}\,dy$

$$\text{total } T = \int_0^{x_2-x_1} \frac{\sqrt{1+y'^2}}{\sqrt{v_0^2-2gy}}\,dy = \int_0^{x_2-x_1} F(x,y,y')\,dy$$

$$\delta T = F(y+\delta y) - F(y)$$

$$= \int_0^{x_2-x_1}\left(\frac{\partial F}{\partial y}\delta y + \frac{\partial F}{\partial y'}\delta y'\right)dx = \int_0^{x_2-x_1}\left(\frac{\partial F}{\partial y} - \frac{d}{dx}\frac{\partial F}{\partial y'}\right)\delta y\,dx$$

Let $\frac{\partial F}{\partial y} - \frac{d}{dx}\frac{\partial F}{\partial y'} = 0$, that is

$$\frac{d}{dx}\left(\frac{\partial F}{\partial y'}y' - F\right) = 0.$$

$$\Rightarrow \frac{d}{dx}\left(\frac{y'^2}{\sqrt{v_0^2 - 2gy}} \cdot \frac{1}{\sqrt{1+y'^2}} - \frac{\sqrt{1+y'^2}}{\sqrt{v_0^2-2gy}}\right) = 0$$

$$\Rightarrow \frac{d}{dx}\left(\frac{1}{\sqrt{(v_0^2 - 2gy)(1+y'^2)}}\right) = 0$$

$$\Rightarrow \frac{d}{dx}\left(\frac{x'}{\sqrt{(v_0^2-2gy)(1+x'^2)}}\right)$$

$$\frac{df}{dx(y)} = \frac{df}{dy}\frac{dy}{dx(y)}$$

$$\frac{d}{dy}\left(\frac{1}{\sqrt{(v_0^2-2gy)(1+x'^2)}}\right) = 0$$

Solution of this function is

$$x = x(y).$$

Pls solve for the function

(4) (a) $L = \frac{1}{2}m\dot{\vec{r}}^2 - q\phi + q\dot{\vec{r}}\cdot\vec{A} = \frac{1}{2}m(\dot{x}^2+\dot{y}^2+\dot{z}^2) - q\phi + q(\dot{x}A_x + \dot{y}A_y + \dot{z}A_z)$

$$\Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0$$

That is $$m\ddot{x} + q\frac{dA_x}{dt} + q\frac{\partial \phi}{\partial x} - q\dot{x}\frac{\partial A_x}{\partial x} - q\dot{y}\frac{\partial A_y}{\partial x} - q\dot{z}\frac{\partial A_z}{\partial x} = 0$$

$$\Rightarrow m\ddot{x} + q\frac{dA_x}{dt} + q\frac{\partial \phi}{\partial x} - q\dot{\vec{r}}\cdot\frac{\partial \vec{A}}{\partial x} = 0$$

$$\Rightarrow m\ddot{x} + q\frac{dA_x}{dt} + q\frac{\partial \phi}{\partial x} - q\frac{\partial}{\partial x}(\dot{\vec{r}}\cdot\vec{A})$$

Similarly, we have a equation for $\vec{y}, \vec{z}$ direction, So

$$\vec{F} = m\ddot{\vec{r}} = -q\frac{\partial \vec{A}}{\partial t} - q\nabla\phi - q\nabla\cdot(\dot{\vec{r}}\cdot\vec{A})$$

$$= -q\frac{\partial \vec{A}}{\partial t} - q\nabla\phi - q\left[\dot{\vec{r}}\times(\nabla\times\vec{A}) + \underbrace{(\dot{\vec{r}}\cdot\nabla)\vec{A} + \vec{A}\times(\nabla\times\dot{\vec{r}}) + (\vec{A}\cdot\nabla)\dot{\vec{r}}}_{=0}\right]$$

$$= -q\frac{\partial \vec{A}}{\partial t} - q\nabla\phi - q\dot{\vec{r}}\times\vec{B} = q(\vec{E} + \vec{v}\times\vec{B})$$

(b) When $\phi \to \phi - \dot{\alpha}$, $\vec{A} \to \vec{A} + \vec{\nabla}\alpha$

$$\vec{E}' = -\nabla\phi' - \partial_t\vec{A}' = -\nabla\phi + \nabla\dot{\alpha} - \frac{\partial \vec{A}}{\partial t} - \frac{\partial}{\partial t}\vec{\nabla}\alpha$$
$$= \vec{E} + \nabla\frac{\partial}{\partial t}\alpha - \frac{\partial}{\partial t}\nabla\alpha = \vec{E}.$$

$$\vec{B}' = \nabla\times\vec{A}' = \nabla\times(\vec{A} + \nabla\alpha) = \nabla\times\vec{A} + \nabla\times\nabla\alpha$$

While $\vec{\nabla}$ and $\vec{\nabla}\alpha$ is in the same direction, $\nabla\times\nabla\alpha = 0$.

So $\vec{B}' = \nabla\times\vec{A} = \vec{B}$ $\qquad \therefore \vec{E}$ and $\vec{B}$ are invariant under gauge transformation.

(c) $S = \int L\,dt = \int \left(\frac{1}{2}m\dot{\vec{r}}^2 - q\phi + q\dot{\vec{r}}\cdot\vec{A}\right) dt$

$\phi' = \phi - \dot{\alpha}$, $\vec{A}' = \vec{A} + \vec{\nabla}\alpha$

$$S' = \int L' dt = \int \left(\frac{1}{2}m\dot{\vec{r}}^2 - q\phi + q\dot{\alpha} + q\dot{\vec{r}}\cdot\vec{A} + q\dot{\vec{r}}\cdot\vec{\nabla}\alpha\right) dt$$

$$= \int \left(\frac{1}{2}m\dot{\vec{r}}^2 - q\phi + q\dot{\vec{r}}\cdot\vec{A}\right) dt + \int \left(q\dot{\alpha} + q\dot{\vec{r}}\cdot\vec{\nabla}\alpha\right) dt$$

$$= \int \left[L + q\left(\frac{\partial\alpha}{\partial t} + \sum_i \frac{\partial r_i}{\partial t}\frac{\partial\alpha}{\partial r_i}\right)\right] dt$$

$$= \int \left(L + q\frac{d\alpha(r,t)}{dt}\right) dt$$

So, $S' = S + \int_{t_1}^{t_2} q\frac{d\alpha(r,t)}{dt}\,dt = S + \alpha(r_2,t_2) - \alpha(r_1,t_1)$

and $\delta S' = 0 \Longleftrightarrow \delta S = 0$ is equal. the action is invariant under gauge transformation.

(5)(a) $L = e^{\gamma t}\left(\frac{1}{2}m\dot{q}^2 - \frac{1}{2}kq^2\right)$

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$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0. \Rightarrow \frac{d}{dt}\left(e^{\gamma t} m\dot{q}\right) + e^{\gamma t} kq = 0.$$

$$m\ddot{q} + \gamma m\dot{q} + kq = 0$$

$$\Rightarrow q = C\cdot e^{\left(-\frac{\gamma}{2} \pm \sqrt{\frac{\gamma^2}{4} - \frac{k}{m}}\right)t}$$ C is constant

$\lambda = \gamma$

$\omega^2 = \frac{k}{m}$

$E = e^{\gamma t}\left(\frac{1}{2}m\dot{q}^2 + \frac{1}{2}kq^2\right)$. $\gamma < 0$ to ensure E will not go ∞.

It might describe an oscillator with friction,

whose energy decay with time. (damped oscillation)

From what we solved for the next problem,

the conserved quantity could be

$$E' = \frac{1}{2}m\dot{s}^2 + \frac{1}{2}\left(k - \frac{m\gamma^2}{4}\right)s^2, \quad \text{with } s = qe^{\frac{\gamma t}{2}} \quad \dot{s} = \frac{\gamma}{2}qe^{\frac{\gamma t}{2}} + \dot{q}e^{\frac{\gamma t}{2}}$$

So that is:

$$E' = \frac{\gamma^2}{4}\cdot\frac{m}{2}q^2e^{\gamma t} + \frac{m}{2}\dot{q}^2e^{\gamma t} + \frac{m}{2}\gamma q\dot{q}e^{\gamma t} + \frac{1}{2}\left(k - \frac{m\gamma^2}{4}\right)q^2e^{\gamma t}$$

$$= \frac{1}{2}kq^2e^{\gamma t} + \frac{m}{2}\gamma q\dot{q}e^{\gamma t} + \frac{m}{2}\dot{q}^2e^{\gamma t}$$

$$= \frac{e^{\gamma t}}{2}\left[kq^2 + m\gamma q\dot{q} + m\dot{q}^2\right]$$

Then

$$\frac{dE'}{dt} = e^{\gamma t}\left[kq\dot{q} + 2m\gamma q\ddot{q} + 2m\gamma\dot{q}^2 + m\dot{q}\ddot{q} + \gamma kq^2 + m\gamma^2 q\dot{q} + \gamma m\dot{q}^2\right]$$

$$= e^{\gamma t}\left(-\gamma^2 m\dot{q}q - \gamma kq^2 - m\gamma\dot{q}^2\right)$$

$$= m\gamma e^{\gamma t}\left(-\dot{q}^2 - \gamma\dot{q}q - \frac{k}{m}q^2\right)$$

While $\dot{q} = \beta q$

$$-\left(\beta^2 + \gamma\beta + \frac{k}{m}\right)q^2 = 0$$

So $\frac{dE'}{dt} = 0$, E' is conserved.

(b) $S = q e^{\frac{\gamma t}{2}}$ $\Rightarrow q = s e^{-\frac{\gamma t}{2}}$ $\dot{q} = -\frac{\gamma s}{2} e^{-\frac{\gamma t}{2}} + \dot{s} e^{-\frac{\gamma t}{2}}$

$$\Rightarrow L = \frac{1}{2} m * \left(\dot{s} - \frac{\gamma s}{2}\right)^2 - \frac{1}{2} k s^2$$

$$= \frac{1}{2} m \left(\dot{s}^2 + \frac{\gamma^2}{4} s^2 - \gamma \dot{s} s\right) - \frac{1}{2} k s^2$$

for that $\dot{s} s = \frac{d(s^2)}{dt} = \frac{d f(s,t)}{dt}$.

in which $f(s,t) = s^2$.

So, the item including $\dot{s} s$ could be discarded,

While $S = \int L\,dt$ only have difference in a constant.

So, $L' = \frac{1}{2} m \dot{s}^2 - \frac{1}{2}\left(k - \frac{m\gamma^2}{4}\right) s^2$.

equation of motion: $m\ddot{s} + \left(k - \frac{m\gamma^2}{4}\right) s = 0$

And $E = \frac{1}{2} m \dot{s}^2 + \frac{1}{2}\left(k - \frac{m\gamma^2}{4}\right) s^2$ is

a conserved quantity.

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(6) (a) $L = \frac{1}{2}m(\dot{x}'^2 + \dot{y}'^2 + \dot{z}'^2)$

$$\dot{x}' = \dot{x}\cos\phi - x\sin\phi\,\dot{\phi} + \dot{y}\sin\phi + y\cos\phi\,\dot{\phi}$$

$$\dot{y}' = -\dot{x}\sin\phi - x\cos\phi\,\dot{\phi} + \dot{y}\cos\phi - y\sin\phi\,\dot{\phi}$$

$$\dot{z}' = \dot{z}.$$

$$\Rightarrow \quad \dot{x}'^2 + \dot{y}'^2 = \dot{x}^2 + x^2\dot{\phi}^2 + \dot{y}^2 + y^2\dot{\phi}^2$$

$$+2(\dot{x}\cos\phi - x\sin\phi\,\dot{\phi})(\dot{y}\sin\phi + y\cos\phi\,\dot{\phi})$$

$$-2(\dot{x}\sin\phi + x\cos\phi\,\dot{\phi})(\dot{y}\cos\phi - y\sin\phi)$$

$$= \dot{x}^2 + x^2\dot{\phi}^2 + \dot{y}^2 + y^2\dot{\phi}^2 + 2\dot{x}y\dot{\phi} - 2x\dot{y}\dot{\phi}$$

So, $L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2 + x^2\dot{\phi}^2 + y^2\dot{\phi}^2 + 2\dot{x}y\dot{\phi} - 2x\dot{y}\dot{\phi})$

(b) $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0 \Rightarrow \frac{d}{dt}(2\dot{x} + 2y\dot{\phi}) - 2x\dot{\phi}^2 + 2\dot{y}\dot{\phi} = 0$

$$\Rightarrow \ddot{x} + \dot{y}\dot{\phi} + y\ddot{\phi} - x\dot{\phi}^2 + \dot{y}\dot{\phi} = 0$$

$$\Rightarrow \ddot{x} - x\dot{\phi}^2 + 2\dot{y}\dot{\phi} + y\ddot{\phi} = 0 \quad (1)$$

$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) - \frac{\partial L}{\partial y} = 0 \Rightarrow \frac{d}{dt}(2\dot{y} - 2x\dot{\phi}) - 2y\dot{\phi}^2 - 2\dot{x}\dot{\phi} = 0$

$$\Rightarrow \ddot{y} - 2\dot{x}\dot{\phi} - x\ddot{\phi} - y\dot{\phi}^2 = 0 \quad (2)$$

① is $\ddot{x} - \omega^2 x + 2\omega\dot{y} + y\dot{\omega} = 0$

② is $\ddot{y} - \omega^2 y - 2\omega\dot{x} - x\dot{\omega} = 0$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{z}}\right) - \frac{\partial L}{\partial z} = 0 \Rightarrow \ddot{z} = 0.$$

$\uparrow \vec{\omega}$

$$\ddot{x}\,\hat{x} + \ddot{y}\,\hat{y} + \ddot{z}\,\hat{z} = \left(\omega^2 x - 2\omega\dot{y} - y\dot{\omega}\right)\hat{x} + \left(\omega^2 y + 2\omega\dot{x} + x\dot{\omega}\right)\hat{y}$$

$$\Rightarrow \ddot{\vec{r}} = \underline{\vec{\omega}\times(\vec{\omega}\times\vec{r})} + \underline{2(\vec{\omega}\times\dot{\vec{r}})} + \underline{\dot{\vec{\omega}}\times\vec{r}}$$

($\vec{r}$ is in x-y plane) miss a minus sign

So, the equation look like a particle acted on by a force.

The items might correspond with

<u>centrifugal force</u>, <u>geostrophic force</u>, <u>tangential force</u>.

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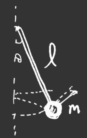
HW 2 Yuxiang Pei yuxiang@uchicago.edu

(1) (a) I'd prefer spherical coordinates with the origin at fixed end of rod.



(b) Spherical coordinates:

$$(x, y, z) = (l\sin\theta\sin\varphi,\ l\sin\theta\cos\varphi,\ l\cos\theta)$$

$$(\dot{x}, \dot{y}, \dot{z}) = (l\cos\theta\sin\varphi\,\dot{\theta} + l\sin\theta\cos\varphi\,\dot{\varphi},\ l\cos\theta\cos\varphi\,\dot{\theta} - l\sin\theta\sin\varphi\,\dot{\varphi},\ -l\sin\theta\,\dot{\theta})$$

$$T = \frac{1}{2}m(\dot{x}^2+\dot{y}^2+\dot{z}^2) = \frac{1}{2}m(l^2\cos^2\theta\,\dot{\theta}^2 + l^2\sin^2\theta\,\dot{\varphi}^2 + l^2\sin^2\theta\,\dot{\theta}^2)$$

$$= \frac{1}{2}m(l^2\dot{\theta}^2 + l^2\sin^2\theta\,\dot{\varphi}^2)$$

$$V = -mgl\cos\theta$$

$$L = T - V = \frac{1}{2}m(l^2\dot{\theta}^2 + l^2\sin^2\theta\,\dot{\varphi}^2) + mgl\cos\theta$$

Cylindrical coordinates:

$$(x, y, z) = (r\sin\theta,\ r\cos\theta,\ h)$$

$$(\dot{x}, \dot{y}, \dot{z}) = (\dot{r}\sin\theta + r\cos\theta\,\dot{\theta},\ \dot{r}\cos\theta - r\sin\theta\,\dot{\theta},\ \dot{h})$$

$$T = \frac{1}{2}m(\dot{x}^2+\dot{y}^2+\dot{z}^2) = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + \dot{h}^2)$$

$V = -mgh$. $\quad L = T - V = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + \dot{h}^2) + mgh$.

(c) $$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = 0.$$

For spherical coordinates: for $q_1 = \theta$, $ml^2\sin\theta\cos\theta\,\dot{\varphi}^2 - mgl\sin\theta = ml^2\ddot{\theta}$

for $q_2 = \varphi$, $0 = \frac{d}{dt}(ml^2\sin^2\theta\,\dot{\varphi})$

For cylindrical coordinates: for $q_1 = r$, $mr\dot{\theta}^2 = \frac{d}{dt}(m\dot{r})$

$q_2 = \theta$, $\frac{d}{dt}(mr^2\dot{\theta}) = 0$

$$q_3 = h, \quad mg = \frac{d}{dt}(m\dot{h})$$

(d) In spherical coordinates, L does not contain φ, (angular symmetry)

So $$\frac{\partial L}{\partial \dot{\varphi}} = ml^2 \sin^2\theta\, \dot{\varphi} = C_1$$

In cylindrical coordinates, L does not contain θ, (angular symmetry)

So, $$\frac{\partial L}{\partial \dot{\theta}} = mr^2 \dot{\theta} = C_2$$

(e) In spherical coordinates, $$\begin{cases} l \sin\theta \cos\theta\, \dot{\varphi}^2 - g\sin\theta = l\ddot{\theta} \\ ml^2 \sin^2\theta\, \dot{\varphi} = C_1 \end{cases}$$

$$\frac{C_1^2}{m^2 l^4 \sin^4\theta} \cdot l \sin\theta \cos\theta - g\sin\theta = l\ddot{\theta}$$

$$\Rightarrow \ddot{\theta} = -\frac{g}{l}\sin\theta + \frac{C_1^2 \cos\theta}{m^2 l^3 \sin^3\theta}$$

This is the equation θ obeys, solve to get $\theta(t)$.

then $$\varphi = \int_{t_1}^{t_2} \frac{C_1\, dt}{ml^2 \sin^2[\theta(t)]} + \varphi_0$$

In cylindrical coordinates, $r\dot{\theta}^2 = \ddot{r}$

$$mr^2\dot{\theta} = C_2$$

$$g = \ddot{h}$$

$\Rightarrow h = \frac{1}{2}gt^2 + V_0 t.$

While $r \cdot \left(\frac{C_2}{mr^2}\right)^2 = \ddot{r} \Rightarrow \ddot{r} = \frac{C_2^2}{m^2 r^3}.$

Solve this function to get $r(t)$.

then $\theta = \int_{t_1}^{t_2} \frac{C_2}{m\,r^2(t)}\,dt + \theta_0$

(2) (a) $G = \sum_i \vec{p}_i \cdot \vec{r}_i$.

$$\frac{dG}{dt} = \sum_i \frac{d\vec{p}_i}{dt} \cdot \vec{r}_i + \sum_i \vec{p}_i \cdot \frac{d\vec{r}_i}{dt}$$

$$= \sum_i m_i \frac{d^2\vec{r}_i}{dt^2} \cdot \vec{r}_i + \sum_i m_i \left(\frac{d\vec{r}_i}{dt}\right)^2$$

$$= \sum_i \vec{F}_i \cdot \vec{r}_i + 2T$$

Consider the time average of this equation.

$$\frac{1}{\tau}\int_0^{\tau} \frac{dG}{dt}\,dt = \overline{\frac{dG}{dt}} = \overline{2T} + \overline{\sum_i \vec{F}_i \cdot \vec{r}_i}$$

A finite and bound system would be that $\vec{r}_i$ and $\vec{p}_i$

are bound. So that $\overline{\left(\frac{dG}{dt}\right)}_\infty = 0$.

We have

$$\overline{2T} + \sum_i \overline{\vec{F}_i \cdot \vec{r}_i} = 0 \qquad (*)$$

While

$$\sum_{i=1}^{N} \vec{r}_i \cdot \vec{F}_i = \sum_{i=1}^{N} \sum_{j<i} (\vec{r}_i - \vec{r}_j) \cdot \vec{F}_{ij}$$

$$\vec{F}_{ij} = -\nabla_i V\left(|\vec{r}_i - \vec{r}_j|\right) = -\frac{dV}{dr} \frac{\vec{r}_i - \vec{r}_j}{r_{ij}}$$

$$V = \sum_{i=1}^{N} \sum_{i<j} V(r_{ij}) = \sum_{i=1}^{N} \sum_{i<j} \alpha |\vec{r}_i - \vec{r}_j|^\beta$$

So,

$$\sum_{i=1}^{N} \vec{r}_i \cdot \vec{F}_i = \sum_{i=1}^{N} \sum_{i<j} -\frac{dV}{dr} \frac{(\vec{r}_i - \vec{r}_j)^2}{r_{ij}}$$

$$= \sum_{i=1}^{N} \sum_{i<j} -\frac{dV}{dr} r_{ij}$$

$$= \sum_{i=1}^{N} \sum_{i<j} -\alpha\beta\, r_{ij}^\beta$$

$$= \sum_{i=1}^{N} \sum_{i<j} -\beta \cdot V_{ij} = -\beta V \qquad (**)$$

Put back into $(*)$, we have

$$\overline{2T} - \overline{\beta V} = 0. \qquad \text{That is } \langle T \rangle = \frac{\beta}{2} \langle V \rangle.$$

(b) If the motion is periodic, over one period we have

$$\frac{1}{\tau}\int_0^\tau \frac{dG}{dt}\,dt = \frac{1}{\tau}\int_0^\tau \left(2T + \sum_i \vec{F}_i\cdot\vec{r}_i\right)dt$$

For a periodic motion, $\frac{1}{\tau}\int_0^\tau \frac{dG}{dt}\,dt = \frac{G(\tau)-G(0)}{\tau}$.

$G(\tau) = G(0)$ because G only depends on $\vec{r}_i, \vec{p}_i$. which should not change after one period.

Still, $\frac{1}{\tau}\int_0^\tau \left(2T+\sum_i \vec{F}_i\cdot\vec{r}_i\right)dt = 0$.

That is, $\langle T\rangle = \frac{\beta}{2}\langle V\rangle$

because (**) still works.

(c) In this case $\beta=-1$, we have

$$\frac{1}{2}m\overline{v^2} = -\frac{1}{2}\cdot\left(\overline{\frac{-\alpha}{r}}\right) \Rightarrow \overline{v^2} = \frac{\alpha}{m}\left(\overline{\frac{1}{r}}\right).$$

It makes sense, because it shows higher $\overline{v}$ for smaller trajectories, and this obeys the laws of motion of celestial bodies

(Kepler 3rd law) $\frac{T^2}{a^3} = C_1$, Consider a circular motion.

We have $\frac{1}{r} = C_2 \cdot v^2$. $\quad (v = \frac{2\pi}{T} r)$

So our above conclusion make sense.

(d) for $V = \alpha r^n$, and circular motion,

We have $L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - \alpha r^n$.

$$\begin{cases} mr\dot{\theta}^2 - \alpha n r^{n-1} = \frac{d}{dt}(m\dot{r}) \\ \frac{d}{dt}(mr^2\dot{\theta}) = 0 \end{cases}$$

$$\Rightarrow \quad mr^2\dot{\theta} = k.$$

$$\ddot{r} = r\dot{\theta}^2 - \frac{\alpha}{m} n r^{n-1}$$

$$= \frac{k^2}{m^2 r^3} - \frac{\alpha}{m} n r^{n-1}$$

For a circular motion $\ddot{r} = 0$.

Then $\frac{k^2}{m^2 r^3} = \frac{\alpha}{m} n r^{n-1}$. $\Rightarrow \frac{k^2}{\alpha m} = n r^{n+2}$.

(***)

$$\langle T \rangle = \frac{1}{\tau}\int_0^{\tau} \frac{1}{2}mr^2\dot{\theta}^2\, dt$$

$$= \frac{1}{\tau}\int_0^\tau \frac{1}{2}\frac{k^2}{mr^2}\,dt = \frac{k^2}{2mr^2}.$$

$$\langle V\rangle = \frac{1}{\tau}\int_0^\tau \alpha r^n\,dt = \alpha r^n.$$

Using (***), we have $\langle T\rangle = \frac{n}{2}\langle V\rangle$.

So virial theorem is satisfied.

[Though, the particle not necessarily move circularly in this potential.]

(3) Noether's theorem

△ If the Lagrangian is not dependent on coordinates x_i,

the system is translationally invariant.

When we perform a spatial translation,

$$x_i' = x_i + \varepsilon_i.$$

$$\delta L_i = \frac{\partial L}{\partial x_i}\delta x_i + \frac{\partial L}{\partial \dot{x}_i}\delta \dot{x}_i$$

$$= \left(\frac{\partial L}{\partial x_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_i}\right)\delta x_i + \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\delta x_i\right)$$

$$= \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\delta x_i\right) = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right)\varepsilon_i$$

So, we have a Noether charge of

$$N_i = \frac{\partial L}{\partial \dot{x}_i}$$

For a mechanical system L could be

$$L = \sum_{i=1}^{N} \left(\frac{1}{2} m_i \dot{\vec{r}}_i^{\,2} - V \right).$$

V should be dependent of $\vec{r}_i$

for $x_i = x$. We have

$$N_x = \sum_{i=1}^{N} m_i \dot{x}_i .$$

$$N_y = \sum_{i=1}^{n} m_i \dot{y}_i$$

$$N_z = \sum_{i=1}^{n} m_i \dot{z}_i .$$

This conforms to momentum conservation in 3 directions.

Under rotations, $\vec{r}_i' = R \vec{r}_i$.

Consider a infinitesimal rotation of

$$R \vec{r}_i = \vec{r}_i + \alpha \times \vec{r}_i$$

$$\delta \vec{r}_i = \vec{r}_i' - \vec{r}_i = \vec{\alpha} \times \vec{r}_i .$$

We have a Noether charge of

$$N_i = \frac{\partial L}{\partial \dot{\vec{r}}_i} \delta \vec{r}_i .$$

For $L = \sum_{i=1}^{N} \left(\frac{1}{2} m_i \dot{\vec{r}}_i^{\,2} - V \right)$

$$N_i = \sum_{i=1}^{N} m \dot{\vec{r}}_i \cdot (\alpha \times \vec{r}_i)$$

$$= \sum_{i=1}^{N} \alpha (m \dot{\vec{r}}_i \times \vec{r}_i)$$

$$= \sum_{i=1}^{N} -\alpha (\vec{r}_i \times \vec{p}_i)$$

It represents the angular momentum,

and is conserved for rotationally invariant systems.

(4) Goldsten 3.19

(a) Using formula 3.12, we could get the equation of motion:

$$m\ddot{r} - \frac{l^2}{mr^3} = -\frac{k}{r^2} e^{-\frac{r}{a}}$$

In the direction of r, the motion could be described as

a 1-D problem in effective potential:

$$V_{eff} = V + \frac{l^2}{2mr^2}$$

only when V_{eff} has a minimum, we could get a confined orbit.

that is, $\frac{dV_{eff}}{dr} = 0 \Rightarrow \frac{dV}{dr} - \frac{l^2}{mr^3} = 0$

$$\Rightarrow \quad e^{-\frac{r}{a}} - \frac{l^2}{kmr} = 0$$

>0

Plot $h_1(r) = e^{-r/a}$ and $h_2(r) = \left(\frac{l^2}{km}\right) \cdot \frac{1}{r}$

$$\left.\frac{d^2V_{eff}}{dr^2}\right|_{r=r_0} = \left.-\frac{1}{a}e^{-r/a} + \frac{l^2}{kmr^2}\right|_{r=r_0} = -\frac{1}{a}\cdot\frac{l^2}{kmr_0} + \frac{l^2}{kmr_0^2}$$

$$= \frac{l^2}{km}\cdot\frac{a-r_0}{ar_0^2}.$$

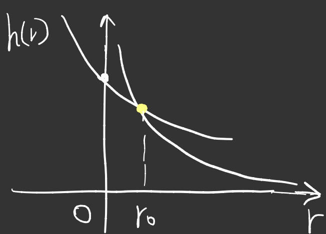
$h_1(r)$ and $h_2(r)$ are closest at the point $r_0 = a$.

$$h_1(a) = e^{-1} \quad \text{and} \quad h_2(a) = \frac{l^2}{kma}.$$

In the case when $h_1(a) < h_2(a)$. they will not cross,

we do not confined orbit for $a < \frac{el^2}{km}$.

When $a > \frac{el^2}{km}$.



they will cross at some point $0 < r_1 < a$.

and another point $r_2 > a$

when $r = r_1$, $\frac{dV_{eff}}{dr} = 0$.

$$\left.\frac{d^2V_{eff}}{dr^2}\right|_{r=r_1} = \frac{l^2}{km}\cdot\frac{a-r_0}{ar_0^2} < 0.$$

for $r_2 > a$. $\left.\frac{d^2V_{eff}}{dr^2}\right|_{r=r_2} > 0$.

So, we will have a stable confined orbit for

$0 < r_1 < a$, $a > \frac{el^2}{km}$. But will not have for other cases.

(b) When the orbit is nearly circular.

Small deviation could be written as

$$u = u_0 + a\cos(\beta\theta)$$

should derive

of which $u = \frac{1}{r}$. $\beta^2 = 3 + \frac{r}{f}\frac{df}{dr}\Big|_{r=r_0}$

$$\beta^2 = 3 - \frac{r^3}{k}e^{\frac{r}{a}} \cdot \left(\frac{2k}{r^3}e^{-\frac{r}{a}} + \frac{k}{ar^2}e^{-\frac{r}{a}}\right)$$

$$= 3 - 2 - \frac{r}{a} = 1 - \frac{r}{a}.$$

So, $\beta = \left(1 - \frac{r}{a}\right)^{\frac{1}{2}} \approx 1 - \frac{r}{2a}$

$$\Rightarrow \frac{1}{r} = \frac{1}{r_0} + a\cos\left[\left(1 - \frac{2r}{a}\right)\theta\right].$$

In this case, we will have the same $r = r_0$

when $\left(1 - \frac{r}{2a}\right)\theta = 2n\pi$.

$$\Rightarrow \theta \approx 2n\pi \cdot \left(1 + \frac{r}{2a} + \cdots\right).$$

$$= 2n\pi + \frac{\pi r}{a} \cdots$$

So, the orbit will precess by $\frac{\pi r}{a}$.

for each revolution.

(5) Goldstein 3.21

$$l\,dt = mr^2\,d\theta \quad \Rightarrow \quad \frac{d}{dt} = \frac{l}{mr^2}\frac{d}{d\theta}.$$

(3.12) gives that
$$m\ddot{r} - \frac{l^2}{mr^3} = f(r).$$

$$\Rightarrow \frac{l}{r^2}\frac{d}{d\theta}\left(\frac{l}{mr^2}\frac{dr}{d\theta}\right) - \frac{l^2}{mr^3} = f(r).$$

$$\Rightarrow \frac{d^2u}{d\theta^2} + u = -\frac{m}{l^2}\frac{d}{du}V\left(\frac{1}{u}\right) = -\frac{m}{l^2}(-k+2hu)$$

$$= \frac{mk}{l^2} - \frac{2mh}{l^2}u$$

So, $$u(\theta) = \frac{mk}{l^2\beta^2}\left(1+\varepsilon\cos\beta(\theta-\theta_0)\right).$$

Of which $\beta^2 = 1 + \frac{2mh}{l^2}$.

Similar to last problem, When $\beta(\theta-\theta_0) = 2n\pi$.

$$\theta - \theta_0 = \frac{2n\pi}{\sqrt{1+\frac{2mh}{l^2}}} \approx 2n\pi\cdot\left(1-\frac{mh}{l^2}\right)$$

is the precessing angle. So, $\dot{\Omega}\tau = 2\pi\cdot\frac{mh}{l^2}$.

$$\Rightarrow \dot{\Omega} = \frac{2\pi mh}{\tau l^2}.$$

For the Mercury, $\eta = \frac{h}{ka} = 7\times10^{-8}$, $\dot{\Omega} = 40''/\text{century}$

$\varepsilon = 0.206$, $\tau = 0.24y$.

$$= \frac{40}{60}\times\frac{1}{60}\times\frac{\pi}{180}\times\frac{1}{100}/y$$

$$= 1.94\times10^{-6}/y$$

While $l^2 = mka(1-\varepsilon^2)$.

We have $$\dot{\Omega} = \frac{2\pi m h}{\tau l^2} = \frac{2\pi\eta}{(1-\varepsilon^2)\tau}.$$

So, $$\eta = \frac{\dot{\Omega}(1-\varepsilon^2)\tau}{2\pi} = \frac{1.94\times10^{-6}\times(1-0.206^2)\times0.24}{2\pi}$$

$$= 7.1\times10^{-8}.$$

conforms to what we suppose in the question.

(6) Goldstein 3.28

(a) $$\vec{F}_B = \dot{\vec{r}}\times\vec{B} = \frac{qb}{r^3}\dot{\vec{r}}\times\vec{r}.$$

So, $$m\frac{d^2\vec{r}}{dt^2} = \frac{qb}{r^3}\dot{\vec{r}}\times\vec{r} + \frac{k}{r^3}\vec{r}$$

Look into $\vec{r}\times\dot{\vec{p}}$.

$$\vec{r}\times\dot{\vec{p}} = \vec{r}\times\vec{F} = \vec{r}\times\frac{qb}{r^3}(\dot{\vec{r}}\times\vec{r}) + \frac{k}{r^3}\overset{0}{\overset{\shortparallel}{\vec{r}\times\vec{r}}}$$

$$= \frac{qb}{r^3}\left(\dot{\vec{r}}|\vec{r}|^2 - \vec{r}(\vec{r}\cdot\dot{\vec{r}})\right) \neq 0.$$

While $\frac{d\vec{L}}{dt} = \frac{d(\vec{r}\times\vec{p})}{dt} = \overset{=0}{\dot{\vec{r}}\times\vec{p}} + \vec{r}\times\dot{\vec{p}}$

So, $\frac{d\vec{L}}{dt}$ is not always 0. $\Rightarrow$ L not conserved.

for $\vec{D} = \vec{L} - \frac{qb}{c}\frac{\vec{r}}{r}$. Where is c from?

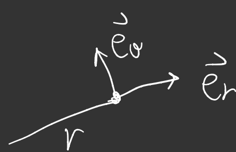
$$\frac{d\vec{D}}{dt} = \frac{d\vec{L}}{dt} - \frac{qb}{c}\frac{1}{r}\dot{\vec{r}} + \frac{qb}{c}\cdot\frac{\dot{r}}{r^2}\vec{r}$$

$$= \frac{qb}{r^3}r^2\dot{\vec{r}} - \frac{qb}{r^3}\vec{r}\,|\vec{r}\cdot\dot{\vec{r}}| - \frac{qb}{c}\cdot\frac{1}{r}\dot{\vec{r}} + \frac{qb}{c}\cdot\frac{\dot{r}}{r^2}\cdot\vec{r}$$

$= 0$. (I just neglect c)

Should be more exact:

what is $\frac{d}{dt}\left(\frac{\vec{r}}{r}\right)$?



$\vec{r} = r\vec{e}_r$

$\dot{\vec{r}} = \dot{r}\vec{e}_r + r\dot{\theta}\vec{e}_\theta$

$$\frac{1}{r}\frac{d\vec{r}}{dt} + \frac{d}{dt}\left(\frac{1}{r}\right)\cdot\vec{r}$$

$$= \frac{1}{r}\cdot\left(\dot{r}\vec{e}_r + \frac{d\vec{e}_r}{dt}r\right) - \frac{\dot{r}}{r^2}\vec{r}$$

$$= \frac{1}{r}\cdot\left(\dot{r}\vec{e}_r + r\dot{\theta}\vec{e}_\theta\right) - \frac{\dot{r}}{r^2}\vec{r} = \frac{\dot{\vec{r}}}{r} - \frac{\dot{r}}{r^2}\vec{r}$$

So, $\vec{D}$ is a conserved vector.

(b) In this case, $\dot{\vec{p}} = f(r)\frac{\vec{r}}{r} + \frac{qb}{r^3}\dot{\vec{r}}\times\vec{r}$

$$\dot{\vec{p}} \times \vec{D} = \left(f(r)\frac{\vec{r}}{r} + \frac{qb}{r^3}\dot{\vec{r}} \times \vec{r} \right) \times \left(\vec{L} - \frac{qb}{c}\frac{\vec{r}}{r} \right)$$

$$= f(r)\frac{\vec{r}}{r} \times \vec{L} + \frac{qb}{r^3}\dot{\vec{r}} \times \vec{r} \times \vec{L} - \frac{q^2b^2}{cr^4}\left(\dot{\vec{r}} \times \vec{r}\right) \times \vec{r}$$

$$= \frac{mf(r)}{r}\left(r\dot{r}\vec{r} - r^2\dot{\vec{r}}\right) - \frac{qb}{r^3}\dot{\vec{r}}\left(\vec{L}\cdot\vec{r}\right) + \frac{qb}{r^3}\vec{r}\left(\vec{L}\cdot\dot{\vec{r}}\right)$$

$$= 0$$

$$+ \frac{q^2b^2}{cr^4}\left(r^2\dot{\vec{r}} - \dot{r}r\vec{r}\right)$$

As $\vec{D}$ is constant, we could rewrite

$$\frac{d}{dt}\left(\vec{p} \times \vec{D}\right) = -\left(mf(r)r^2 - \frac{q^2b^2}{cr}\right)\frac{d}{dt}\left(\frac{\vec{r}}{r}\right)$$

$$\text{if } f(r) = -\frac{k}{r^2} + \frac{1}{m}\left(\frac{qb}{c}\right)^2\frac{1}{r^3}$$

We should have $\frac{d}{dt}\left(\vec{p} \times \vec{D}\right) = \frac{d}{dt}\left(\frac{mk\vec{r}}{r}\right)$.

So, $\vec{A} = \vec{p} \times \vec{D} - mk\frac{\vec{r}}{r}$.

that is similar to Runge–Lenz vector.

HW 3 Yuxiang Pei yuxiang@uchicago.edu

(1) Vector Identities

$$\varepsilon_{ijk} = \begin{cases} 1, & \text{when } i,j,k = 1,2,3;\ 2,3,1;\ 3,1,2; \\ -1, & \text{when } i,j,k = 1,3,2;\ 2,1,3;\ 3,2,1; \\ 0, & \text{other cases.} \end{cases}$$

That is, $$\varepsilon_{ijk} = \begin{vmatrix} \delta_{1i} & \delta_{2i} & \delta_{3i} \\ \delta_{1j} & \delta_{2j} & \delta_{3j} \\ \delta_{1k} & \delta_{2k} & \delta_{3k} \end{vmatrix}$$

So, $$\varepsilon_{ijk}\varepsilon_{imn} = \begin{vmatrix} \delta_{1i} & \delta_{2i} & \delta_{3i} \\ \delta_{1j} & \delta_{2j} & \delta_{3j} \\ \delta_{1k} & \delta_{2k} & \delta_{3k} \end{vmatrix} \cdot \begin{vmatrix} \delta_{1i} & \delta_{2i} & \delta_{3i} \\ \delta_{1m} & \delta_{2m} & \delta_{3m} \\ \delta_{1n} & \delta_{2n} & \delta_{3n} \end{vmatrix}$$

$$= \begin{vmatrix} \delta_{1i} & \delta_{2i} & \delta_{3i} \\ \delta_{1j} & \delta_{2j} & \delta_{3j} \\ \delta_{1k} & \delta_{2k} & \delta_{3k} \end{vmatrix} \cdot \begin{vmatrix} \delta_{1i} & \delta_{1m} & \delta_{1n} \\ \delta_{2i} & \delta_{2m} & \delta_{2n} \\ \delta_{3i} & \delta_{3m} & \delta_{3n} \end{vmatrix}$$

$$= \begin{vmatrix} 1 & \delta_{im} & \delta_{in} \\ \delta_{ij} & \delta_{mj} & \delta_{jn} \\ \delta_{ik} & \delta_{mk} & \delta_{kn} \end{vmatrix} = \delta_{mj}\delta_{kn} - \delta_{mk}\delta_{jn}$$

Suppose that we have vector

$$\vec{e_i} = \delta_{1i}\vec{e_1} + \delta_{2i}\vec{e_2} + \delta_{3i}\vec{e_3}$$

$$\vec{e_j} = \delta_{1j}\vec{e_1} + \delta_{2j}\vec{e_2} + \delta_{3j}\vec{e_3}$$

$$\vec{e_k} = \delta_{1k}\vec{e_1} + \delta_{2k}\vec{e_2} + \delta_{3k}\vec{e_3}$$

So, $$\varepsilon_{ijk} = (\vec{e_i} \times \vec{e_j}) \cdot \vec{e_k}.$$

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(1) Vector Identities

$$\varepsilon_{ijk} = \begin{cases} 1, & \text{when } i,j,k = 1,2,3;\ 2,3,1;\ 3,1,2; \\ -1, & \text{when } i,j,k = 1,3,2;\ 2,1,3;\ 3,2,1; \\ 0, & \text{other cases.} \end{cases}$$

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So, $$\varepsilon_{ijk}\varepsilon_{imn} = \begin{vmatrix} \delta_{1i} & \delta_{2i} & \delta_{3i} \\ \delta_{1j} & \delta_{2j} & \delta_{3j} \\ \delta_{1k} & \delta_{2k} & \delta_{3k} \end{vmatrix} \cdot \begin{vmatrix} \delta_{1i} & \delta_{2i} & \delta_{3i} \\ \delta_{1m} & \delta_{2m} & \delta_{3m} \\ \delta_{1n} & \delta_{2n} & \delta_{3n} \end{vmatrix}$$

$$= \begin{vmatrix} \delta_{1i} & \delta_{2i} & \delta_{3i} \\ \delta_{1j} & \delta_{2j} & \delta_{3j} \\ \delta_{1k} & \delta_{2k} & \delta_{3k} \end{vmatrix} \cdot \begin{vmatrix} \delta_{1i} & \delta_{1m} & \delta_{1n} \\ \delta_{2i} & \delta_{2m} & \delta_{2n} \\ \delta_{3i} & \delta_{3m} & \delta_{3n} \end{vmatrix}$$

$$= \begin{vmatrix} 1 & \delta_{im} & \delta_{in} \\ \delta_{ij} & \delta_{mj} & \delta_{jn} \\ \delta_{ik} & \delta_{mk} & \delta_{kn} \end{vmatrix} = \delta_{mj}\delta_{kn} - \delta_{mk}\delta_{jn}$$

For the first equality,

$$\vec{a}\times(\vec{b}\times\vec{c}) = \vec{a}\times\left(\sum_i \varepsilon_{ijk} b_j c_k \vec{e}_i\right)$$

The sum operation for m, n, k ... is omitted.

$$= \sum_l \varepsilon_{lmn} a_m \varepsilon_{njk} b_j c_k \vec{e}_l$$

$$= \sum_l \varepsilon_{lmn}\varepsilon_{njk} a_m b_j c_k \vec{e}_l$$

$$= \sum_l \left(\delta_{lj}\delta_{mk} - \delta_{lk}\delta_{jm}\right) a_m b_j c_k \vec{e}_l$$

$$= \sum_l \delta_{lj}\delta_{mk} a_m c_k b_j \vec{e}_l - \sum_l \delta_{lk}\delta_{jm} a_m b_j c_k \vec{e}_l$$

$$= \sum_l a_k c_k \cdot b_l \vec{e}_l - \sum_l a_m b_m \cdot c_l \vec{e}_l$$

$$= (\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})\vec{c}$$

For the second equality.

$$\vec{a}\cdot(\vec{b}\times\vec{c}) = \vec{a}\cdot\left(\sum_i \varepsilon_{ijk} b_j c_k \vec{e}_i\right)$$

$$= \sum_i \varepsilon_{ijk} a_i b_j c_k \vec{e}_i$$

$$= \sum_i \varepsilon_{jki} a_j b_k c_i \vec{e}_i$$

$$= \vec{c}\cdot(\vec{a}\times\vec{b}) \qquad \varepsilon_{ijk} a_j b_k c_i$$

$$= \sum_i \varepsilon_{kij} a_k b_i c_j \vec{e}_i \qquad = \varepsilon_{ijk} a_i b_j c_k$$

$$= \vec{b}\cdot(\vec{c}\times\vec{a}) \qquad a_j b_k c_i$$

(2)

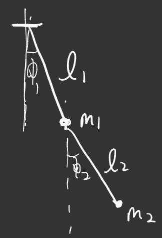
(a) $\vec{r}_{m_1} = (l_1 \sin\phi_1 \sin\varphi_1,\ l_1 \sin\phi_1 \cos\varphi_1,\ l_1 \cos\phi_1)$

$\vec{r}_{m_2} = l_1 \sin\phi_1 \sin\varphi_1 + l_2 \sin\phi_2 \sin\varphi_2,\ l_1 \sin\phi_1 \cos\varphi_1 + l_2 \sin\phi_2 \cos\varphi_2,\ l_1 \cos\phi_1 + l_2 \cos\phi_2)$

To make this case simpler, let $\varphi_1 = 0$.

Then $\vec{r}_{m_1} = (l_1 \sin\phi_1,\ l_1 \cos\phi_1)$

$\vec{r}_{m_2} = (l_1 \sin\phi_1 + l_2 \sin\phi_2,\ l_1 \cos\phi_1 + l_2 \cos\phi_2)$.



$\dot{\vec{r}}_{m_1} = (l_1 \cos\phi_1 \dot{\phi}_1,\ -l_1 \sin\phi_1 \dot{\phi}_1)$

$\dot{\vec{r}}_{m_2} = (l_1 \cos\phi_1 \dot{\phi}_1 + l_2 \cos\phi_2 \dot{\phi}_2,\ -l_1 \sin\phi_1 \dot{\phi}_1 - l_2 \sin\phi_2 \dot{\phi}_2)$

$$L = \frac{1}{2} m_1 l_1^2 \dot{\phi}_1^2 + \frac{1}{2} m_2 \left(l_1^2 \dot{\phi}_1^2 + l_2^2 \dot{\phi}_2^2 + 2 l_1 l_2 \cos(\phi_1 - \phi_2) \dot{\phi}_1 \dot{\phi}_2 \right)$$
$$+ m g l_1 \cos\phi_1 + m_2 g l_1 \cos\phi_1 + m_2 g l_2 \cos\phi_2$$

(b)

$$\frac{d}{dt}\left[m_1 l_1^2 \dot{\phi}_1 + m_2 \left(l_1^2 \dot{\phi}_1 + l_1 l_2 \cos(\phi_1 - \phi_2) \dot{\phi}_2 \right) \right] = -m_2 l_1 l_2 \sin(\phi_1 - \phi_2) \dot{\phi}_1 \dot{\phi}_2$$
$$- (m_1 + m_2) g l \sin\phi_1$$

$$(m_1 + m_2) l_1^2 \ddot{\phi}_1 + m_2 l_1 l_2 \sin(\phi_1 - \phi_2) \dot{\phi}_2^2 + m_2 l_1 l_2 \cos(\phi_1 - \phi_2) \ddot{\phi}_2 + (m_1 + m_2) g l \sin\phi_1 = 0$$

$$\frac{d}{dt}\left[m_2 l_2^2 \dot{\phi}_2 + m_2 l_1 l_2 \cos(\phi_1 - \phi_2) \dot{\phi}_1 \right] = m_2 l_1 l_2 \sin(\phi_1 - \phi_2) \dot{\phi}_1 \dot{\phi}_2$$
$$- m_2 g l \sin\phi_2$$

$$m_2 l_2^2 \ddot{\phi}_2 + m_2 l_1 l_2 \cos(\phi_1-\phi_2)\ddot{\phi}_1 - m_2 l_1 l_2 \sin(\phi_1-\phi_2)\dot{\phi}_1^2 + m_2 g l_2 \sin\phi_2 = 0$$

$$\Rightarrow \quad l_2 \ddot{\phi}_2 + l_1 \cos(\phi_1-\phi_2)\ddot{\phi}_1 - l_1 \sin(\phi_1-\phi_2)\dot{\phi}_1^2 + g\sin\phi_2 = 0$$

$$E_g = \sum_\alpha \frac{\partial L}{\partial \dot{q}_\alpha} - L$$

general energy

$$= \frac{\partial L}{\partial \dot{\phi}_1}\dot{\phi}_1 + \frac{\partial L}{\partial \dot{\phi}_2}\dot{\phi}_2 - L$$

$$= (m_1+m_2) l_1^2 \dot{\phi}_1^2 + m_2 l_1 l_2 \cos(\phi_1-\phi_2)\dot{\phi}_1\dot{\phi}_2$$

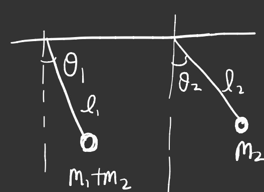
$$+ \; m_2 l_2^2 \dot{\phi}_2^2 + m_2 l_1 l_2 \cos(\phi_1-\phi_2)\dot{\phi}_2^2 - L$$

$$= \frac{1}{2}(m_1+m_2) l_1^2 \dot{\phi}_1^2 + \frac{1}{2} m_2 l_2^2 \dot{\phi}_2^2 + m_2 l_1 l_2 \cos(\phi_1-\phi_2)\dot{\phi}_2^2$$

$$- (m_1+m_2) g l_1 \cos\phi_1 - m_2 g l_2 \cos\phi_2$$

E_g is not conserved.

E_g is similar to a configuration where



So, we get the lowest energy when

$K = -(m_1+m_2) g l_1 - m_2 g l_2$ is lowest.

That is

Interaction Energy

$m_2 l_1 l_2 \cos(\phi_1-\phi_2)\dot{\phi}_2^2$

$$-\delta K = (\delta m_1 + \delta m_2) g l_1 + \delta m_2 g l_2$$

$$+ (m_1+m_2) g \,\delta l_1 + m_2 g \,\delta l_2 = 0$$

$$\theta_1 - \theta_2 = \frac{\pi}{2}.$$

is the lowest energy configuration. ?

(c) Because ϕ_1, ϕ_2 are small. $\cos\phi_1 \approx 1 - \frac{\phi_1^2}{2}$. $\cos\phi_2 = 1 - \frac{\phi_2^2}{2}$.

$$L = \frac{1}{2}(m_1+m_2) l_1^2 \dot{\phi}_1^2 + \frac{1}{2} m_2 l_2^2 \dot{\phi}_2^2 + m_2 l_1 l_2 \dot{\phi}_1 \dot{\phi}_2$$

$$- \frac{1}{2}(m_1+m_2) g l_1 \phi_1^2 + \frac{1}{2} m_2 g l_2 \phi_2^2$$

(d) $\Rightarrow$

$$\begin{cases} (m_1+m_2) l_1^2 \ddot{\phi}_1 + m_2 l_1 l_2 \ddot{\phi}_2 + (m_1+m_2) g l_1 \alpha_1 = 0. \\ m_2 l_1 l_2 \ddot{\phi}_1 + m_2 l_2 \ddot{\phi}_2 + m_2 g l_2 \phi_2 = 0 \end{cases}$$

$$\Rightarrow \begin{bmatrix} (m_1+m_2) l_1^2 & m_2 l_1 l_2 \\ m_2 l_1 l_2 & m_2 l_2^2 \end{bmatrix} \begin{bmatrix} \ddot{\phi}_1 \\ \ddot{\phi}_2 \end{bmatrix} + \begin{bmatrix} (m_1+m_2) g l_1 & \\ & m_2 g l_2 \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} = 0.$$

Thus, solve for

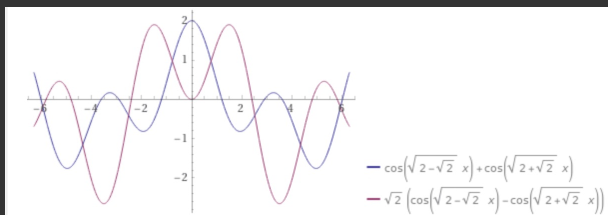
$$\begin{vmatrix} (m_1+m_2) l_1^2 - \omega^2 (m_1+m_2) g l_1 & m_2 l_1 l_2 \\ m_2 l_1 l_2 & m_2 l_2^2 - \omega^2 m_2 g l_2 \end{vmatrix} = 0$$

$$\Rightarrow \quad \omega_{1,2} = \sqrt{\frac{g}{l}} \left(2 \pm \sqrt{2}\right)^{\frac{1}{2}}$$

And $\phi_1(t) = A\cos(\omega_1 t+\varphi_1) + B\cos(\omega_2 t+\varphi_2)$

$$\phi_2(t) = -\sqrt{2}\,A\cos(\omega_1 t+\varphi_1) + \sqrt{2}\,B\cos(\omega_2 t+\varphi_2)$$

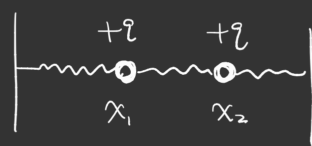
The resulting motion is a sum of two harmonics, as is shown by an example plot of $\phi_1(t)$, $\phi_2(t)$ below.



We could interpret the motion in oscillation of $\sqrt{\frac{g}{l}}(2+\sqrt{2})^{\frac{1}{2}}$ but modulated by a frequency of $\sqrt{\frac{g}{l}}(2-\sqrt{2})^{\frac{1}{2}}$.

(3) Goldstein 6.13

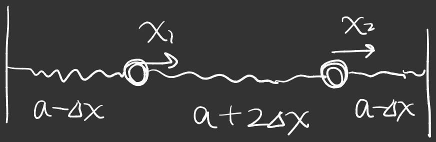
First we look for the equilibrium position.



$$\frac{k_e q^2}{(a+2\Delta x)^2} = 3k\Delta x \quad (*)$$

Suppose that it has a solution Δx, and x_1, x_2 are relative to equilibrium.

$$L = \frac{1}{2}m(\dot{x}_1^2+\dot{x}_2^2) - \frac{1}{2}k\left[(x_1^2+x_2^2)+(x_1-x_2)^2\right] - \frac{k_e q^2}{(a+2\Delta x+x_1-x_2)^2}$$



$$m\ddot{x}_1 = k\cdot(2\Delta x + x_2 - x_1) + k(\Delta x - x_1) - \frac{k_e q^2}{(a+2\Delta x+x_1-x_2)^2}$$

$$= kx_2 - 2kx_1 + \frac{k_e q^2 \cdot 2\cdot(x_1-x_2)}{(a+2\Delta x)^3}$$

$$m\ddot{x}_2 = k(\Delta x + x_2) + k(2\Delta x + x_2 - x_1) + \frac{k_e q^2}{(a+2\Delta x+x_1-x_2)^2}$$

$$= 2kx_2 - kx_1 - \frac{k_e q^2\cdot 2(x_1-x_2)}{(a+2\Delta x)^3}$$

Let $\dfrac{2k_e q^2}{(a+2\Delta x)^3} = k'$. (**)

We have $m\ddot{x}_1 = kx_2 - 2kx_1 + k'x_1 - k'x_2$

$$= (k'-2k)x_1 + (k-k')x_2$$

$$m\ddot{x}_2 = 2kx_2 - kx_1 - k'(x_1-x_2)$$

$$= -(k+k')x_1 + (2k+k')x_2.$$

$x_1 = u_1 e^{i\omega t}$, $x_2 = u_2 e^{i\omega t}$. $\Rightarrow$

$$-mu_1\omega^2 = (k'-2k)u_1 + (k-k')u_2$$

$$-mu_2\omega^2 = -(k+k')u_1 + (2k+k')u_2.$$

$$\Rightarrow \begin{bmatrix} k'-2k+m\omega^2 & k-k' \\ -(k+k') & m\omega^2+2k+k' \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = 0$$

So, $$(m\omega^2)^2 + 2k'm\omega^2 + k'^2 - 4k^2 + k^2 - k'^2 = 0$$

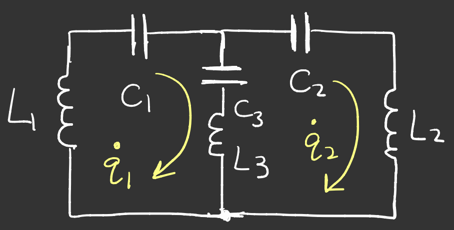
$$m^2\omega^4 + 2k'm\omega^2 - 3k^2 = 0.$$

$$(m\omega^2 + k')^2 = 3k^2 + k'^2.$$

$$\Rightarrow \quad \omega = \left(\frac{\sqrt{3k^2+k'^2} - k'^2}{m}\right)^{\frac{1}{2}}.$$

Of which k' is shown by (*) and (**).

(4) Goldstein 6.14



$$T = \frac{1}{2}L_1\dot{q}_1^2 + \frac{1}{2}L_2\dot{q}_2^2 + \frac{1}{2}L_3(\dot{q}_1 - \dot{q}_2)^2$$

$$V = \frac{q_1^2}{2C} + \frac{q_2^2}{2C} + \frac{(q_1+q_2)^2}{2C}$$

So, $$L = T - V = \frac{1}{2}L_1\dot{q}_1^2 + \frac{1}{2}L_2\dot{q}_2^2 + \frac{1}{2}L_3(\dot{q}_1 - \dot{q}_2)^2 - \frac{q_1^2}{2C_1} - \frac{q_2^2}{2C_2} - \frac{(q_1+q_2)^2}{2C_3}$$

We have secular equation:

$$\begin{bmatrix} L_1+L_3 & -L_3 \\ -L_3 & L_2+L_3 \end{bmatrix} \omega^2 - \begin{bmatrix} \frac{1}{C_1}+\frac{1}{C_3} & \frac{1}{C_3} \\ \frac{1}{C_3} & \frac{1}{C_2}+\frac{1}{C_3} \end{bmatrix} = 0.$$

Thus, we get $A\omega^4 - B\omega^2 + C = 0.$

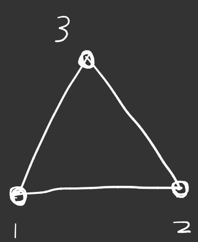
$$w = \left(\frac{-B \pm \sqrt{B^2-4AC}}{2A}\right)^{\frac{1}{2}}.$$

Of which $A = L_1L_2 + L_2L_3 + L_1L_3$

$$B = \frac{L_1}{C_2} + \frac{L_1}{C_3} + \frac{L_2}{C_3} + \frac{L_2}{C_1} + \frac{L_3}{C_1} + \frac{L_3}{C_2}.$$

$$C = \frac{1}{C_1C_2} + \frac{1}{C_2C_3} + \frac{1}{C_1C_3}.$$

(5) Goldstein 6.17



(a) In 2D motion, each mass has 2 degrees of freedom. There could be 6 normal modes however, due to the restrictions in the 3 force connections between them, there should be 3 zero modes.

(b) $\vec{r}_1 = (x_1, y_1)$. $\vec{r}_2 = (a+x_2, y_2)$

$$\vec{r}_3 = \left(\frac{a}{2} + x_3, \frac{\sqrt{3}}{2}a + y_3\right)$$

$$T = \frac{m}{2}\left(\dot{x}_1^2 + \dot{y}_1^2 + \dot{x}_2^2 + \dot{y}_2^2 + \dot{x}_3^2 + \dot{y}_3^2\right).$$

$$V = \frac{k}{2}\left[(a + x_2 - x_1)^2 + (y_2 - y_1)^2\right] + \frac{k}{2}\left[\left(\frac{a}{2} + x_3 - x_1\right)^2 + \left(\frac{\sqrt{3}}{2}a + y_3 - y_1\right)^2\right]$$

$$+ \frac{k}{2}\left[\left(x_3 - x_2 - \frac{a}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}a + y_3 - y_2\right)^2\right].$$

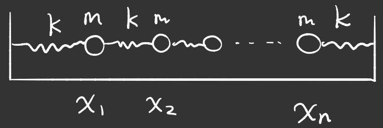
$$\approx \frac{k}{2}(x_2 - x_1)^2 + \frac{k}{8}(x_2 - x_3 - \sqrt{3}y_2 + \sqrt{3}y_3)^2 + \frac{k}{8}(x_3 - x_1 - \sqrt{3}y_3 + \sqrt{3}y_1)^2$$

$|V - \omega^2 T| = 0$ gives

$$\begin{vmatrix} \frac{5}{4}k - \omega^2 m & \frac{\sqrt{3}}{4}k & -k & 0 & -\frac{k}{4} & -\frac{\sqrt{3}}{4}k \\ \frac{\sqrt{3}}{4}k & \frac{3}{4}k - \omega^2 m & 0 & 0 & -\frac{\sqrt{3}}{4}k & -\frac{3}{4}k \\ -k & 0 & \frac{5}{4}k - \omega^2 m & -\frac{\sqrt{3}}{4}k & -\frac{k}{4} & \frac{\sqrt{3}}{4}k \\ 0 & 0 & -\frac{\sqrt{3}}{4}k & \frac{3}{4}k - \omega^2 m & \frac{\sqrt{3}}{4}k & -\frac{3}{4}k \\ -\frac{k}{4} & -\frac{\sqrt{3}}{4}k & -\frac{k}{4} & \frac{\sqrt{3}}{4}k & \frac{k}{2} - \omega^2 m & 0 \\ -\frac{\sqrt{3}}{4}k & -\frac{3}{4}k & \frac{\sqrt{3}}{4}k & -\frac{3}{4}k & 0 & \frac{3}{2}k - \omega^2 m \end{vmatrix} = 0$$

HW 4 Yuxiang Pei yuxiang@uchicago.edu

(1) I will first revise the condition discussed in class:



$$L = T - V. \qquad T = \frac{m}{2}\sum_{j=1}^{n} \dot{x}_j^2 .$$

$$V = \frac{1}{2}kx_1^2 + \frac{1}{2}k\sum_{j=1}^{n-1}(x_{j+1}-x_j)^2 + \frac{1}{2}kx_n^2$$

$$= \frac{1}{2}k\sum_{j=0}^{n}(x_{j+1}-x_j)^2 \qquad (x_0 = 0.\ x_{n+1} = 0)$$

$$\Rightarrow \quad m\ddot{x}_j = -k(x_j - x_{j+1}) - k(x_j - x_{j-1}) \qquad (j = 1, \ldots n)$$

Let $x_j = A_j e^{i\omega t}$, and get

$$m\omega^2 A_j = k(A_j - A_{j+1}) + k(A_j - A_{j-1})$$

$A_j = Ae^{i\alpha j}$. Then

$$\frac{m\omega^2}{k} e^{i\alpha j} = e^{i\alpha j} - e^{i\alpha(j+1)} + e^{i\alpha j} - e^{i\alpha(j-1)}$$

$$\Rightarrow \quad \frac{m\omega^2}{k} = 4\sin^2\frac{\alpha}{2}. \qquad \omega = 2\sqrt{\frac{k}{m}}\sin\frac{\alpha}{2}.$$

$$x_j = A_+ e^{i(\alpha j + \omega t)} + A_- e^{i(-\alpha j + \omega t)}$$

Because $A_+ + A_- = 0$ at $x_0 = 0$.

So, $x_j = \hat{A} e^{i\omega t} \sin \alpha j$

$x_{n+1} = 0$ so $\sin (n+1)\alpha = 0$. $\Rightarrow \alpha = \frac{\pi}{n+1}, \frac{2\pi}{n+1}, \cdots \frac{n\pi}{n+1}$.

Take the real part to get $\underline{x_j = \sum_{\ell=1}^{n} C_\ell \sin \alpha_\ell j \cos(\omega_\ell t + \theta_\ell)}$

⚠ For periodic boundary conditions,

Let $x_j = A_j e^{i\omega t}$ and get EL eqn:

$$m\omega^2 A_j = k(A_j - A_{j+1}) + k(A_j - A_{j-1})$$

$$A_j = A e^{i\alpha j}. \quad \Rightarrow \quad x_j = A_+ e^{i(\alpha j + \omega t)} + A_- e^{i(-\alpha j + \omega t)}.$$

$$\underline{\omega = 2\sqrt{\frac{k}{m}} \sin \frac{\alpha}{2}}.$$

Because $x_{n+1} = x_1$, we have

$$A_+ e^{in\alpha} + A_- e^{-in\alpha} = A_+ + A_- \qquad (A_+, A_- \text{ are real})$$

$$\begin{cases} A_+ \cos n\alpha + A_- \cos n\alpha = A_+ + A_- \\ i(A_+ \sin n\alpha - A_- \sin n\alpha) = 0 \end{cases} \Rightarrow \begin{cases} A_+ = A_- = \frac{A}{2} \\ \cos n\alpha = 1. \end{cases}$$

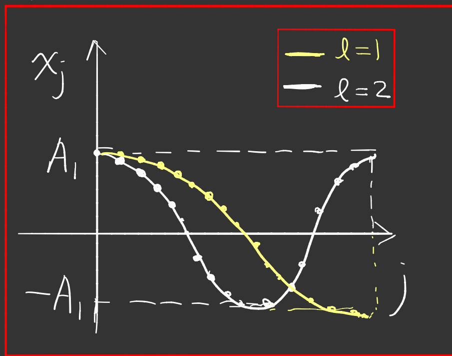
$$\Rightarrow \quad \alpha = \frac{\pi}{n}, \frac{2\pi}{n} \cdots \frac{(n-1)}{n}\pi, \pi. \qquad \omega_\ell = 2\sqrt{\frac{k}{m}} \sin\left(\frac{\ell\pi}{2n}\right).$$

$$x_j = \sum_{\ell=1}^{n} A_\ell \cos(\alpha_\ell j) \cos(\omega_\ell t + \theta_0).$$

(b) x_j versus j.

For the mode of $\ell = 1$.

$$x_j = A_1 \cos\left(\frac{\pi}{n} j\right) \cos(\omega t + \theta_0)$$



We could see that x_j is cosinoidal depending on j, which is dependent on the position of each oscillator. This means each oscillator will not only oscillate with time, but the phase will also depend on its position. The oscillation will propagate from one oscillator to next oscillator. (group velocity)

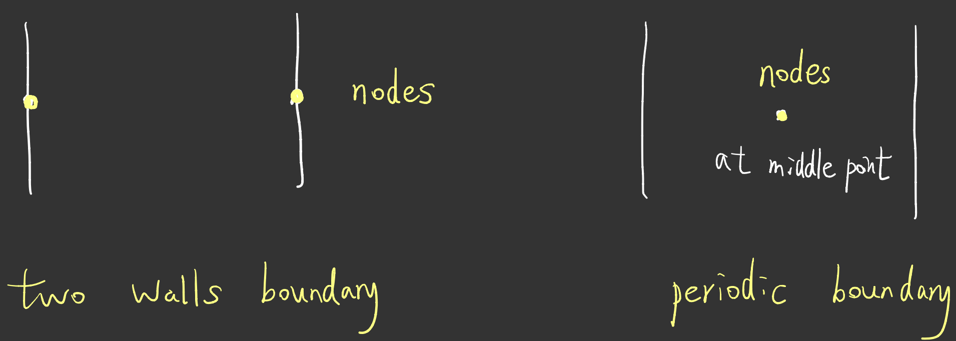
(c) We might already noticed that the condition of walls at two ends gives nodes at boundaries.

But in the periodic condition, as ploted above, gives antinodes at the "boundaries".

When $n \to \infty$, the oscillators form a continous medium, and will form a standing wave within the "boundary".

$$x_{\frac{n}{2}} = \sum_{l=1}^{n} A_l \cos\left(\frac{l\pi}{2n}\right) \cos(\omega_l t + \theta_0) = 0 \quad \text{when } n \to \infty .$$

So, the middle of these n osillators will also become an antinode.



(2) (a) $L = T - V$. $\quad T = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$

$$V = V(\vec{r}) \quad r = \sqrt{x^2 + y^2 + z^2}$$

$$H = \frac{1}{2m}(P_x^2 + P_y^2 + P_z^2) + V(r)$$

pass to spherical coordinates:

$$\begin{cases} x = r\sin\theta\cos\phi \\ y = r\sin\theta\sin\phi \\ z = r\cos\theta \end{cases}$$

$$L = \frac{1}{2}m\left(\dot{r}^2 + r^2\sin^2\theta\,\dot{\phi}^2 + r^2\dot{\theta}^2\right) - V(\vec{r})$$

$$P_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r}$$

$$P_\theta = \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta}$$

$$P_\phi = \frac{\partial L}{\partial \dot{\phi}} = mr^2\sin^2\theta\,\dot{\phi}$$

$$\Rightarrow \quad H = \frac{1}{2m}\left(P_r^2 + \frac{P_\theta^2}{r^2} + \frac{P_\phi^2}{r^2\sin^2\theta}\right) + V(\vec{r})$$

(b) $\dot{P}_r = -\frac{\partial H}{\partial r}$ $\qquad \dot{r} = \frac{\partial H}{\partial P_r}$

$\dot{P}_\theta = -\frac{\partial H}{\partial \theta}$ $\qquad \dot{\theta} = \frac{\partial H}{\partial P_\theta}$

$\dot{P}_\phi = -\frac{\partial H}{\partial \phi}$ $\qquad \dot{\phi} = \frac{\partial H}{\partial P_\phi}$

We could see that $\dot{P}_\phi = -\frac{\partial H}{\partial \phi} = 0$

But $\dot{P}_\theta = -\frac{\partial H}{\partial \theta} = \frac{P_\phi^2}{2mr^2} \cdot \frac{-2\cos\theta}{\sin^3\theta} = \frac{-P_\phi^2\cos\theta}{mr^2\sin^3\theta}$.

So, P_ϕ is conserved but P_θ is not.

When we perform a spatial transition,

$$\phi' = \phi + \Delta\phi.$$

$\delta L = 0.$ So we have Noether charge P_ϕ.

But when we perform

$\theta' = \theta + \Delta\theta.$ $\delta L \neq 0.$ P_θ is not conserved.

(3) Goldstein Chapter 8. Problem 16.

$$H = \frac{p^2}{2A} - Bqpe^{-\alpha t} + \frac{AB}{2} q^2 e^{-\alpha t}(\alpha + Be^{-\alpha t}) + \frac{kq^2}{2}.$$

(a) Because $H(q,p) = p\dot{q}(q,p) - L(q,\dot{q}(q,p))$.

We have $L = \dot{q}p - H$

$$\dot{q} = \frac{\partial H}{\partial p} = \frac{p}{A} - Bqe^{-\alpha t} \quad ①$$

$$\dot{p} = -\frac{\partial H}{\partial q} = Bpe^{-\alpha t} - ABqe^{-\alpha t}(\alpha + Be^{-\alpha t}) - kq. \quad ②$$

From ① I get

$$p = A\dot{q} + ABqe^{-\alpha t}.$$

So, $L = \dot{q}p - H$

$$= A\dot{q}^2 + ABq\dot{q}e^{-\alpha t} - H$$

$$H = \frac{(A\dot{q} + ABqe^{-\alpha t})^2}{2A} - Bq(A\dot{q} + ABqe^{-\alpha t})e^{-\alpha t}$$

$$+ \frac{AB}{2}q^2 e^{-\alpha t}(\alpha + Be^{-\alpha t}) + \frac{kq^2}{2}$$

$$= \frac{1}{2}A\dot{q}^2 + \underline{\frac{1}{2}AB^2q^2e^{-2\alpha t}} + \underline{ABq\dot{q}e^{-\alpha t}}$$

$$\underline{-ABq\dot{q}e^{-\alpha t}} \underline{- AB^2q^2e^{-2\alpha t}} + \frac{1}{2}ABq^2\alpha e^{-\alpha t}$$

$$\underline{+ \frac{1}{2}AB^2q^2e^{-2\alpha t}} + \frac{1}{2}kq^2$$

$$= \frac{1}{2}A\dot{q}^2 + \frac{1}{2}ABq^2\alpha e^{-\alpha t} + \frac{1}{2}kq^2.$$

So, $L = A\dot{q}^2 + ABq\dot{q}e^{-\alpha t} - H$

$$= \underline{\frac{1}{2}A\dot{q}^2 + ABq\dot{q}e^{-\alpha t} - \frac{1}{2}ABq^2\alpha e^{-\alpha t} - \frac{1}{2}kq^2}$$

To verify this result,

Let $\dot{q} = \frac{P}{A} - Bq e^{-\alpha t} \Rightarrow$

$$L = \frac{p^2}{2A} - Bpq e^{-\alpha t} + Bq e^{-\alpha t} \cdot \left(P - ABq e^{-\alpha t}\right)$$

$$+ \frac{1}{2} AB^2 q^2 e^{-2\alpha t} - \frac{1}{2} ABq^2 \alpha e^{-\alpha t} - \frac{1}{2} kq^2$$

$$= \frac{p^2}{2A} - \frac{1}{2} AB^2 q^2 e^{-\alpha t}\left(e^{-\alpha t} + \frac{\alpha}{B}\right) - \frac{1}{2} kq^2$$

$$= \left(\frac{P}{A} - Bq e^{-\alpha t}\right) P - H.$$ It is correct.

(b) Because $L' = L + \frac{df(t)}{dt}$

Let $f(t) = -\frac{1}{2} ABq^2 e^{-\alpha t}$

$$\frac{df(t)}{dt} = \frac{1}{2} ABq^2 \alpha e^{-\alpha t} - ABq\dot{q} e^{-\alpha t}$$

So, $L' = L + \frac{df(t)}{dt} = \frac{1}{2} A\dot{q}^2 - \frac{1}{2} kq^2.$

(c) $H' = p\dot{q} - L' = \frac{p^2}{A} - Bpq e^{-\alpha t} - L'$

$$\boxed{\dot{q} = \frac{p}{A} - Bq e^{-\alpha t}}$$

$$\Rightarrow L' = \frac{1}{2}A\left(\frac{p^2}{A^2} - \frac{2pqBe^{-\alpha t}}{A} + B^2q^2e^{-2\alpha t}\right) - \frac{1}{2}kq^2$$

$$= \frac{p^2}{2A} - Bpq e^{-\alpha t} + \frac{1}{2}AB^2q^2e^{-2\alpha t} - \frac{1}{2}kq^2.$$

So, $H' = \frac{p^2}{2A} + \frac{1}{2}kq^2 - \frac{1}{2}AB^2q^2e^{-2\alpha t}$

Maybe I should not use this

$H' = p\dot{q} - L$

$$p = \frac{\partial L'}{\partial \dot{q}} = A\dot{q}$$

So, $H' = p \cdot \frac{p}{A} - L = \frac{p^2}{A} - \frac{1}{2}A \cdot \left(\frac{p}{A}\right)^2 + \frac{1}{2}kq^2$

$$= \frac{p^2}{2A} + \frac{1}{2}kq^2.$$

HW 5 Yuxiang Pei yuxiang@uchicago.edu

(1)

(a)
$$-[g,f] = -\sum_{i=1}^{n}\left(\frac{\partial g}{\partial q_i}\frac{\partial f}{\partial p_i} - \frac{\partial g}{\partial p_i}\frac{\partial f}{\partial q_i}\right) = \sum_{i=1}^{n}\left(\frac{\partial g}{\partial p_i}\frac{\partial f}{\partial q_i} - \frac{\partial g}{\partial q_i}\frac{\partial f}{\partial p_i}\right)$$
$$= \sum_{i=1}^{n}\left(\frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i}\right) = 0$$

(b)
$$[af+bg, h] = \sum_{i=1}^{n}\left(\frac{\partial(af+bg)}{\partial q_i}\frac{\partial h}{\partial p_i} - \frac{\partial(af+bg)}{\partial p_i}\frac{\partial h}{\partial q_i}\right)$$
$$= \sum_{i=1}^{n}\left(a\frac{\partial f}{\partial q_i}\frac{\partial h}{\partial p_i} + b\frac{\partial f}{\partial q_i}\frac{\partial h}{\partial p_i} - a\frac{\partial f}{\partial p_i}\frac{\partial h}{\partial q_i} - b\frac{\partial g}{\partial p_i}\frac{\partial h}{\partial q_i}\right)$$
$$= a\sum_{i=1}^{n}\left(\frac{\partial f}{\partial q_i}\frac{\partial h}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial h}{\partial q_i}\right) + b\sum_{i=1}^{n}\left(\frac{\partial f}{\partial q_i}\frac{\partial h}{\partial p_j} - \frac{\partial f}{\partial p_i}\frac{\partial h}{\partial p_i}\right)$$
$$= a[f,h] + b[g,h].$$

(c)
$$[fg, h] = \sum_{i=1}^{n}\left(\left(\frac{\partial f}{\partial q_i}g + f\frac{\partial g}{\partial q_i}\right)\cdot\frac{\partial h}{\partial p_i} - \left(\frac{\partial f}{\partial p_i}g + f\frac{\partial g}{\partial p_i}\right)\cdot\frac{\partial h}{\partial q_i}\right)$$
$$= \sum_{i=1}^{n}\left(\frac{\partial f}{\partial q_i}\frac{\partial h}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial h}{\partial q_i}\right)g + \sum_{i=1}^{n} f\left(\frac{\partial g}{\partial q_i}\frac{\partial h}{\partial p_i} - \frac{\partial g}{\partial p_i}\frac{\partial h}{\partial q_i}\right)$$
$$= [f,h]g + [f,g]h.$$

(d)
$$[q_i, q_j] = \sum_{k=1}^{n}\left(\frac{\partial q_i}{\partial q_k}\frac{\partial q_j}{\partial p_k} - \frac{\partial q_i}{\partial p_k}\frac{\partial q_j}{\partial q_k}\right) = \frac{\partial q_j}{\partial p_i} - \frac{\partial q_i}{\partial p_j} = 0.$$

(p_i, q_i are independent)

$$[p_i, p_j] = \sum_{k=1}^{n} \left(\frac{\partial p_i}{\partial q_k} \frac{\partial p_j}{\partial p_k} - \frac{\partial p_i}{\partial p_k} \frac{\partial q_j}{\partial q_k} \right) = \frac{\partial p_j}{\partial q_j} - \frac{\partial q_j}{\partial p_i} = 0$$

$$[q_i, p_j] = \sum_{k=1}^{n} \left(\frac{\partial q_i}{\partial q_k} \frac{\partial p_j}{\partial p_k} - \frac{\partial q_i}{\partial p_k} \frac{\partial p_j}{\partial q_k} \right) = \frac{\partial p_j}{\partial p_i} - \frac{\partial q_i}{\partial p_j} = \delta_{ij}.$$

(e)

$$df = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial q_i} dq_i + \frac{\partial f}{\partial p_i} dp_i \right) + \frac{\partial f}{\partial t} dt$$

$$\frac{df}{dt} = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial q_i} \frac{dq_i}{dt} + \frac{\partial f}{\partial p_i} \frac{dp_i}{dt} \right) + \frac{\partial f}{\partial t} dt.$$

While $[f, H] = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial q_i} \frac{\partial H}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial H}{\partial q_i} \right)$

$$= \sum_{i=1}^{n} \left(\frac{\partial f}{\partial q_i} \dot{q}_i + \frac{\partial f}{\partial p_i} \dot{p}_i \right) = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial q_i} \frac{dq_i}{dt} + \frac{\partial f}{\partial p_i} \frac{dp_i}{dt} \right)$$

So, $\frac{df}{dt} = [f, H] + \frac{\partial f}{\partial t}.$

(f)

$$H = \frac{p^2}{2m} - \frac{k^2}{2q^2}, \quad D = -2Ht + pq$$

$$[H, D] = [H, -2Ht] + [H, pq]$$

$$= -2t \underbrace{[H, H]}_{=0} + \underbrace{[H, -2t]}_{=0} H + [H, pq]$$

$$= \left[\frac{p^2}{2m}, pq\right] - \left[\frac{k}{2q^2}, pq\right]$$

$$= p\left[\frac{p^2}{2m}, q\right] + \left[\frac{p^2}{2m}, p\right]q - \left[\frac{k}{2q^2}, p\right]q - p\left[\frac{k}{2q^2}, q\right]$$

$$= p\cdot\left(-\frac{p}{m}\right) + \frac{k^2}{q^3}\cdot q = \frac{-p^2}{m} + \frac{k}{q^2}$$

So,
$$\frac{dD}{dt} = [D, H] + \frac{\partial D}{\partial t} = -[H, D] + \frac{\partial D}{\partial t}$$

$$= \frac{p^2}{m} - \frac{k}{q^2} + (-2H)$$

$$= \frac{p^2}{m} - \frac{k}{q^2} - \frac{p^2}{m} + \frac{k}{q^2} = 0.$$

I did not find the corresponding symmetry at first, but after some searching I know that its scale transformation.

(will show it on the last page)

(g) $N \times N$ matrices F, G.

$$[F, G] = FG - GF = GF - FG = -[G, F].$$

$$[aF + bG, H] = (aF + bG)H - H(aF + bG)$$

$$= a(FH - HF) + b(GH - HG)$$

$$= a[F, H] + b[G, H]$$

$$[FG,H] = FGH - HFG = FGH - FHG + FHG - HFG$$
$$= F(GH-HG) + (FH-HF)G = F[G,H] + [F,H]G.$$

Then I show why it is reasonable to replace ?

$$[f,g] = \sum_{i=1}^{n}\left(\frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i}\right) \quad \text{with } [F,G] = FG - GF$$

(2)
(a) $\vec{L} = \vec{r} \times \vec{p}$. $\qquad L_i = \varepsilon_{ijk} x_j p_k$.

$$[x_i, L_j] = [x_i, \varepsilon_{ijk} x_k p_i] = \sum_{l=1}^{3} \frac{\partial x_i}{\partial x_l}\frac{\partial(\varepsilon_{ijk} x_k p_i)}{\partial p_l} - \sum_{l=1}^{3}\frac{\partial x_i}{\partial p_l}\frac{\partial(\varepsilon_{ijk} x_k p_i)}{\partial x_l}$$
$$= \sum_{l=1}^{3} \varepsilon_{ijk}\delta_{il} x_k = \varepsilon_{ijk} x_k$$

$$[p_i, L_j] = [p_i, \varepsilon_{j\alpha\beta} x_\alpha p_\beta] = \sum_{l=1}^{3}\left(-\varepsilon_{j\alpha\beta} p_\beta \delta_{li}\delta_{l\alpha}\right) = -\varepsilon_{j\alpha\beta} p_\beta$$

$$[L_i, L_j] = \sum_{l=1}^{3}\frac{\partial L_i}{\partial x_l}\cdot\frac{\partial L_j}{\partial p_l} - \sum_{l=1}^{3}\frac{\partial L_i}{\partial p_l}\frac{\partial L_j}{\partial x_l}$$

$$\boxed{L_i = \varepsilon_{ith} x_t p_h, \quad L_j = \varepsilon_{j\alpha\beta} x_\alpha p_\beta}$$

$$[L_i, L_j] = \sum_l \delta_{lt}\,\delta_{l\beta}\,\varepsilon_{ith}\,p_h\,\varepsilon_{j\alpha\beta}\,x_\alpha$$

$$-\sum_l \delta_{lh}\,\delta_{l\alpha}\,\varepsilon_{ith}\,x_t\,\varepsilon_{j\alpha\beta}\,p_\beta$$

$$= \varepsilon_{ilh}\,\varepsilon_{j\alpha l}\,x_\alpha p_h - \varepsilon_{itl}\,\varepsilon_{jl\beta}\,x_t\,p_\beta$$

$$= (\delta_{hj}\,\delta_{i\alpha} - \delta_{h\alpha}\,\delta_{ij})\,x_\alpha p_h - (\delta_{i\beta}\,\delta_{tj} - \delta_{ij}\,\delta_{t\beta})\,x_t\,p_\beta$$

$$= x_i p_j - \delta_{ij}\,\delta_{h\alpha}\,x_\alpha p_h - x_j p_i + \delta_{ij}\,\delta_{t\beta}\,x_t p_\beta$$

$$= x_i p_j - p_i x_j$$

(b) $[\vec{r}^{\,2}, L_j] = [\sum_i x_i^2, L_j]$

$$= \sum_l \frac{\partial(\sum_i x_i^2)}{\partial x_l}\,\frac{\partial(\varepsilon_{ith}\,x_t p_h)}{\partial p_l} - \sum_l \frac{\partial(\sum_i x_i^2)}{\partial p_l}\,\frac{\partial(\varepsilon_{ith}\,x_t p_h)}{\partial x_l}$$

$$= \sum_l \sum_i \delta_{il}\cdot 2x_l \cdot \delta_{lh}\,\varepsilon_{ith}\,x_t - \sum_l \sum_i \cdot 0$$

$$= \sum_l \sum_i \delta_{il}\cdot 2x_l \cdot \varepsilon_{itl}\,x_t$$

$$= \sum_i 2x_i\,x_t \cdot \varepsilon_{iti} = 0\,.$$

$$[\vec{p}^2, L_j] = \sum_l \left(\frac{\partial(\sum_i p_i^2)}{\partial x_l} \frac{\partial(\varepsilon_{ith} x_t p_h)}{\partial p_l} - \frac{\partial(\sum_i p_i^2)}{\partial p_l} \frac{\partial(\varepsilon_{ith} x_t p_h)}{\partial x_l} \right)$$

$$= -\sum_l \sum_i \delta_{il} \cdot 2p_i \cdot \varepsilon_{ith} \delta_{lt} p_h$$

$$= -\sum_i 2p_i \cdot \varepsilon_{iih} p_h = 0.$$

$$[\vec{L}^2, L_j] = \left[\sum_i L_i^2, L_j\right] = \left[L_j^2 + L_\alpha^2 + L_\beta^2, L_j\right]$$

$$= [L_j^2, L_j] + [L_\alpha^2, L_j] + [L_\beta^2, L_j]$$

$$= 0 + L_\alpha [L_\alpha, L_j] + [L_\alpha, L_j] L_\alpha + 2L_\beta [L_\beta, L_j]$$

$j\ \alpha\ \beta$

$$= 2L_\alpha [L_\alpha, L_j] + 2L_\beta [L_\beta, L_j]$$

$$= -2L_\alpha L_\beta + 2L_\beta L_\alpha = 0$$

Physical meaning: We might find from (a) that $[x_i, L_j]$ or $[p_i, L_j]$ stands for the changes in x_i or p_i after rotate around j axis. So, $[\vec{r}^2, L_j]$, $[\vec{p}^2, L_j]$, $[\vec{L}^2, L_j] = 0$ means $\vec{r}^2$, $\vec{p}^2$ and $\vec{L}^2$ are invariant under the rotation operation around j axis.

(c) $H = \frac{\vec{p}^2}{2m} + U(\vec{r})$.

$$[H, L_j] = \left[\sum_i \frac{p_i^2}{2m} + U(\vec{r}),\ L_j\right] = \left[\sum_i \frac{p_i^2}{2m},\ L_j\right] + \left[U(\vec{r}), L_j\right]$$

$$= 0 + \sum_l \left(\frac{\partial U}{\partial x_l}\frac{\partial L_j}{\partial p_l} - \frac{\partial U}{\partial p_l}\frac{\partial L_j}{\partial x_l}\right)$$

$$= \frac{\partial U}{\partial x_\beta}\varepsilon_{j\alpha\beta}x_\alpha - \frac{\partial U}{\partial p_\alpha}\varepsilon_{j\alpha\beta}p_\beta = \frac{\partial U}{\partial x_\beta}\varepsilon_{j\alpha\beta}x_\alpha.$$

(d) $U = -\frac{k}{r}$. $\quad \vec{A} = \vec{p}\times\vec{L} - mk\frac{\vec{r}}{r}$

$$A_i = \varepsilon_{ijk}p_jL_k - mk\frac{x_i}{r}$$

$$[A_i, H] = \left[\varepsilon_{ijk}p_jL_k,\ \frac{p^2}{2m}\right] - mk\left[\frac{x_i}{r},\ \frac{\vec{p}^2}{2m}\right]$$

$$+ \left[\varepsilon_{ijk}p_jL_k,\ U(\vec{r})\right] - mk\left[\frac{x_i}{r},\ U(r)\right]$$

$$= \varepsilon_{ijk}p_j\left[L_k, \frac{\vec{p}^2}{2m}\right] + \varepsilon_{ijk}L_k\left[p_j, \frac{\vec{p}^2}{2m}\right] - mk\left[\frac{x_i}{r}, \frac{\vec{p}^2}{2m}\right]$$

$$+ \varepsilon_{ijk}p_j\left[L_k, U(\vec{r})\right] + \varepsilon_{ijk}L_k\left[p_j, U(\vec{r})\right]$$

$$- mk\left[\frac{x_i}{r}, U(\vec{r})\right]$$

$$= 0 + 0 - mk\left[\frac{x_i}{r}, \frac{p_i^2}{2m}\right] - mk\left[\frac{x_i}{r}, U(\vec{r})\right]$$

$$- \varepsilon_{ijk}\, p_j \frac{\partial U}{\partial x_\beta} \varepsilon_{k\alpha\beta}\, x_\alpha - \varepsilon_{ijk} L_k \frac{\partial U(\vec{r})}{\partial x_j}$$

kij

$k\alpha\beta$

$$= -mk \frac{\partial(x_i/r)}{\partial x_i} \cdot \frac{p_i}{m} - \left(\delta_{i\alpha}\delta_{j\beta} - \delta_{i\beta}\delta_{j\alpha}\right) \frac{\partial U}{\partial x_\beta} x_\alpha p_j$$

$$- \varepsilon_{ijk}\, \varepsilon_{kmn}\, x_m p_n \frac{\partial U}{\partial x_j}$$

$$= -mk \frac{\partial(x_i/r)}{\partial x_i} \cdot \frac{p_i}{m} - \frac{\partial U}{\partial x_j} x_i p_j + \frac{\partial U}{\partial x_i} x_j p_j$$

$$- \left(\delta_{im}\delta_{jn} - \delta_{in}\delta_{jm}\right) \frac{\partial U}{\partial x_j} x_m p_n$$

$$= -k \frac{\partial(x_i/r)}{\partial x_i} \cdot p_i \underline{+ \frac{\partial U}{\partial x_j} x_i p_j} + \frac{\partial U}{\partial x_i} x_j p_j$$

$$\underline{- \frac{\partial U}{\partial x_j} x_i p_j} + \frac{\partial U}{\partial x_j} x_j p_i$$

$$= -k \cdot \frac{r^2 - x_i^2}{r^3} \cdot p_i + \frac{\partial U}{\partial x_i} x_j p_j - \frac{\partial U}{\partial x_j} x_j p_i$$

$$U = -\frac{k}{r}.$$

$$\frac{\partial U}{\partial x_j} = -k \cdot \frac{-1 \cdot \frac{x_j}{r}}{r^2} = \frac{k x_j}{r^3}.$$

$$= -\frac{k}{r} p_i + \frac{k x_i^2}{r^3} p_i + \frac{k x_i}{r^3} x_j p_j$$

$$- \frac{k x_j^2}{r^3} p_i$$

$$= k \frac{r^2 - \sum' x_j^2}{r^3} p_i = k \frac{r^2 - r^2}{r^3} p_i = 0.$$

Significance: It means the vector $\vec{A}$ is constant for $U(r) = -\frac{k}{r}$.

Which suggests a higher level symmetry for this central force system.

---

(1)(f) If the system go through a scale transformation,

$$q \to Q = \lambda q, \; t \to \tau = \lambda^2 t, \; p \to P = \frac{p}{\lambda}.$$

For the original system we have

$$\dot{q} = \frac{\partial H}{\partial p} = \frac{p}{m}, \qquad \dot{p} = -\frac{\partial H}{\partial q} = \frac{k}{q^3}.$$

We could see that the equations of motion:

$$\dot{Q} = \frac{dQ}{d\tau} = \frac{\lambda dq}{\lambda^2 dt} = \frac{\dot{q}}{\lambda} = \frac{p}{m\lambda} = \frac{P}{m}$$

$$\dot{P} = \frac{dP}{d\tau} = \frac{\frac{1}{\lambda} dp}{\lambda^2 dt} = \frac{1}{\lambda^3}\dot{p} = \frac{1}{\lambda^3}\cdot\frac{k}{q^3} = \frac{k}{Q^3}.$$

Which means the system is invariant under

Scale transformation.

HW 6 Yuxiang Pei yuxiang@uchicago.edu

Goldstein Chapter 9. 25. 30. Chapter 13. 1.4.

Chapter 9 Canonical Transformations

25. (a) $H = \frac{1}{2}\left(\frac{1}{q^2} + p^2 q^4\right)$.

equations of motion: $\dot{q} = \frac{\partial H}{\partial p} = pq^4$.

$\dot{p} = -\frac{\partial H}{\partial q} = \frac{1}{q^3}$.

(b) $p\,dq - H\,dt = P\,dQ - K\,dt + dF$.

$dF = p\,dq - P\,dQ + (K - H)\,dt$

$\Rightarrow p = \frac{\partial F}{\partial q}$. $P = -\frac{\partial F}{\partial Q}$. $K = H + \frac{\partial F}{\partial t}$.

For a harmonic oscillator we have $\dot{Q} = P$, $\dot{P} = -Q$.

$K = \frac{1}{2}(P^2 + Q^2)$. We could let $K = 0$ (not depend on time)

So, $p\,dq - P\,dQ + \left[\frac{1}{2}(P^2 + Q^2) - \frac{1}{2}\left(\frac{1}{q^2} + p^2 q^4\right)\right]dt = 0$.

$p\dot{q} - P\dot{Q} + \frac{1}{2}(P^2 + Q^2) - \frac{1}{2}\left(\frac{1}{q^2} + p^2 q^4\right) = 0$.

$$\Rightarrow \quad \frac{1}{2}p^2q^4 + \frac{1}{2}(Q^2 - P^2) - \frac{1}{2q^2} = 0.$$

$$Q^2 - \frac{1}{q^2} + p^2q^4 - P^2 = 0.$$

We could have $Q = \pm\frac{1}{q}$ and $P = \pm pq^2$.

In this case $H = \frac{1}{2}(Q^2 + P^2)$

$$\dot{Q} = \frac{\partial H}{\partial P} = P. \qquad \dot{P} = -\frac{\partial H}{\partial Q} = -Q.$$

For a valid canonical transformation we should have

$Q = -\frac{1}{q}$, $P = pq^2$ or $Q = \frac{1}{q}$, $P = -pq^2$.

Because the possion bracket should be the same.

So, $Q = A\cos(\omega t + \varphi)$. $P = -\omega A\sin(\omega t + \varphi)$

$$q = -\frac{1}{Q}, \quad p = \frac{P}{q^2} = PQ^2.$$

or $q = \frac{1}{Q}$. $p = -\frac{P}{q^2} = -PQ^2$.

In both cases we have

$$\dot{q} = \mp\dot{\left(\frac{1}{Q}\right)} = \pm\frac{1}{Q^2}\dot{Q} = \pm\frac{\omega\sin(\omega t + \varphi)}{A\cos^2(\omega t + \varphi)}$$

$$= \pm(PQ^2)\cdot\left(-\frac{1}{Q}\right)^4 = pq^4.$$

Which conforms to the equation of motion we got in (a).

30. (a) $$[A,B] = \sum_i \left(\frac{\partial A}{\partial q_i}\frac{\partial B}{\partial p_i} + \frac{\partial A}{\partial p_i}\frac{\partial B}{\partial q_i} \right)$$

For constants of motion, $\frac{dA}{dt} = 0$, $\frac{dB}{dt} = 0$.

that is, $[A,H] + \frac{\partial A}{\partial t} = 0$. $[B,H] + \frac{\partial B}{\partial t} = 0$.

$$\frac{d[A,B]}{dt} = [[A,B],H] + \frac{\partial [A,B]}{\partial t}$$

$$= [[A,B],H] + \left[\frac{\partial A}{\partial t}, B\right] + \left[A, \frac{\partial B}{\partial t}\right]$$

$$= [[A,B],H] - [[A,H],B] - [A,[B,H]]$$

$$= [[A,B],H] + [[H,A],B] + [[B,H],A] = 0.$$

So, $[A,B]$ itself is a constant of motion.

(b) $\frac{dH}{dt} = 0$. $\frac{dF}{dt} = 0$. show $\frac{d}{dt}\left(\frac{\partial^n F}{\partial t^n}\right) = 0$.

$[F,H] + \frac{\partial F}{\partial t} = 0$. $\Rightarrow$ $\frac{\partial F}{\partial t} = -[F,H]$.

$$\frac{d}{dt}\left(\frac{\partial^n F}{\partial t^n}\right) = \left[\frac{\partial^n F}{\partial t^n}, H\right] + \frac{\partial}{\partial t}\left(\frac{\partial^n F}{\partial t^n}\right)$$

From (a) we know that $\frac{d[F,H]}{dt} = [[F,H],H] + \frac{\partial [F,H]}{\partial t} = 0$

$$[H,H] + \frac{\partial H}{\partial t} = 0. \Rightarrow \frac{\partial H}{\partial t} = 0.$$

So, $\frac{d}{dt}\left(\frac{\partial^n F}{\partial t^n}\right) = \left[\frac{\partial^n F}{\partial t^n}, H\right] + \frac{\partial}{\partial t}\left(\frac{\partial^n F}{\partial t^n}\right)$

$$= \left[\frac{\partial^n F}{\partial t^n}, H\right] + \frac{\partial^n}{\partial t^n}\left([F,H]\right)$$

$$= \left[\frac{\partial^n F}{\partial t^n}, H\right] + \left[\frac{\partial^n H}{\partial t^n}, F\right] + \left[H, \frac{\partial^n F}{\partial t^n}\right]$$

$$= \left[\underbrace{\frac{\partial^n H}{\partial t^n}}_{=0}, F\right] = 0.$$

So, $\frac{\partial^n F}{\partial t^n}$ is a constant of motion.

(c) $\frac{\partial F}{\partial t} = \cancel{\frac{\partial x}{\partial t}} - \frac{p}{m}.$

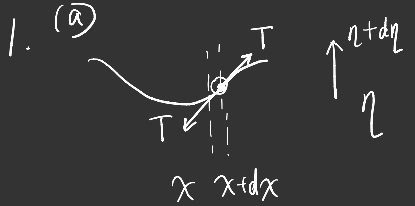
$$[H,F] = \frac{\partial F}{\partial t} - \frac{dF}{dt}. \qquad H = \frac{p^2}{2m}.$$

$$[H,F] = \left[\frac{p^2}{2m}, x - \frac{pt}{m}\right] = 0 + \left(\frac{-p}{m}\right) = \frac{-p}{m}.$$

So, $\frac{\partial F}{\partial t} = [H,F].$

Chapter 13 Formulations of Continuous Systems and Fields

1. (a)

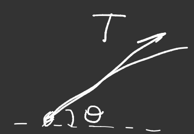


$$dT = \frac{1}{2}(\mu \cdot dx)\left(\frac{\partial \eta}{\partial t}\right)^2$$

$$dV = \frac{1}{2}k(d\eta)^2 .$$

While the horizontal component is always much larger than the vertical one (the angle θ is small) we have

$$k\,d\eta = T \cdot \frac{d\eta}{dx}$$



$$\Rightarrow T = k\,dx$$

So, $dV = \frac{1}{2}k(d\eta)^2 = \frac{1}{2}\left(\frac{T}{dx}\right)(d\eta)^2 = \frac{1}{2}T\left(\frac{\partial \eta}{\partial x}\right)^2 dx .$

$$\Rightarrow$$

$$L = \int dT - dV = \frac{1}{2}\int \underbrace{\left[\mu\left(\frac{\partial \eta}{\partial t}\right)^2 - T\left(\frac{\partial \eta}{\partial x}\right)^2\right]}_{\ell} dx .$$

(b) Lagrangian: $\frac{d}{dt}\left(\frac{\partial \ell}{\partial \dot\eta}\right) - \frac{\partial \ell}{\partial \eta} = 0 .$

$$\Rightarrow \frac{d}{dt}(\mu \dot\eta) + \frac{T}{2}\frac{\partial}{\partial \eta}\left(\frac{\partial \eta}{\partial x}\right)^2 = 0$$

$$\mu\ddot{\eta} + \frac{T}{2}\cdot 2\cdot\frac{\partial}{\partial x}\left(\frac{\partial\eta}{\partial x}\right)\cdot\left(\frac{\partial\eta}{\partial x}\right)\cdot\frac{\partial x}{\partial\eta} = 0$$

$$\Rightarrow \quad \mu\ddot{\eta} - T\cdot\frac{\partial^2\eta}{\partial x^2} = 0$$

4. $$\mathcal{L} = \frac{h^2}{8\pi^2 m}\nabla\psi\cdot\nabla\psi^* + V\psi^*\psi + \frac{h}{4\pi i}(\psi^*\dot{\psi} - \psi\dot{\psi}^*)$$

$$\Rightarrow \quad \frac{\partial\mathcal{L}}{\partial\dot{\psi}^*} = \frac{ih}{4\pi}\psi \quad \Rightarrow \quad \frac{\partial}{\partial t}\left(\frac{\partial\mathcal{L}}{\partial\dot{\psi}^*}\right) = \frac{ih}{4\pi}\dot{\psi}$$

$$\frac{\partial\mathcal{L}}{\partial(\nabla\psi^*)} = \frac{h^2}{8\pi^2 m}\nabla\psi \quad \Rightarrow \quad \nabla\cdot\left[\frac{\partial\mathcal{L}}{\partial(\nabla\psi^*)}\right] = \frac{h^2}{8\pi^2 m}\nabla^2\psi$$

$$\frac{\partial\mathcal{L}}{\partial\psi^*} = V\psi + \frac{h}{4\pi i}\dot{\psi}$$

Because that

$$\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{\psi}^*}\right) + \nabla\cdot\left(\frac{\partial\mathcal{L}}{\partial(\nabla\psi^*)}\right) - \frac{\partial\mathcal{L}}{\partial\psi^*} = 0.$$

We have $$\frac{ih}{4\pi}\dot{\psi} + \frac{h^2}{8\pi^2 m}\nabla^2\psi - V\psi + \frac{ih}{4\pi}\dot{\psi} = 0.$$

$$\Rightarrow \quad \frac{ih}{2\pi}\dot{\psi} = -\frac{h^2}{8\pi^2 m}\nabla^2\psi + V\psi.$$

Mid-term exam Yuxiang Pei yuxiang@uchicago.edu

(1) (a)

$$T = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + \dot{z}^2)$$

$$z = \alpha r^{2n} \rightarrow \dot{z} = 2n\alpha r^{2n-1}\dot{r}$$

$$V = mgz. \qquad \dot{\theta} = \omega$$

$$\Rightarrow L = T - V = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + 4n^2\alpha^2 r^{4n-2}\dot{r}^2) - mg\alpha r^{2n}.$$

$$= \frac{1}{2}m(\dot{r}^2 + \omega^2 r^2 + 4n^2\alpha^2 r^{4n-2}\dot{r}^2) - mg\alpha r^{2n}$$

(b) $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) = \frac{\partial L}{\partial r}. \Rightarrow$

$$mr\omega^2 + 2n^2\alpha^2 \cdot (4n-2) \cdot r^{4n-3} \cdot \dot{r}^2 - 2nmg\alpha r^{2n-1} = m(1 + 4n^2\alpha^2 r^{4n-2})\ddot{r}$$

(*)

(c) L doesn't contain t. Energy is conserved.

$$E = \sum_\alpha \frac{\partial L}{\partial \dot{q}_\alpha} - L$$

$$= T + V = \frac{1}{2}m\left[(1 + 4n^2\alpha^2 r^{4n-2})\dot{r}^2 + \omega^2 r^2\right] + mg\alpha r^{2n}.$$

(d) When z is fixed. r is also fixed.

$\ddot{r} = 0$. $\dot{r} = 0$. Use (*) we get

$$mR\omega^2 = 2nmg\alpha R^{2n-1} \Rightarrow \alpha = \frac{\omega^2}{2ngR^{2n-2}}$$

(2) (a) $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = \frac{\partial L}{\partial x}$, $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) = \frac{\partial L}{\partial y}$.

$$\Rightarrow \frac{d}{dt}\left(m\dot{x} - \frac{1}{2}qBy\right) = \frac{1}{2}qB\dot{y} \qquad \Rightarrow \qquad m\ddot{x} = qB\dot{y}$$

$$\frac{d}{dt}\left(m\dot{y} + \frac{1}{2}qBx\right) = -\frac{1}{2}qB\dot{x} \qquad\qquad m\ddot{y} = -qB\dot{x}$$

For particle in $\vec{B}$ field of z direction,

we have $m\ddot{\vec{r}} = q\,\vec{v}\times\vec{B}$ $\qquad \vec{v} = \dot{x}\hat{x} + \dot{y}\hat{y}$. $\quad \vec{B} = B\hat{z}$.

that is $(m\ddot{\vec{r}})_x = q(\vec{v}\times\vec{B})_x \qquad \Rightarrow \qquad m\ddot{x} = qB\dot{y}$

$(m\ddot{\vec{r}})_y = q(\vec{v}\times\vec{B})_y \qquad\qquad m\ddot{y} = -qB\dot{x}$

(b) Let $x = A_1\cos(\omega t + \varphi_1) + x_0$

$y = A_2 \sin(\omega t + \varphi_2) + y_0$

$$\Rightarrow \quad -m\omega^2 A_1 \cos(\omega t + \varphi_1) = qB\omega A_2 \cos(\omega t + \varphi_2)$$

$$-m\omega^2 A_2 \sin(\omega t + \varphi_2) = qB\omega A_1 \sin(\omega t + \varphi_1)$$

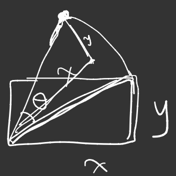
So $\omega = \frac{qB}{m}$. $\quad A_2 = -A_1$. $\quad \varphi_1 = \varphi_2$

$$\Rightarrow \begin{cases} x = A\cos\left(\frac{qB}{m}t + \varphi\right) + x_0 \\ y = -A\sin\left(\frac{qB}{m}t + \varphi\right) + y_0 . \end{cases} \qquad (**)$$

The orbit look like a circle in x-y plane.

It is periodic. $T = \frac{2\pi}{\omega} = \frac{2\pi m}{qB}$.

(c) After the rotation.



$$x' = x\cos\theta - y\sin\theta$$

$$y' = x\sin\theta + y\cos\theta$$

$$\dot{x}' = \dot{x}\cos\theta - \dot{y}\sin\theta$$

$$\dot{y}' = \dot{x}\sin\theta + \dot{y}\cos\theta.$$

$$L' = \frac{1}{2}m\left(\dot{x}'^2 + \dot{y}'^2\right) + \frac{1}{2}qB\left(x'\dot{y}' - y'\dot{x}'\right)$$

$$= \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) + \frac{1}{2}qB\Big(\underline{x\dot{x}\sin\theta\cos\theta} + x\dot{y}\cos^2\theta - \dot{x}y\sin^2\theta \underline{- y\dot{y}\sin\theta\cos\theta} \underline{- x\dot{x}\sin\theta\cos\theta} - \dot{x}y\cos^2\theta + x\dot{y}\sin^2\theta \underline{+ y\dot{y}\sin\theta\cos\theta}\Big)$$

$$= \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}qB\left(x\dot{y} - y\dot{x}\right)$$

L' is invariant under this rotation.

(d) L could be written in (r, θ):

$$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + \frac{1}{2}qBr^2\dot{\theta}$$

$$\left(\begin{array}{ll} x = r\cos\theta & \dot{x} = \dot{r}\cos\theta - r\sin\theta\,\dot{\theta} \\ y = r\sin\theta & \dot{y} = r\cos\theta\,\dot{\theta} + \dot{r}\sin\theta \\ \multicolumn{2}{l}{x\dot{y} - y\dot{x} = r^2\cos^2\theta\,\dot{\theta} + r\dot{r}\sin\theta\cos\theta - r\dot{r}\sin\theta\cos\theta + r^2\sin^2\theta\,\dot{\theta}} \\ \multicolumn{2}{l}{\qquad = r^2\dot{\theta}} \end{array}\right)$$

We could see that L is independent of θ.

So $$\ell = \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta} + \frac{1}{2}qBr^2$$

is conserved.

Verify: $\ell = m(x^2+y^2)\dot{\theta} + \frac{1}{2}qB(x^2+y^2)$.

Plug in (**), we could get

$$\frac{d\ell}{dt} = 2m(\dot{x}x + \dot{y}y)\dot{\theta} + m(x^2+y^2)\ddot{\theta} + qB(x\dot{x}+y\dot{y}) = 0.$$

Interpretation: total angular momentum is conserved.

particle + magnetic field

$mr^2\dot{\theta}$ $\qquad \frac{1}{2}qBr^2$

(3) (a) $f(r) = -\frac{dV(r)}{dr} = \frac{2A}{r^3}$.

When $A>0$. $\vec{f}(r)$ is the same direction of $\vec{r}$. repulsive.

$A<0$. $\vec{f}(r)$ is opposite direction of $\vec{r}$. attractive.

(b) $m\ddot{r} - \frac{l^2}{mr^3} = f(r)$.

$\Rightarrow m\ddot{r} - \frac{l^2}{mr^3} = \frac{2A}{r^3}$. While $l\,dt = mr^2 d\theta$.

$$\frac{dr}{dt} = \sqrt{\frac{2}{m}\left(E - V - \frac{l^2}{2mr^2}\right)}$$

$$\Rightarrow d\theta = \frac{l\,dr}{mr^2\sqrt{\frac{2}{m}\left(E - V - \frac{l^2}{2mr^2}\right)}}$$

$$= \frac{l\,dr}{mr^2\sqrt{\frac{2}{m}\left(E + \frac{2mA - l^2}{2mr^2}\right)}}$$

So, $l^2 - 2mA > 0$

$\Rightarrow A < \frac{l^2}{2m}$.

$$d\theta = \frac{dr}{r^2\sqrt{\frac{2m}{l^2}E + \frac{2mA - l^2}{l^2}\frac{1}{r^2}}}$$

$$= \frac{dr}{r^2\sqrt{C + B\cdot\frac{1}{r^2}}}$$

$$\theta - \theta_0 = \frac{\tan^{-1}\sqrt{\frac{r^2\cdot C}{-B} - 1}}{\sqrt{\frac{-B}{C}}}$$

That is, $\sqrt{r^2 \cdot \dfrac{2mE}{\ell^2 - 2mA} - 1} = \tan\left[\sqrt{\dfrac{\ell^2 - 2mA}{2mE}}(\theta - \theta_0)\right]$

So, $r = \sqrt{\dfrac{\ell^2 - 2mA}{2mE}\left[\tan^2\sqrt{\dfrac{\ell^2 - 2mA}{2mE}}(\theta - \theta_0) + 1\right]}$

Let $\underbrace{\dfrac{\ell^2 - 2mA}{2mE} = \xi .}$

$$r = \sqrt{\xi}\sqrt{\tan^2\left[\sqrt{\xi}(\theta - \theta_0)\right] + 1}$$

$$= \frac{\sqrt{\xi}}{\cos\sqrt{\xi}(\theta - \theta_0)}$$

Final Yuxiang Pei yuxiang@uchicago.edu

1. (a) Coordinate of M: $(x + L\sin\phi, -L\cos\phi)$

$$T = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}M(\dot{x}^2 + L^2\cos^2\phi\,\dot{\phi}^2 + 2\dot{x}L\cos\phi\,\dot{\phi} + L^2\sin^2\phi\,\dot{\phi}^2)$$

$$V = \frac{1}{2}kx^2 - MgL\cos\phi$$

$$L = T - V = \frac{1}{2}(m+M)\dot{x}^2 + \frac{1}{2}ML^2\dot{\phi}^2 + ML\cos\phi\,\dot{x}\dot{\phi} + MgL\cos\phi - \frac{1}{2}kx^2$$

(b) $$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\alpha} - \frac{\partial L}{\partial q_\alpha} = 0.$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = 0 \Rightarrow \frac{d}{dt}\left[(m+M)\dot{x} + ML\cos\phi\,\dot{\phi}\right] + kx = 0$$

$$\Rightarrow (m+M)\ddot{x} - ML\sin\phi\,\dot{\phi}^2 + ML\cos\phi\,\ddot{\phi} + kx = 0$$

ϕ is small $\Rightarrow (m+M)\ddot{x} - ML\phi\cdot\dot{\phi}^2 + ML(1-\frac{\phi^2}{2})\ddot{\phi} + kx = 0.$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\phi}} - \frac{\partial L}{\partial \phi} = 0 \Rightarrow \frac{d}{dt}\left[ML\dot{\phi} + ML\cos\phi\,\dot{x}\right] + MgL\sin\phi = 0.$$

$$\Rightarrow ML\ddot{\phi} - ML\sin\phi\,\dot{\phi}\dot{x} + ML\cos\phi\,\ddot{x} + MgL\sin\phi = 0$$

ϕ is small $\Rightarrow ML\ddot{\phi} - ML\phi\dot{\phi}\dot{x} + ML(1-\frac{\phi^2}{2})\ddot{x} + MgL\phi = 0$

Let $\cos\phi \approx 1$.

We have
$$(m+M)\ddot{x} - ML\phi\dot{\phi}^2 + ML\ddot{\phi} + kx = 0$$

$$ML\ddot{\phi} - ML\phi\dot{\phi}\dot{x} + ML\ddot{x} + MgL\phi = 0$$

(c) $(K-\omega^2 M)u = 0.$ $\qquad V = MgL\left(1-\frac{\phi^2}{2}\right) - \frac{1}{2}kx^2$

$$T = \frac{1}{2}(m+M)\dot{x}^2 + \frac{1}{2}ML^2\dot{\phi}^2 + ML\dot{x}\dot{\phi}$$

$$M = \begin{bmatrix} m+M & ML \\ ML & ML^2 \end{bmatrix} \qquad K = \begin{bmatrix} k & \\ & MgL \end{bmatrix}$$

$$\begin{pmatrix} k - \omega^2(m+M) & \omega^2 ML \\ \omega^2 ML & MgL - \omega^2 ML^2 \end{pmatrix} u = 0$$

$$(ML^2\omega^2 - MgL)((m+M)\omega^2 - k) - \omega^4 M^2 L^2 = 0.$$

$$mML^2\omega^4 - \left[(m+M)MgL + kML^2\right]\omega^2 + kMgL = 0$$

$$\omega^4 - \left[\frac{(m+M)g}{mL} + \frac{k}{m}\right]\omega^2 + \frac{kg}{mL} = 0.$$

$$\Rightarrow \quad \omega_1 = \left[\frac{(M+m)g + kL + \sqrt{[(M+m)g+kL]^2 - 4kmgL}}{2mL}\right]^{\frac{1}{2}}$$

$$\omega_2 = \left[\frac{(M+m)g + kL - \sqrt{[(M+m)g+kL]^2 - 4kmgL}}{2mL}\right]^{\frac{1}{2}}$$

Just noticed the numbers.

$$\omega^4 - 4\omega^2 + 2 = 0 \Rightarrow \begin{array}{l} \omega_1 = \sqrt{2+\sqrt{2}} \\ \omega_2 = \sqrt{2-\sqrt{2}} \end{array}$$

(d) Should be $x = u_1 \cos(\omega t + \alpha)$

$$\theta = u_2 \cos(\omega t + \alpha).$$

use the numbers above.

$$\begin{pmatrix} 2-2\omega^2 & \omega^2 \\ \omega^2 & 1-\omega^2 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = 0.$$

① for $\omega_1 = \sqrt{2+\sqrt{2}}$, $\begin{pmatrix} -2-2\sqrt{2} & 2+\sqrt{2} \\ 2+\sqrt{2} & -1-\sqrt{2} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = 0$

We have $u_1 = a$; $u_2 = \sqrt{2}a$.

So, $x_1 = a \cdot \cos(\sqrt{2+\sqrt{2}}\,t + \alpha_1)$

$$\theta_1 = \sqrt{2}a \cos((2+\sqrt{2})t + \alpha_1)$$

The motion is that m and M move together to the same direction. in the same phase, move in a relatively high frequency.

② for $\omega_2 = \sqrt{2-\sqrt{2}}$.
$$\begin{pmatrix} 2+2\sqrt{2} & 2-\sqrt{2} \\ 2-\sqrt{2} & \sqrt{2}-1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = 0$$

$$u_1 = b, \quad u_2 = -\sqrt{2}b.$$

$$x_2 = b\cos\left(\sqrt{2-\sqrt{2}}\,t + \alpha_2\right)$$

$$\theta_2 = -\sqrt{2}b\cos\left(\sqrt{2-\sqrt{2}}\,t + \alpha_2\right)$$

they move towards and opposite each other (in 180° phase of each other), move in relatively low frequency.

(e) From above calculations,

$$x = a_1\cos(\omega_1 t + \alpha) + b_1\cos(\omega_2 t + \beta)$$

$$\phi = a_2\cos(\omega_1 t + \alpha) + b_2\cos(\omega_2 t + \beta)$$

$$x_0 = a_1\cos\alpha + b_1\cos\beta, \quad \phi_0 = a_2\cos\alpha + b_2\cos\beta.$$

Also, $\dot{x}(t=0)=0$, $\dot{\phi}(t=0)=0$.

So,
$$-\omega_1 a_1 \sin\alpha - \omega_2 b_1 \sin\beta = 0$$
$$-\omega_1 a_2 \sin\alpha - \omega_2 b_2 \sin\beta = 0.$$

$\Rightarrow \alpha = \beta = 0.$

$$x_0 = a_1 + b_1.$$
$$\phi_0 = a_2 + b_2.$$

Use the numbers we have

$$\frac{a_1}{a_2} = \frac{L}{\sqrt{2}}. \qquad \frac{b_1}{b_2} = -\frac{L}{\sqrt{2}}.$$

So,
$$a_2 = \left(\frac{x_0}{\sqrt{2}L} - \frac{\phi_0}{2}\right)\cdot L$$
$$b_2 = \frac{x_0}{\sqrt{2}L} + \frac{\phi_0}{2}$$

$$\Rightarrow \boxed{\begin{aligned} x(t) &= \left(\frac{x_0}{\sqrt{2}} - \frac{\phi_0 L}{2}\right)\cos\left(\sqrt{2+\sqrt{2}}\,t\right) + \left(\frac{x_0}{\sqrt{2}} + \frac{\phi_0 L}{2}\right)\cos\left(\sqrt{2-\sqrt{2}}\,t\right) \\ \phi(t) &= \left(\frac{\phi_0}{2} - \frac{x_0}{\sqrt{2}L}\right)\cos\left(\sqrt{2+\sqrt{2}}\,t\right) + \left(\frac{\phi_0}{2} + \frac{x_0}{\sqrt{2}L}\right)\cos\left(\sqrt{2-\sqrt{2}}\,t\right) \end{aligned}}$$

(2) (a) $[H, S_1] = \frac{\partial H}{\partial x}\frac{\partial S_1}{\partial p_x} - \frac{\partial H}{\partial p_x}\frac{\partial S_1}{\partial x}$

$$+ \frac{\partial H}{\partial y}\frac{\partial S_1}{\partial p_y} - \frac{\partial H}{\partial p_y}\frac{\partial S_1}{\partial y}$$

$$= kx \cdot \frac{p_x}{m} - \frac{p_x}{m} \cdot kx + ky \cdot \frac{p_y}{m} - \frac{p_y}{m} \cdot ky$$

$$= 0.$$

$$[H, S_2] = \frac{\partial H}{\partial x}\frac{\partial S_2}{\partial p_x} - \frac{\partial H}{\partial p_x}\frac{\partial S_2}{\partial x} + \frac{\partial H}{\partial y}\frac{\partial S_2}{\partial p_y} - \frac{\partial H}{\partial p_y}\frac{\partial S_2}{\partial y}$$

$$= kx \cdot \frac{p_y}{m} - \frac{p_x}{m} \cdot ky + ky \cdot \frac{p_x}{m} - \frac{p_y}{m} \cdot kx$$

$$= 0.$$

$$[H, S_3] = kx \cdot (-\omega y) - \frac{p_x}{m} \cdot \omega p_y + ky \cdot \omega x - \frac{p_y}{m} \cdot (-\omega p_x)$$

$$= 0.$$

(b) $dq_\alpha = \frac{\partial G}{\partial p_\alpha} d\lambda, \quad dp_\alpha = -\frac{\partial G}{\partial q_\alpha} d\lambda.$

$G = S_1$. we have

$$dx = \frac{\partial S_1}{\partial p_x} d\lambda = \frac{p_x}{m}\varepsilon. \qquad dy = \frac{p_y}{m}\varepsilon$$

$$dp_x = -\frac{\partial S_1}{\partial x} d\lambda = -kx\varepsilon. \qquad dp_y = -ky\varepsilon.$$

So, $df = f(q+dq, p+dp, t+dt) - f(q,p,t)$

$$= d\lambda \sum_\alpha \left(\frac{\partial f}{\partial q_\alpha}\frac{\partial S_1}{\partial p_\alpha} - \frac{\partial f}{\partial p_\alpha}\frac{\partial S_1}{\partial q_\alpha} \right) + \frac{\partial f}{\partial t} dt$$

$$= \varepsilon \left(\frac{\partial f}{\partial x}\frac{\partial S_1}{\partial p_x} - \frac{\partial f}{\partial p_x}\frac{\partial S_1}{\partial x} \right)$$

$$+ \varepsilon \left(\frac{\partial f}{\partial y}\frac{\partial S_1}{\partial p_y} - \frac{\partial f}{\partial p_y}\frac{\partial S_1}{\partial y} \right) + \frac{\partial f}{\partial t} dt$$

$$= \varepsilon \cdot \left(\frac{p_x}{m}\frac{\partial f}{\partial x} + kx\frac{\partial f}{\partial p_x} \right)$$

$$+ \varepsilon \left(\frac{p_y}{m}\frac{\partial f}{\partial y} + ky\frac{\partial f}{\partial p_y} \right) + \frac{\partial f}{\partial t} dt.$$

Let $F_2(q,p,t) = \sum_\alpha q_\alpha P_\alpha + \varepsilon S_1.$

$$= xP_x + yP_y + \varepsilon\left(\frac{P_x^2}{2m} - \frac{P_y^2}{2m} \right.$$

$$\left. + \frac{k}{2}(x^2 - y^2) \right).$$

We have

$$\boxed{(x, y, p_x, p_y) \Longrightarrow \left(x + \frac{p_x}{m}\varepsilon,\ y + \frac{p_y}{m}\varepsilon,\ p_x - kx\varepsilon,\ p_y - ky\varepsilon\right)}$$

(c) $$[S_1, S_2] = \frac{\partial S_1}{\partial x}\frac{\partial S_2}{\partial p_x} - \frac{\partial S_1}{\partial p_x}\frac{\partial S_2}{\partial x} + \frac{\partial S_1}{\partial y}\frac{\partial S_2}{\partial p_y} - \frac{\partial S_1}{\partial p_y}\frac{\partial S_2}{\partial y}$$

$$= kx \cdot \frac{p_y}{m} - \frac{p_x}{m} \cdot ky + (-ky) \cdot \frac{p_x}{m} - \left(\frac{-p_y}{m}\right) \cdot kx$$

$$= 2kx \cdot \frac{p_y}{m} - 2ky \cdot \frac{p_x}{m}.$$

$$= \frac{2k}{m}(xp_y - yp_x) = 2\omega S_3$$

$$[S_2, S_3] = ky \cdot (-\omega y) - \frac{p_y}{m} \cdot \omega p_y + kx \cdot \omega x - \frac{p_x}{m} \cdot (-\omega p_x)$$

$$= \frac{\omega}{m}(p_x^2 - p_y^2) + k\omega(x^2 - y^2) = 2\omega S_1.$$

$$[S_1, S_3] = kx \cdot (-\omega y) - \frac{p_x}{m} \cdot \omega p_y + (-ky) \cdot \omega x - \left(\frac{-p_y}{m}\right) \cdot (-\omega p_x)$$

$$= -\frac{2\omega}{m} p_x p_y - 2\omega kxy = -2\omega S_2.$$

they should all be constants of motion.

but none of them are new. (additional)

(d) $H^2 = S_1^2 + S_2^2 + S_3^2$.

$$S_1^2 + S_2^2 + S_3^2 = \left(\frac{p_x^2 - p_y^2}{2m}\right)^2 + \left[\frac{k(x^2-y^2)}{2}\right]^2$$

$$+ 2\,\frac{p_x^2 - p_y^2}{2m}\cdot\frac{k(x^2-y^2)}{2}$$

$$+ \left(\frac{p_x p_y}{m} + kxy\right)^2 + \frac{k}{m}(xp_y - yp_x)^2$$

$$= \left(\frac{p_x^2 + p_y^2}{2m}\right)^2 - \frac{4p_x^2 p_y^2}{(2m)^2} + \left[\frac{k(x^2+y^2)}{2}\right]^2 - \frac{4k x^2 y^2}{4}$$

$$+ \frac{k}{2m}(p_x^2 - p_y^2)(x^2-y^2) + \frac{k}{m}(xp_y - yp_x)^2$$

$$+ \left(\frac{p_x p_y}{m} + kxy\right)^2$$

$$= \left(\frac{p_x^2+p_y^2}{2m}\right)^2 + \left(\frac{k(x^2+y^2)}{2}\right)^2$$

$$+ \frac{k}{2m}\cdot\left(x^2p_x^2 + y^2p_y^2 - x^2p_y^2 - y^2p_x^2 + 2x^2p_y^2 + 2y^2p_x^2 - 4xy\,p_xp_y\right)$$

$$+ \frac{p_x^2p_y^2}{m^2} + k^2x^2y^2 + 2p_xp_y\,xy\cdot\frac{k}{m} - \frac{p_x^2p_y^2}{m^2} - kx^2y^2$$

$$= \left(\frac{p_x^2+p_y^2}{2m}\right)^2 + \left(\frac{k(x^2+y^2)}{2}\right)^2 + \frac{k}{2m}\left(x^2p_x^2 + x^2p_y^2 + y^2p_x^2 + y^2p_y^2\right)$$

$$= \left[\frac{p_x^2+p_y^2}{2m} + \frac{k}{2}(x^2+y^2)\right]^2 = H^2$$

(e) $\dot{x} = \frac{\partial H}{\partial p_x} = \frac{p_x}{m}$ $\qquad$ $\dot{p}_x = -\frac{\partial H}{\partial x} = -kx$

$\dot{y} = \frac{\partial H}{\partial p_y} = \frac{p_y}{m}$ $\qquad$ $\dot{p}_y = -\frac{\partial H}{\partial y} = -ky$

$$H = \dot{x} p_x + \dot{y} p_y - L.$$

$$\Rightarrow \quad L = \dot{x} p_x + \dot{y} p_y - H = \frac{p_x^2 + p_y^2}{2m} - \frac{k}{2}(x^2 + y^2)$$

$$= \underline{\frac{1}{2} m(\dot{x}^2 + \dot{y}^2) - \frac{k}{2}(x^2 + y^2)}$$

EL eqn: $\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = 0$, $\frac{d}{dt}\frac{\partial L}{\partial \dot{y}} - \frac{\partial L}{\partial y} = 0$

$$\Rightarrow \quad m\ddot{x} + kx = 0. \qquad m\ddot{y} + ky = 0.$$

Solution: $\underline{x = x_0 \cos(\omega t + \varphi_1)}$ $\qquad \omega = \sqrt{\frac{k}{m}}$.

$\underline{y = y_0 \cos(\omega t + \varphi_2)}$.

$$S_3 = \omega(x p_y - y p_x)$$

$$= \omega(x m\dot{y} - y m\dot{x})$$

$$= \omega m(x\dot{y} - y\dot{x})$$

While $x\dot{y} = -\omega x_0 y_0 \cos(\omega t + \varphi_1)\sin(\omega t + \varphi_2)$

$$y\dot{x} = -\omega x_0 y_0 \sin(\omega t + \varphi_1)\cos(\omega t + \varphi_2)$$

$$x\dot{y} - y\dot{x} = \omega x_0 y_0 \left[\sin(\omega t + \varphi_1)\cos(\omega t + \varphi_2) - \cos(\omega t + \varphi_1)\sin(\omega t + \varphi_2)\right]$$

$$= \omega x_0 y_0 \sin(\varphi_1 - \varphi_2)$$

So, $S_3 = \omega^2 m x_0 y_0 \sin(\varphi_1 - \varphi_2)$

$$= k x_0 y_0 \sin(\varphi_1 - \varphi_2)$$