HW 1 Chapter 2 Electrostatics yuxiang@uchicago.edu

1. $\nabla^2 \phi(\vec{x}) = 0$.

Multiply by $\phi(\vec{x})$ then do integral

$$\int_V \phi(\vec{x}) \underbrace{\nabla^2 \phi(\vec{x})}_{=0} d^3x = \int_V \nabla \cdot \left[\phi(\vec{x}) \vec{\nabla} \phi(\vec{x})\right] d^3x - \int_V |\nabla \phi(\vec{x})|^2 d^3x$$

Apply Gauss' Theorem to V,

$$\int_S \phi(\vec{x}) \vec{\nabla} \phi(\vec{x}) \cdot \hat{n} \, dS - \int_V |\nabla \phi(\vec{x})|^2 d^3x = 0.$$

$$\int_S \underbrace{-f(\vec{x}) |\nabla \phi(\vec{x}) \cdot \hat{n}|^2 dS}_{\leq 0} \underbrace{- \int_V |\nabla \phi(\vec{x})|^2 d^3x}_{\leq 0} = 0$$

These two terms are both ≤ 0.

So, $\int_V |\nabla \phi(\vec{x})|^2 d^3x = 0$.

Which implies $\nabla \phi(\vec{x}) = 0$ within V.

and $\phi(\vec{x}) = 0$ within V for $\phi \to 0$ at ∞.

2. (a) $\vec{E} = -\vec{\nabla}\phi$

$$3(\vec{p}\cdot\hat{r})\hat{r} - \vec{p} \over r^3$$

$$= -\nabla \frac{\vec{p}\cdot(\vec{x}-\vec{x}')}{|\vec{x}-\vec{x}'|^3}$$

$$= -(\vec{p}\cdot\nabla)\frac{\vec{x}-\vec{x}'}{|\vec{x}-\vec{x}'|^3} + \underbrace{\left(\frac{\vec{x}-\vec{x}'}{|\vec{x}-\vec{x}'|^3}\cdot\vec{\nabla}\right)\vec{p}}_{=0}$$

$$+ \vec{p}\times\underbrace{\left(\vec{\nabla}\times\frac{\vec{x}-\vec{x}'}{|\vec{x}-\vec{x}'|^3}\right)}_{=0} + \frac{\vec{x}-\vec{x}'}{|\vec{x}-\vec{x}'|^3}\times\underbrace{(\vec{\nabla}\times\vec{p})}_{=0}$$

$$= -(\vec{p}\cdot\nabla)\frac{\vec{x}-\vec{x}'}{|\vec{x}-\vec{x}'|^3}$$

For $(\vec{p}\cdot\nabla)\frac{\vec{r}}{r^3} = \left(p_x\frac{\partial}{\partial x} + p_y\frac{\partial}{\partial y} + p_z\frac{\partial}{\partial z}\right)\frac{\vec{r}}{r^3}$

$$= \vec{r}\cdot p_x\frac{\partial}{\partial x}\left(\frac{1}{r^3}\right) + \vec{r}\cdot p_y\frac{\partial}{\partial y}\left(\frac{1}{r^3}\right) + \vec{r}\cdot p_z\frac{\partial}{\partial z}\left(\frac{1}{r^3}\right)$$

$$+ \frac{p_x}{r^3}\frac{\partial}{\partial x}(\vec{r}) + \frac{p_y}{r^3}\frac{\partial}{\partial y}(\vec{r}) + \frac{p_z}{r^3}\frac{\partial}{\partial z}(\vec{r})$$

$$= \vec{r}\cdot\frac{-3(xp_x + yp_y + zp_z)}{r^5} + \frac{p_x\hat{x} + p_y\hat{y} + p_z\hat{z}}{r^3}$$

$$= \frac{1}{r^5} \cdot \left[r^2 \cdot \vec{p} - 3(\vec{r} \cdot \vec{p})\vec{r} \right].$$

So, $$\vec{E} = \frac{3(\vec{r} \cdot \vec{p})\vec{r} - r^2 \vec{p}}{r^5}$$

$$= \frac{3\left[(\vec{x} - \vec{x}') \cdot \vec{p}\right](\vec{x} - \vec{x}') - (\vec{x} - \vec{x}')^2 \vec{p}}{|\vec{x} - \vec{x}'|^5}$$

$$\vec{E}^{ext} = \frac{3\left[(\vec{x}_2 - \vec{x}_1) \cdot \vec{p}_1\right](\vec{x}_2 - \vec{x}_1) - (\vec{x}_2 - \vec{x}_1)^2 \vec{p}_1}{|\vec{x}_2 - \vec{x}_1|^5}$$

$$\vec{F} = \int \rho(\vec{x})\, \vec{E}^{ext}(\vec{x})\, d^3x$$

Suppose $\vec{p}_2$ is small in extent,

$$\vec{F} = \underset{\underset{0}{\shortparallel}}{q} \vec{E}^{ext} + (\vec{p} \cdot \vec{\nabla}) \vec{E}^{ext} + \cdots$$

$$= (\vec{p}_2 \cdot \vec{\nabla}) \vec{E}^{ext} = (\vec{p}_2 \cdot \vec{\nabla}) \left\{ \frac{3\left[(\vec{x}_2 - \vec{x}_1) \cdot \vec{p}_1\right](\vec{x}_2 - \vec{x}_1)}{|\vec{x}_2 - \vec{x}_1|^5} \right\}$$

$$- (\vec{p}_2 \cdot \vec{\nabla})\, \vec{p} / |\vec{x}_2 - \vec{x}_1|^3$$

For $(\vec{p}_2 \cdot \vec{\nabla}) \left[\dfrac{3(\vec{r} \cdot \vec{p}_1)\vec{r} - r^2 \vec{p}_1}{r^5} \right]$

$$= (\vec{p}_2 \cdot \vec{\nabla}) \frac{3(\vec{r} \cdot \vec{p}_1)\vec{r}}{r^5} - (\vec{p}_2 \cdot \vec{\nabla}) \frac{\vec{p}_1}{r^3}$$

$$= \frac{3(\vec{r} \cdot \vec{p}_1)}{r^6} \left[\vec{p}_2 - \hat{r} \cdot (\vec{p}_2 \cdot \hat{r}) \right] + \hat{r} (\vec{p}_2 \cdot \hat{r}) \frac{\partial}{\partial r} \left[\frac{3(\vec{r} \cdot \vec{p}_1)}{r^5} \right] + \frac{3(\hat{r} \cdot \vec{p}_2)\vec{p}_1}{r^5}$$

$$= \frac{3(\hat{r} \cdot \vec{p}_1)\vec{p}_2}{r^5} - \frac{15(\vec{p}_1 \cdot \hat{r})(\vec{p}_2 \cdot \hat{r})\hat{r}}{r^5} + \frac{3(\hat{r} \cdot \vec{p}_2)\vec{p}_1}{r^5}$$

$$= \frac{3\left[(\vec{p}_1 \cdot \hat{r})\vec{p}_2 + (\vec{p}_2 \cdot \hat{r})\vec{p}_1\right] - 15(\vec{p}_1 \cdot \hat{r})(\vec{p}_2 \cdot \hat{r})\hat{r}}{r^5}$$

So,

$$\boxed{\vec{F} = \frac{3\left[\vec{p}_1 \cdot (\vec{x}_2 - \vec{x}_1)\right]\vec{p}_2 + \left[\vec{p}_2 \cdot (\vec{x}_2 - \vec{x}_1)\right]\vec{p}_1}{|\vec{x}_2 - \vec{x}_1|^6} - \frac{15\left[\vec{p}_1 \cdot (\vec{x}_2 - \vec{x}_1)\right] \cdot \left[\vec{p}_2 \cdot (\vec{x}_2 - \vec{x}_1)\right](\vec{x}_2 - \vec{x}_1)}{|\vec{x}_2 - \vec{x}_1|^8}}$$

(b) Interaction energy

$$\mathcal{E}^{int} = \int \rho \phi^{ext} d^3x$$

$$= \underbrace{q\phi^{ext}}_{=0} + \vec{p}\cdot\nabla\phi^{ext} + \cdots$$

$$= \vec{P}_2 \cdot \vec{E}^{ext}$$

$$= \frac{3\left[(\vec{x}_2-\vec{x}_1)\cdot\vec{P}_1\right]\left[(\vec{x}_2-\vec{x}_1)\cdot\vec{P}_2\right] - (\vec{x}_2-\vec{x}_1)^2\,\vec{P}_1\cdot\vec{P}_2}{|\vec{x}_2-\vec{x}_1|^5}$$

3. $$\mathcal{E}^{self} = \int \frac{|\vec{E}^{self}|^2}{8\pi} d^3x \qquad \rho = \frac{3e}{4\pi R^3}$$

$$\vec{E}^{self} = -\nabla\phi^{self}(\vec{x}) = \int \frac{\rho(\vec{x}')}{|\vec{x}-\vec{x}'|^3}(\vec{x}-\vec{x}')\, d^3x'$$

$$= \int_0^r \frac{\rho}{x'^3}\cdot 4\pi x'^2\, \vec{x}'\, dx' = \frac{4}{3}\rho\pi r \qquad (r \leq R)$$

$$\mathcal{E}^{self}_{(r\leq R)} = \int_0^R \frac{1}{8\pi}\cdot\left(\frac{4}{3}\rho\pi r\right)^2 \cdot 4\pi r^2 dr$$

$$= \int_0^R \rho^2 \cdot \frac{8}{9}\pi^2 r^4 dr = \frac{8\rho^2}{9}\cdot\frac{\pi^2 R^5}{5}$$

$$= \frac{8}{9}\times\frac{9e^2}{16\pi^2 R^6}\times\frac{\pi^2 R^5}{5} = \frac{e^2}{10R}$$

$$\mathcal{E}^{self} = \mathcal{E}^{self}_{(r\leq R)} + \mathcal{E}^{self}_{(r>R)}$$

$$= \frac{e^2}{10R} + \int_R^\infty \frac{1}{8\pi}\cdot\left(\frac{e}{x^2}\right)^2 \cdot 4\pi x^2 dx$$

$$= \frac{e^2}{10R} + \frac{e^2}{2R} = \boxed{\frac{3e^2}{5R}}$$

Plugging in the numbers,

$$\mathcal{E}^{self} = \frac{(4.8\times10^{-10}\times 3.3356\times10^{-10})^2\times 3}{5\times10^{-15}}\times 9\times10^9$$

$$= 1.38\times10^{-13}\ \text{J}$$

mass of proton : 1.67×10^{-27} kg.

$$E_{self} = m_p c^2 = 1.67 \times 10^{-27} \times (3 \times 10^8)^2$$

$$= 1.51 \times 10^{-10} \text{ J}$$

fraction : $E_{self} / \mathcal{E}^{self} = 9.2 \times 10^{-4}$.

(b)

$$\mathcal{E}^{int} = \int \rho_2 \Phi_1 \, d^3x$$

$$\Phi_1 = \Phi_p = \frac{e}{r} \qquad \rho_2 = \rho_e = -e|\psi|^2.$$

$$\mathcal{E}^{int} = \int_0^\infty \frac{e}{r} \cdot (-e) \frac{1}{\pi a^3} \cdot e^{-\frac{2r}{a}} \cdot 4\pi r^2 dr$$

$$= \frac{-4e^2}{a^3} \int_0^\infty e^{-2r/a} \cdot r \, dr$$

$$= -\frac{4e^2}{a} \int_0^\infty e^{-2x} \cdot x \, dx$$

$$= -\frac{e^2}{a}$$

Ground state energy of hydrogen atom:

$$E_G = \frac{-e^2}{2a} = -0.5\frac{e^2}{a}.$$

We could notice that

$$E_G = \frac{1}{2}\mathcal{E}^{int}.$$

It is understandable because taking into account the kinetic energy of electron we could get the total energy.

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4. This kind of charge density satisfies that $\rho \to 0$ sufficiently rapidly at infinity, the unique solution to Poisson's eqn with $\phi \to 0$ at infinity is given by

$$\phi(\vec{x}) = \int \frac{\rho(\vec{x}')}{|\vec{x}-\vec{x}'|} d^3x'$$

We are interested in solution at $|\vec{x}| > R$, $|\vec{x}| > |\vec{x}'|$.

By spherical harmonic function expansion we could get

$$\phi(\vec{x}) = \int G(\vec{x}, \vec{x}') \rho(\vec{x}') d^3x'$$

$$= \sum_{lm} \frac{4\pi}{2l+1} \frac{q_{lm}}{r^{l+1}} Y_{lm}(\theta, \varphi)$$

$$q_{lm} = \int \rho(\vec{x}') r'^{l} Y^*_{lm}(\theta', \varphi') d^3x'.$$

While $\rho(\vec{x}') = (R-r)\left[1-\cos(\theta)\right]^2$

$$= (R-r)\cdot\left(1+\cos^2\theta - 2\cos\theta\right)$$

$$= (R-r)\cdot\left(\sqrt{\frac{\pi}{5}}\cdot\frac{4}{3}Y_{20} - \sqrt{\frac{\pi}{3}}\cdot 4Y_{10} + \frac{8}{3}\sqrt{\pi}\, Y_{00}\right)$$

$$q = \int \rho(\vec{x}')\, r'^{l}\, Y^*_{lm}(\theta',\varphi')\, r'^2 \sin\theta \, dr' d\theta' d\varphi'$$

$$= (R-r_0)\left[\int r'^{\,l+2}\, dr' \int \left(\frac{4}{3}\sqrt{\frac{\pi}{5}}\, Y_{20} Y^*_{20} - 4\sqrt{\frac{\pi}{3}}\, Y_{10} Y^*_{10} + \frac{8}{3}\sqrt{\pi}\, Y_{00} Y^*_{00}\right) \sin\theta' d\theta' d\varphi'\right]$$

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$$= (R-r_0)\cdot\left(\frac{4}{3}\sqrt{\frac{\pi}{5}} \cdot \int_0^R r'^4 dr' - 4\sqrt{\frac{\pi}{3}}\int_0^R r'^3 dr' + \frac{8}{3}\sqrt{\pi}\int_0^R r'^2 dr'\right)$$

So,

$$q_{20} = (R-r_0)\cdot \frac{4}{3}\sqrt{\frac{\pi}{5}} \cdot \frac{R^5}{5}.$$

$$q_{10} = -(R-r_0)\cdot \sqrt{\frac{\pi}{3}}\, R^4.$$

$$q_{00} = (R-r_0)\cdot \frac{8}{9}\sqrt{\pi}\, R^3.$$

Then

$$\phi(\vec{x}) = \frac{4\pi}{5}\frac{q_{20}}{r^3} Y_{20}(\theta,\varphi) + \frac{4\pi}{3}\frac{q_{10}}{r^2} Y_{10}(\theta,\varphi) + 4\pi \cdot \frac{q_{00}}{r} Y_{00}(\theta,\varphi)$$

$$= \frac{4\pi}{5}\cdot\frac{\cancel{4}}{15}\cancel{\sqrt{\frac{\pi}{5}}}\,\frac{R^5}{r^3}\cdot\frac{1}{\cancel{4}}\cancel{\sqrt{\frac{5}{\pi}}}\left(3\cos^2\theta - 1\right)(R-r_0)$$

$$- \frac{4\pi}{3}\cancel{\sqrt{\frac{\pi}{3}}}\,\frac{R^4}{r^2}\cdot\frac{1}{2}\cancel{\sqrt{\frac{3}{\pi}}}\cos\theta\cdot(R-r_0)$$

$$+ 4\pi\cdot\frac{8}{9}\cancel{\sqrt{\pi}}\,\frac{R^3}{r}\cdot\frac{1}{2\cancel{\sqrt{\pi}}}\cdot(R-r_0)$$

$$= \left[\frac{4\pi}{75}\left(3\cos^2\theta - 1\right)\cdot\frac{R^5}{r^3} - \frac{2\pi}{3}\cdot\frac{R^4}{r^2}\cos\theta + \frac{16\pi}{9}\frac{R^3}{r}\right](R-r_0)$$

$$= \left(\frac{4\pi}{25}\cos^2\theta \cdot \frac{R^5}{r^3} - \frac{4\pi}{75}\cdot\frac{R^5}{r^3} - \frac{2\pi}{3}\cos\theta \cdot \frac{R^4}{r^2} + \frac{16\pi}{9}\frac{R^3}{r}\right)\cdot(R-r_0)$$

$$(r \geq R)$$

(It is too complex, did I make faults in calculation?)

It is complex !!

6. (a) $\phi(\vec{x}) = \int G(\vec{x},\vec{x}')\rho(\vec{x}')d^3x'$

$$= \sum_{lm} \frac{4\pi}{2l+1}\frac{q_{lm}}{r^{l+1}} Y_{lm}(\theta,\varphi)$$

$$q_{lm} = \int \rho(\vec{x}') r'^l Y^*_{lm}(\theta',\varphi')d^3x'.$$

$$q_{lm} = \int \sigma(\theta,\varphi)\,\delta(r-R)\,r'^l\, Y^*_{lm}(\theta',\varphi')\, r'^2 \sin\theta\, dr'd\theta'd\varphi'$$

$$= \int \sigma(\theta,\varphi) R^{l+2} Y^*_{lm} \sin\theta' d\theta' d\varphi'$$

While $\phi(\vec{x}=R) = \sum_{lm} \frac{4\pi}{2l+1}\left(\int \sigma(\theta',\varphi) R\, Y^*_{lm} \sin\theta' d\theta' d\varphi'\right)\cdot Y_{lm}$

$$= \alpha\cos\theta = \alpha \cdot 2\cdot\sqrt{\frac{3}{\pi}}\, Y_{10}$$

We have $\frac{4\pi}{3}\int \sigma(\theta',\varphi) R\, Y^*_{10} \sin\theta' d\theta' d\varphi' = \alpha \cdot 2\sqrt{\frac{3}{\pi}}$

$$\Rightarrow \int \sigma(\theta',\varphi')\, Y_{10}^{*} \sin\theta'\, d\theta'\, d\varphi' = \frac{3\alpha}{2\pi R}\sqrt{\frac{3}{\pi}}$$

$$\cancel{\frac{1}{2}\sqrt{\frac{3}{\pi}}} \int \sigma(\theta',\varphi') \cdot \cos\theta' \sin\theta'\, d\theta'\, d\varphi' = \frac{3\alpha}{\pi R} \cdot \cancel{\frac{1}{2}\sqrt{\frac{3}{\pi}}}$$

Not enough to define $\sigma(\theta',\varphi')$ barely from boundary.

So, try to find the whole potential first.

For that $\phi(x) = \int G(\vec{x},\vec{x}')\rho(\vec{x}')\, d^3x'$

$$= \sum_{lm} \frac{4\pi}{2l+1}\left[\alpha_{lm}(r)\, r^{l} + \frac{\beta_{lm}(r)}{r^{l+1}}\right] Y_{lm}(\theta,\varphi)$$

$$= 2\cdot\sqrt{\frac{3}{\pi}}\,\alpha\, Y_{10} \qquad (r=R)$$

Suppose that we integrate from $\boxed{R^- \text{ to } \infty}$.

$$\alpha_{10}(r) = \int_0^{2\pi}\int_0^{\pi}\int_{R^-}^{\infty} \frac{\sigma(\theta,\varphi)\,\delta(r-R)}{r'^{\,l+1}}\, Y_{10}^{*}(\theta',\varphi')\, r'^2 \sin\theta'\, dr'\, d\theta'\, d\varphi'$$

$$= \int_0^{2\pi}\int_0^{\pi} \frac{\sigma(\theta,\varphi)}{R^{l-1}}\, Y_{10}^{*}(\theta',\varphi') \sin\theta'\, d\theta'\, d\varphi'$$

As could be seen that $\alpha_{lm}(r)$ does not depend on r and it should be a constant.

So, $\phi(\vec{x}) = \frac{4\pi}{3}\alpha_{10}\cdot r\, Y_{10} = \sqrt{\frac{4\pi}{3}}\,\alpha_{10}\, r\cos\theta$

For that $\phi(|\vec{x}|=R) = \alpha\cos\theta$.

$$\phi(\vec{x}) = \alpha\frac{r}{R}\cos\theta. \quad (r<R)$$

Similarly, $\phi(\vec{x}) = \frac{4\pi}{3}\frac{\beta_{10}}{r^2}Y_{10} = \sqrt{\frac{4\pi}{3}}\frac{\beta_{10}}{r^2}\cos\theta$

So, $\phi(\vec{x}) = \alpha\frac{R^2}{r^2}\cos\theta. \qquad (r>R)$

$$\boxed{\phi(\vec{x}) = \begin{cases} \alpha\frac{r}{R}\cos\theta, & (r<R) \\ \alpha\cos\theta. & (r=R) \\ \alpha\frac{R^2}{r^2}\cos\theta, & (r>R) \end{cases}}$$

(b) As written above.

(c) We know that $\sqrt{\frac{4\pi}{3}}\alpha_{10}\cdot\frac{1}{R} = \alpha\cdot\frac{1}{R}$.

$$\Rightarrow \alpha_{10} = \sqrt{\frac{3}{4\pi}}$$

So, $\int_0^{2\pi}\int_0^{\pi} \sigma(\theta,\varphi)\, Y_{10}^*(\theta,\varphi) \sin\theta' \, d\theta' d\varphi' = \sqrt{\frac{3}{4\pi}} \frac{\alpha}{R}$

From the orthogonality of spherical harmonics

we know that if

$$\sigma(\theta,\varphi) \cdot \sqrt{\frac{4\pi}{3}} = Y_{10}(\theta,\varphi) \frac{\alpha}{R}$$

The equation will be satisfied.

$$\Rightarrow \boxed{\sigma(\theta,\varphi) = \frac{3}{4\pi} \frac{\alpha\cos\theta}{R}}$$

(d) From the potential we could know the

electric field :

$$\vec{E}(\vec{x}) = -\nabla\phi(\vec{x})$$

$$= -\left(\frac{\partial}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial}{\partial\theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\right)\phi(\vec{x})$$

So, the field is :

$$\vec{E}(\vec{x}) = \begin{cases} -\frac{\alpha}{R}\cos\theta\,\hat{r} + \frac{\alpha}{R}\sin\theta\,\hat{\theta} & (r<R) \\ \frac{2\alpha R^2}{r^3}\cos\theta\,\hat{r} + \frac{\alpha R^2}{r^3}\sin\theta\,\hat{\theta} & (r>R) \end{cases}$$

With energy density $\mathcal{E} = \frac{|\vec{E}|^2}{8\pi}$

$$\mathcal{E}(\vec{x}) = \begin{cases} \frac{\alpha^2}{8\pi R^2} & (r<R) \\ \frac{\alpha^2 R^4}{8\pi r^6}(1+3\cos^2\theta) & (r>R) \end{cases}$$

The total electrostatic energy is given by

$$\mathcal{E} = \int \mathcal{E}(\vec{x})\, d^3x = \int_0^R \frac{\alpha^2}{8\pi R^2}\cdot 4\pi r^2 dr$$

$$+ \int_0^\pi \int_R^\infty \frac{\alpha^2 R^4}{8\pi r^6}(1+3\cos^2\theta)\, r^2 \sin\theta\, dr\, d\theta \cdot 2\pi$$

$$= \frac{\alpha^2 R}{6} + \underbrace{4\cdot\int_R^\infty \frac{\alpha^2 R^4}{8\pi r^4}\cdot 2\pi\, dr}_{\frac{\alpha^2 R}{3}} = \frac{1}{2}\alpha^2 R$$

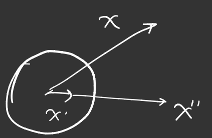
8. (a) We know that

$$G_D(\vec{x}, \vec{x}') = \frac{1}{|\vec{x}-\vec{x}'|} + \frac{\alpha}{|\vec{x}-\vec{x}''|}$$

with $\vec{x}'' = \vec{x}' \cdot \left(\frac{R}{|\vec{x}'|}\right)^2$. $\alpha = -\frac{R}{|\vec{x}'|}$.

With this form the potential inside shell is given by

$$\phi(\vec{x}) = \int_v G_D(\vec{x}, \vec{x}_2)\, \rho(\vec{x}_2)\, d^3\vec{x}_2$$



inside V : $\vec{x}_2$

$$= \int_v G_D(\vec{x}, \vec{x}') \cdot q\,\delta(\vec{x}_2 - \vec{x}')\, d^3\vec{x}_2$$

$$= \int_v \left(\frac{1}{|\vec{x}-\vec{x}'|} + \frac{\alpha}{|\vec{x}-\vec{x}''|}\right) \cdot q\,\delta(\vec{x}_2 - \vec{x}')\, d^3\vec{x}_2$$

$$= \frac{q}{|\vec{x}-\vec{x}'|} + \frac{\alpha q}{|\vec{x}-\vec{x}'|} .$$

So, the electric field is given by

$$\vec{E}(\vec{x}) = -\nabla\phi(\vec{x})$$

$$= \frac{q(\vec{x}-\vec{x}')}{|\vec{x}-\vec{x}'|^3} + \frac{\alpha q(\vec{x}-\vec{x}'')}{|\vec{x}-\vec{x}''|^3} .$$

Then we could use $\vec{E} \cdot \hat{n} = 4\pi\sigma$ to get σ.

For $\vec{E}(|\vec{x}|=R) = \vec{E}(\theta, \varphi)$. $\vec{x} = (R\sin\theta\cos\varphi,\ R\sin\theta\sin\varphi,\ R\cos\theta)$.

While $\vec{E}(|\vec{x}|=R)\cdot\hat{n} = \vec{E}(|\vec{x}|=R)\cdot\hat{x}$

$$E_r = \frac{q(\vec{x}-\vec{x}')\cdot\hat{x}}{|\vec{x}-\vec{x}'|^3} + \frac{\alpha q(\vec{x}-\vec{x}'')\cdot\hat{x}}{|\vec{x}-\vec{x}''|^3}$$

$$= \frac{q(R-\vec{x}'\cdot\hat{x})}{|\vec{x}-\vec{x}'|^3} - \frac{R}{|\vec{x}'|}\cdot\frac{q\left(R-\vec{x}'\cdot\frac{R^2}{|\vec{x}'|^2}\cdot\hat{x}\right)}{\left|\vec{x}-\vec{x}'\cdot\frac{R^2}{|\vec{x}'|^2}\right|^3}$$

If $\vec{x}' = (a, b, c)$, we have

$$E_r = \frac{q(R - a\sin\theta\sin\varphi - b\sin\theta\sin\varphi - c\cos\theta)}{\left[\sqrt{(R\sin\theta\cos\varphi-a)^2+(R\sin\theta\sin\varphi-b)^2+(R\cos\theta-c)^2}\right]^3}$$

$$-\frac{R}{\sqrt{a^2+b^2+c^2}}\cdot\frac{q\left[R-\frac{R^2}{a^2+b^2+c^2}\cdot(a\sin\theta\sin\varphi+b\sin\theta\sin\varphi+c\cos\theta)\right]}{\left[\left(R\sin\theta\cos\varphi-\frac{R^2}{a^2+b^2+c^2}a\right)^2+\left(R\sin\theta\sin\varphi-\frac{R^2}{a^2+b^2+c^2}b\right)^2+\left(R\cos\theta-\frac{R^2}{a^2+b^2+c^2}c\right)^2\right]^{\frac{3}{2}}}$$

And $\sigma(\theta,\varphi)$ is given by $\frac{E_r}{4\pi}$.

(b) $\vec{F}_e = q\vec{E}_{ext}$

$$= q \cdot \frac{\alpha q(\vec{x} - \vec{x}'')}{|\vec{x} - \vec{x}''|^3}$$

$$= -q^2 \frac{R \cdot \left(\frac{\vec{x}}{|\vec{x}'|} - \frac{R^2}{|\vec{x}'|^3}\vec{x}'\right)}{\left|\vec{x} - \frac{R^2}{|\vec{x}'|^2}\vec{x}'\right|^3}$$

In order to hold the charge,

Let $|\vec{x}'| = r$

$$\vec{F} = -\vec{F}_e = q^2 \frac{R \cdot \left(\frac{\vec{x}}{|\vec{x}'|} - \frac{R^2}{|\vec{x}'|^3}\vec{x}'\right)}{\left|\vec{x} - \frac{R^2}{|\vec{x}'|^2}\vec{x}'\right|^3}$$

$$= q^2 \frac{\frac{R}{r}\vec{x} - \frac{R^2}{r^3}\vec{x}'}{\left|\vec{x} - \frac{R^2}{r^2}\vec{x}'\right|^3}$$

$$= \frac{Rq^2}{r} \cdot \frac{\left(\vec{x} - \frac{R^2}{r^2}\vec{x}'\right)}{\left|\vec{x} - \frac{R^2}{r^2}\vec{x}'\right|^3}$$

9. (a) $\phi(\vec{x}) = \int_{\mathcal{V}} G_D(\vec{x}, \vec{x}')\rho(\vec{x}_2)\,d^3\vec{x}_2$

In Green's Theorem, let $\phi_1 = \phi$. $\phi_2 = \frac{1}{|\vec{x}-\vec{x}'|}$

$$\nabla^2\phi_1 = \nabla^2\phi = 0.$$

$$\nabla^2\phi_2 = -4\pi\rho_2 = -4\pi\,\delta(\vec{x}-\vec{x}')$$

So, $\int_S \hat{n}\cdot(\phi\vec{\nabla}\phi_2 - \phi_2\vec{\nabla}\phi)\,ds = -4\pi\int_{\mathcal{V}} \phi\rho_2\,d^3x$

$$= 4\pi\underbrace{\int_{\mathcal{V}} \phi\,\delta(\vec{x}-\vec{x}')\,d^3x}_{=\phi(\vec{x})}$$

$$\text{Left} = \int_S \vec{n}\cdot\phi\vec{\nabla}\phi_2\,ds - \int_S \frac{\hat{n}\cdot\vec{\nabla}\phi}{|\vec{x}-\vec{x}'|}\,ds$$

For $\mathcal{V}$ a sphere of radius R and $S = \partial\mathcal{V}$.

$$\text{Left} = \int_S \phi\cdot\frac{1}{R^2}\,dS - \int_S \frac{\hat{n}\cdot\nabla\phi}{R}\,dS = 4\pi\,\phi(\vec{x})$$

That is,

$$\frac{1}{4\pi R^2}\int_S \phi\,dS - \frac{1}{4\pi R}\int_S \hat{n}\cdot\vec{\nabla}\phi\,ds = \phi(\vec{x})$$

While $\int_S \hat{n}\cdot\vec{\nabla}\phi \, dS = \int_V \vec{\nabla}\cdot\vec{\nabla}\phi \, d^3x = 0.$

We get $\frac{1}{4\pi R^2}\int_S \phi \, dS = \phi(\vec{x})$

Which is the mean value theorem.

(b) In a neighborhood of R, for point charge q at $\vec{x}$.

$$\phi(\vec{x}) = \frac{1}{4\pi R^2}\int_S \phi \, dS.$$

Find the condition for a charge to be in stable equilibrium,

Gets a little com

We should have $\nabla^2\phi > 0$ for $\nabla\phi = 0$ points.

But we already know that $\nabla^2\phi = 0$ for this source-free space, so the charge can not be in stable equilibrium.

△ For neutral equilibrium we should have $\nabla^2\phi = 0$.

For $\nabla\phi = 0$ points.

This means that the field at point charge

$$\vec{\nabla}\phi(\vec{x}) = \vec{\nabla}\left(\frac{1}{4\pi R^2}\int_S \phi\, dS\right) = 0.$$

Also with no charge (source free) we have

$$\int_S \vec{E}\cdot d\vec{S} = 0 = \int_S -\vec{\nabla}\phi\cdot d\vec{S} = 0.$$

We have to prove that $\vec{E} = 0$ on the surface of this region.

$$\vec{\nabla}\left(\frac{1}{4\pi R^2}\int_S \phi\, dS\right) = \frac{1}{4\pi R^2}\int_S (\vec{\nabla}\phi)\, dS = \vec{0}.$$

That is, $\int_S \vec{E}\, dS = 0$. and $\int_S \vec{E}\cdot d\vec{S} = 0$

Only when $\vec{E} = 0$ can both equations be satisfied.

HW 3 Dielectrics

1. Since we have Poisson equation inside of dielectric medium,

$$\nabla^2 \langle \phi \rangle = -4\pi \langle \rho \rangle.$$

We could still have the form of electric potential:

$$\langle \phi(r) \rangle = \begin{cases} \sum_{l,m} A_{lm} r^l Y_{lm}(\theta, \varphi) & (r \leq R) \\ \sum_{l,m} B_{lm} \frac{1}{r^{l+1}} Y_{lm}(\theta, \varphi) & (r > R) \end{cases}$$

Since $\langle \vec{P} \rangle = \chi \langle \vec{E} \rangle$, $\langle \vec{D} \rangle = \langle \vec{E} \rangle + 4\pi \langle \vec{P} \rangle$.

We know the total field inside the ball should also be uniform and be in $\hat{z}$ direction. So, the potential inside ball should be of form

$$\langle \phi(\vec{r}) \rangle = a + b \cdot \vec{E}^{ext} \cdot \vec{r} = \underline{a + b\, C_0 r \cos\theta}.$$

Potential outside of ball should be of form

$$\langle \phi(\vec{r}) \rangle = \underline{a' + b' C_0 r \cos\theta + \sum_{l,m} B_{lm} \frac{1}{r^{l+1}} Y_{lm}(\theta, \varphi)}$$

Since the potential at infinity is uniform, to be

$$\phi(\infty) = d - C_o r\cos\theta.$$

Let $d=0$ we get $\Rightarrow a'=0,\ b'=-1.$

$$a + bC_oR\cos\theta = -C_oR\cos\theta + \sum_{lm} B_{lm}\frac{1}{r^{l+1}}Y_{lm}(\theta,\varphi)\Big|_{r=R} \quad ①$$

(continuity of potential at $r=R$)

gives $\langle\phi(\vec{r})\rangle = -C_o r\cos\theta \quad (r\leq R)$

$$\varepsilon\frac{\partial\langle\phi(\vec{r}^-)\rangle}{\partial r}\Big|_{r=R} = \varepsilon_o\frac{\partial\langle\phi(\vec{r}^+)\rangle}{\partial r}\Big|_{r=R}$$

(boundary condition at $r=R$).

So,

$$\varepsilon C_o\cos\theta = \varepsilon_o\cdot(-C_o\cos\theta) + \sum_{l,m} B_{lm}\frac{-(l+1)}{r^{l+2}}Y_{lm}(\theta,\varphi)\Big|_{r=R} \quad ②$$

With ①, ②, and let $l=1, m=0$. $l=0, m=0$.

$$\Rightarrow\begin{cases} a + bC_oR\cos\theta = -C_oR\cos\theta + \dfrac{B_{00}}{R}\cdot\dfrac{1}{2\sqrt{\pi}} + \dfrac{B_{10}}{R^2}\cdot\dfrac{1}{2}\sqrt{\dfrac{3}{\pi}}\cos\theta \\ \varepsilon bC_o\cos\theta = -C_o\varepsilon_o\cos\theta - \dfrac{\varepsilon_o B_{00}}{R^2}\cdot\dfrac{1}{2\sqrt{\pi}} - \varepsilon_o B_{10}\cdot\dfrac{2}{R^3}\cdot\dfrac{1}{2}\sqrt{\dfrac{3}{\pi}}\cos\theta \end{cases}$$

Solve to get $B_{00}=0$. $a=0$.

$$b = \frac{-3\varepsilon_o}{\varepsilon+2\varepsilon_o} \qquad B_{10} = \frac{\varepsilon-\varepsilon_o}{\varepsilon+2\varepsilon_o}C_oR^3\sqrt{\frac{4\pi}{3}}$$

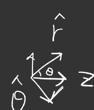
Thus,

$$\langle \phi(r) \rangle = \begin{cases} \dfrac{-3\varepsilon_0}{\varepsilon + 2\varepsilon_0} C_0 r \cos\theta. & (r \leq R) \\[2ex] -C_0 r \cos\theta + \dfrac{\varepsilon - \varepsilon_0}{\varepsilon + 2\varepsilon_0} \dfrac{C_0 R^3}{r^2} \cos\theta. & (r > R) \end{cases}$$

$$\langle E(r) \rangle = -\nabla \langle \phi(r) \rangle$$

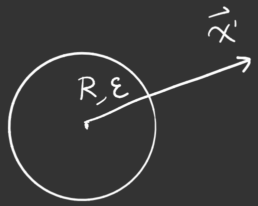
$$\langle \vec{E}(r) \rangle = \begin{cases} \dfrac{-3\varepsilon_0}{\varepsilon + 2\varepsilon_0} C_0 \cos\theta \, \hat{r} + \dfrac{3\varepsilon_0}{\varepsilon + 2\varepsilon_0} C_0 \sin\theta \, \hat{\theta} & (r \leq R) \\[2ex] \left(-C_0 \cos\theta - \dfrac{2(\varepsilon - \varepsilon_0)}{\varepsilon + 2\varepsilon_0} \dfrac{C_0 R^3}{r^3} \cos\theta \right) \hat{r} & \\[2ex] + \left(C_0 + \dfrac{\varepsilon - \varepsilon_0}{\varepsilon + 2\varepsilon_0} \dfrac{C_0 R^3}{r^3} \right) \sin\theta \, \hat{\theta} & (r > R) \end{cases}$$

It could be again written into



$$\langle \vec{E}(r) \rangle = \begin{cases} \dfrac{3\varepsilon_0}{\varepsilon + 2\varepsilon_0} C_0 \hat{z} & (r \leq R) \\[2ex] C_0 \hat{z} + \dfrac{\varepsilon - \varepsilon_0}{\varepsilon + 2\varepsilon_0} \dfrac{C_0 R^3}{r^3} \hat{z} - \dfrac{\varepsilon - \varepsilon_0}{\varepsilon + 2\varepsilon_0} \dfrac{C_0 R^3}{r^3} \cos\theta \, \hat{r} & (r > R) \end{cases}$$

2.



We have poisson equation:

$$\nabla^2 \Phi(\vec{x}) = -4\pi\,\delta(\vec{x}-\vec{x}')$$

Suppose the potential in each region take the form:

$$\phi(\vec{x}) = \sum_{l,m} \frac{4\pi}{2l+1}\left[\alpha_{lm}(r)\, r^{l} + \frac{\beta_{lm}(r)}{r^{l+1}}\right] Y_{lm}(\theta,\varphi).$$

For $|\vec{x}|>d$ we have $\phi_3(\vec{x}) = \sum_{l,m} \frac{d_{lm}(r)}{r^{l+1}} Y_{lm}(\theta,\varphi)$

For $R \leq |\vec{x}| \leq d$ we have

$$\phi_2(\vec{x}) = \sum_{l,m} \frac{4\pi}{2l+1}\left[b_{lm}(r)\, r^{l} + \frac{C_{lm}(r)}{r^{l+1}}\right] Y_{lm}(\theta,\varphi).$$

For $|\vec{x}|<R$ we have

$$\phi_1(\vec{x}) = \sum_{l,m} \frac{4\pi}{2l+1} a_{lm}(r)\, r^{l}\, Y_{lm}(\theta,\varphi).$$

We have boundary condition at $|\vec{x}| = d$,

$$b_{lm} d^{l} + \frac{C_{lm}}{d^{l+1}} = \frac{d_{lm}}{d^{l+1}} \quad ①$$

$$b_{lm} \cdot l\, d^{l-1} - (l+1)\, C_{lm} \cdot \frac{1}{d^{l+2}} = -(l+1)\, d_{lm} \cdot \frac{1}{d^{l+2}} + \frac{1}{d^{2}} \quad ②$$

The boundary condition at $|\vec{x}|=R$,

$$a_{\ell m} R^{\ell} = b_{\ell m} R^{\ell} + \frac{C_{\ell m}}{R^{\ell+1}} \quad ③$$

Also, from the continuity of $\vec{D}$.

$$\varepsilon \frac{\partial \phi_1}{\partial r}\bigg|_{r=R} = \frac{\partial \phi_2}{\partial r}\bigg|_{r=R}$$

$$\Rightarrow \quad \varepsilon\, a_{\ell m} \cdot \ell R^{\ell-1} = b_{\ell m} \cdot \ell R^{\ell-1} + C_{\ell m} \cdot \frac{-(\ell+1)}{R^{\ell+2}} \quad ④$$

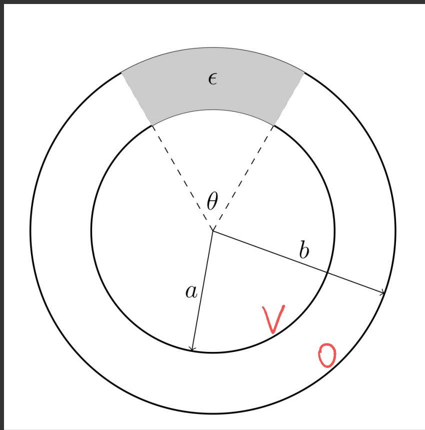
Combining ①②③④, solve to get

$$\begin{cases} a_{\ell m} = \dfrac{1}{(\varepsilon+1)\ell+1} \cdot \dfrac{1}{d^{\ell}} \\[2ex] b_{\ell m} = \dfrac{-1}{2\ell+1} \cdot \dfrac{1}{d^{\ell}} \\[2ex] C_{\ell m} = \dfrac{-1}{2\ell+1} \cdot \dfrac{(\varepsilon-1)\ell}{(\varepsilon+1)\ell+1} \cdot \dfrac{R^{2\ell+1}}{d^{\ell}} \\[2ex] d_{\ell m} = \dfrac{-1}{2\ell+1} \cdot \dfrac{(\varepsilon-1)\ell}{(\varepsilon+1)\ell+1} \cdot \dfrac{R^{2\ell+1}}{d^{\ell}} + \dfrac{1}{2\ell+1} d^{\ell} \end{cases}$$

So, we have potential

$$\phi(\vec{x}) = \begin{cases} \sum_{l,m} \frac{4\pi}{2l+1} \frac{1}{(\varepsilon+1)l+1} \cdot \frac{r^l}{d^l} Y_{lm}(\theta,\varphi) & (|\vec{x}|<R) \\ \sum_{l,m} \left[\frac{4\pi}{(2l+1)^2} \frac{r^l}{d^l} Y_{lm}(\theta,\varphi) - \frac{4\pi}{(2l+1)^2} \cdot \frac{(\varepsilon-1)l}{(\varepsilon+1)l+1} \cdot \frac{R^{2l+1}}{d^l r^{l+1}} Y_{lm}(\theta,\varphi) \right] & (R \leq |\vec{x}| \leq d) \\ \sum_{l,m} \left[\frac{-1}{2l+1} \cdot \frac{(\varepsilon-1)l}{(\varepsilon+1)l+1} \cdot \frac{R^{2l+1}}{d^l r^{l+1}} + \frac{1}{2l+1} \frac{d^l}{r^{l+1}} \right] Y_{lm}(\theta,\varphi) & (|\vec{x}|>d) \end{cases}$$

3.



(a) Suppose the potential in $a<r<b$ take the form:

$$\phi(\vec{x}) = \sum_{l,m} \frac{4\pi}{2l+1}\left[\alpha_{lm}(r) r^l + \frac{\beta_{lm}(r)}{r^{l+1}}\right] Y_{lm}(\theta,\varphi).$$

it's a little dangerous to use this basis but the "suppose" helped !!

Boundary conditions are:

$$V = \sum_{l,m} \frac{4\pi}{2l+1}\cdot\left(\alpha_{lm}\cdot a^l + \frac{\beta_{lm}}{a^{l+1}}\right) Y_{lm}(\theta,\varphi)$$

$$0 = \sum_{l,m} \frac{4\pi}{2l+1}\cdot\left(\alpha_{lm}\cdot b^l + \frac{\beta_{lm}}{b^{l+1}}\right) Y_{lm}(\theta,\varphi)$$

$$\varepsilon \frac{\partial\phi}{\partial\theta}\Big|_{\theta=0^+} = \frac{\partial\phi}{\partial\theta}\Big|_{\theta=0^-}. \Rightarrow \frac{\partial\phi}{\partial\theta}=0.$$

So, $l=0$, $m=0$.

$$V = 4\pi \cdot \left(\alpha_{00} + \frac{\beta_{00}}{a}\right) \cdot \frac{1}{\sqrt{4\pi}}$$

$$0 = 4\pi \cdot \left(\alpha_{00} + \frac{\beta_{00}}{b}\right) \cdot \frac{1}{\sqrt{4\pi}}$$

$$\Rightarrow \begin{cases} \alpha_{00} = \dfrac{-a}{b-a} \cdot \dfrac{V}{\sqrt{4\pi}} \\ \beta_{00} = \dfrac{ab}{b-a} \cdot \dfrac{V}{\sqrt{4\pi}} \end{cases}$$

Potential in $a<r<b$ is given by

$$\phi(\vec{r}) = -\frac{a}{b-a} V + \frac{ab}{(b-a)r} V .$$

(b)

$$\sigma(r=a) = \left.\frac{\partial \phi}{\partial r}\right|_{r=a} = \frac{-b}{(b-a)a} ,$$

$$\sigma(r=b) = \left.\frac{\partial \phi}{\partial r}\right|_{r=b} = \frac{-a}{(b-a)b} .$$

(c) $\vec{E}$ should be radial, so is $\vec{D}$. $\vec{P}$.

So, there will be no polarization charge density at angle

θ boundary.

For boundary $r=a$. Since $\vec{D} = \vec{E} + 4\pi\vec{P}$.

In homogeneous material $\vec{P} = (\varepsilon-1)\vec{E}/4\pi$

$$\vec{E}(r=a^-) = 0 \qquad \vec{E}(r=a^+) = \frac{-b}{(b-a)a}\hat{r}.$$

So, $\sigma_p(r=a) = \vec{E}(r=a^+)\cdot\hat{r} - \vec{E}(r=a^-)\cdot\hat{r}$

$$= \frac{\varepsilon-1}{4\pi}\cdot\frac{-b}{(b-a)a}$$

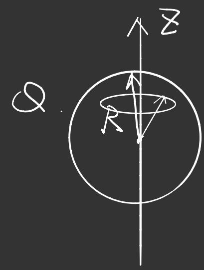
$$\vec{E}(r=b^-) = \frac{-a}{(b-a)b}\hat{r}. \qquad \vec{E}(r=b^+) = 0.$$

So, $\sigma_p(r=b) = \vec{E}(r=b^+)\cdot\hat{r} - \vec{E}(r=b^-)\cdot\hat{r}$

$$= \frac{\varepsilon-1}{4\pi}\cdot\frac{-a}{(b-a)b}.$$

HW 4 yuxiang@uchicago.edu

Chapter 4. 2. 3. 6



2. First, look into the current distribution.

$$\rho = \frac{Q}{V} = \frac{3Q}{4\pi R^3}.$$

$\dfrac{\rho \cdot dr \cdot \overset{R}{d\theta} \cdot \overset{2\pi}{r\sin\theta}}{\frac{2\pi}{\omega}}$ r^2?

$\dfrac{\cancel{2\pi} r\sin\theta \cdot \omega}{\cancel{2\pi}}$ $dr\,d\theta$

$\cos\theta$?

$2\pi r \cdot r\sin\theta$

At (r, θ, φ), the current is given by

$$d\vec{J} = \rho\omega r\sin\theta\, dr\, d\theta.$$

$$= -\sqrt{\frac{4\pi}{3}}\, \rho\omega r \cdot \vec{r} \times \vec{\nabla} Y_{10}(\theta,\varphi)\, dr\, d\theta.$$

So, $\vec{J} = -\sqrt{\frac{4\pi}{3}}\, \rho\omega r \cdot \vec{r} \times \vec{\nabla} Y_{10}(\theta,\varphi)$.

J is electric parity current distribution.

Magnetic potential should be in the form:

$$\vec{A} = \alpha^M_{\ell m}(r)\, \vec{r} \times \vec{\nabla} Y_{\ell m} = \alpha^M_{10}(r)\, \vec{r} \times \vec{\nabla} Y_{10}$$

$$\alpha^M_{10}(r) = \frac{4\pi}{c} \cdot \frac{1}{(2\ell+1)} \left[\frac{1}{r^{\ell+1}} \int_0^r j^M_{\ell m}(r') r'^{\ell} r'^2 dr' + r^{\ell} \int_r^\infty j^M_{\ell m}(r') \frac{1}{r'^{\ell+1}} r'^2 dr' \right]$$

$$= \frac{4\pi}{c} \frac{1}{(2+1)} \left[\frac{1}{r^2} \int_0^r j^M_{\ell m}(r') r'^3 dr' + r \int_r^\infty j^M_{\ell m}(r') dr' \right]$$

Since $j_{\ell m}^{M}(r') = 0$ for $r > R$, $j_{\ell m}^{M}(r) = -\sqrt{\frac{4\pi}{3}}\rho\omega r$ for $r \leq R$.

$$\alpha_{10}^{M}(r) = \frac{4\pi}{c}\cdot\frac{1}{3}\cdot\frac{1}{r^2}\cdot\left(-\sqrt{\frac{4\pi}{3}}\rho\omega\cdot\frac{r^5}{5}\right)$$

$$+\frac{4\pi}{c}\cdot\frac{1}{3}\cdot r\cdot\left[-\sqrt{\frac{4\pi}{3}}\rho\omega\frac{(R^2-r^2)}{2}\right]$$

$$= \frac{4\pi}{3c}\cdot\sqrt{\frac{4\pi}{3}}\rho\omega\left(\frac{r^3}{10}-\frac{rR^2}{2}\right). \qquad (r \leq R)$$

$$\alpha_{10}^{M}(r) = \frac{4\pi}{c}\cdot\frac{1}{3}\cdot\frac{1}{r^2}\cdot\left(-\sqrt{\frac{4\pi}{3}}\rho\omega\cdot\frac{R^5}{5}\right)$$

$$= -\frac{4\pi}{3c}\sqrt{\frac{4\pi}{3}}\rho\omega\cdot\frac{R^5}{5r^2} \qquad (r > R)$$

$$\vec{A}(r \leq R) = \frac{4\pi}{3c}\cdot\cancel{\sqrt{\frac{4\pi}{3}}}\rho\omega\left(\frac{r^3}{10}-\frac{rR^2}{2}\right)\vec{r}\times\left[\cancel{\sqrt{\frac{3}{4\pi}}}\cdot(-\sin\theta)\,\hat{\theta}\right]\cdot\frac{1}{r}$$

$$= \frac{4\pi}{3c}\rho\omega\left(\frac{rR^2}{2}-\frac{r^3}{10}\right)\sin\theta\cdot\hat{\varphi}. \qquad (r \leq R)$$

$$\vec{A}(r > R) = -\frac{4\pi}{3c}\cancel{\sqrt{\frac{4\pi}{3}}}\rho\omega\cdot\frac{R^5}{5r^2}\,\vec{r}\times\frac{1}{r}\cdot\left[\cancel{\sqrt{\frac{3}{4\pi}}}(-\sin\theta)\,\hat{\theta}\right]$$

$$= \frac{4\pi}{3c}\rho\omega\cdot\frac{R^5}{5r^2}\sin\theta\cdot\hat{\varphi}$$

Magnetic field is then given by

$$\vec{B} = \vec{\nabla} \times \vec{A}.$$

$$= \frac{1}{r\sin\theta}\left(\frac{\partial(A_\varphi \sin\theta)}{\partial\theta} - \frac{\partial A_\theta}{\partial\varphi}\right)\hat{r} + \frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial A_r}{\partial\varphi} - \frac{\partial(rA_\varphi)}{\partial r}\right)\hat{\theta}$$

$$+ \frac{1}{r}\left(\frac{\partial(rA_\theta)}{\partial r} - \frac{\partial A_r}{\partial\theta}\right)\hat{\varphi}$$

$$= \frac{1}{r\sin\theta}\frac{\partial(A_\varphi\sin\theta)}{\partial\theta}\hat{r} - \frac{1}{r}\frac{\partial(rA_\varphi)}{\partial r}\hat{\theta}$$

$$\vec{B}(r \leq R) = \frac{4\pi}{3c}\rho\omega\left(\frac{rR^2}{2} - \frac{r^3}{10}\right) \cdot 2\cancel{\sin\theta}\cos\theta \cdot \frac{1}{r\cancel{\sin\theta}}\hat{r}$$

$$-\frac{1}{r}\cdot\frac{4\pi}{3c}\rho\omega\left(rR^2 - \frac{2}{5}r^3\right)\sin\theta\,\hat{\theta}$$

$$= \frac{4\pi}{3c}\rho\omega\left(R^2 - \frac{r^2}{5}\right)\cos\theta\,\hat{r} - \frac{4\pi}{3c}\rho\omega\left(R^2 - \frac{2}{5}r^2\right)\sin\theta\,\hat{\theta}$$

$$\vec{B}(r > R) = \frac{4\pi}{3c}\rho\omega\cdot\frac{2R^5}{5r^3}\cos\theta\,\hat{r} + \frac{4\pi}{3c}\rho\omega\cdot\frac{R^5}{5r^3}\sin\theta\cdot\hat{\varphi}$$

Put ρ into Q we could get the final result.

Magnetic potential is given by

$$\vec{A}(\vec{r}) = \begin{cases} \dfrac{Q\omega}{cR^3}\left(\dfrac{rR^2}{2} - \dfrac{r^3}{10}\right)\sin\theta \cdot \hat{\varphi} & (r \leq R) \\[2ex] \dfrac{Q\omega}{cR^3} \cdot \dfrac{R^5}{5r^2}\sin\theta \cdot \hat{\varphi} & (r > R) \end{cases}$$

$$\vec{B}(\vec{r}) = \begin{cases} \dfrac{Q\omega}{cR^3}\left(R^2 - \dfrac{r^2}{5}\right)\cos\theta\,\hat{r} - \dfrac{Q\omega}{cR^3}\left(R^2 - \dfrac{2}{5}r^2\right)\sin\theta\,\hat{\theta} & (r \leq R) \\[2ex] \dfrac{Q\omega}{cR^3} \cdot \dfrac{2R^5}{5r^3}\cos\theta\,\hat{r} + \dfrac{Q\omega}{cR^3} \cdot \dfrac{R^5}{5r^3}\sin\theta \cdot \hat{\varphi} & (r > R) \end{cases}$$

Sorry after all these works I found that this is uniformly charged shell instead of sphere.

Let me try to give another solution below.

2. For the case of shell.

First, look into the current distribution.

$$\sigma = \frac{Q}{4\pi R^2}.$$

At (r, θ, φ), the current is given by

$$d\vec{J} = R\sigma\omega \sin\theta \cdot \delta(r-R) \cdot \hat{\varphi}$$

$$= -\sqrt{\frac{4\pi}{3}}\, R\sigma\omega\, \delta(r-R) \cdot \vec{r} \times \vec{\nabla} Y_{10}(\theta, \varphi).$$

So, $\vec{J} = -\sqrt{\frac{4\pi}{3}}\, R\sigma\omega\, \delta(r-R) \cdot \vec{r} \times \vec{\nabla} Y_{10}(\theta, \varphi).$

J is electric parity current distribution.

Magnetic potential should be in the form:

$$\vec{A} = \alpha^M_{lm}(r)\, \vec{r} \times \vec{\nabla} Y_{lm} = \alpha^M_{10}(r)\, \vec{r} \times \vec{\nabla} Y_{10}$$

$$\alpha^M_{10}(r) = \frac{4\pi}{c} \cdot \frac{1}{(2l+1)} \left[\frac{1}{r^{l+1}} \int_0^r j^M_{lm}(r') r'^l r'^2 dr' + r^l \int_r^\infty j^M_{lm}(r') \frac{1}{r'^{l+1}} r'^2 dr' \right]$$

$$= \frac{4\pi}{c} \cdot \frac{1}{(2+1)} \left[\frac{1}{r^2} \int_0^r j^M_{lm}(r') r'^3 dr' + r \int_r^\infty j^M_{lm}(r') dr' \right]$$

Since $j_{em}^{M}(r') = -\sqrt{\frac{4\pi}{3}} R\sigma\omega\,\delta(r'-R)$

$$\alpha_{10}^{M}(r) = \frac{4\pi}{c}\cdot\frac{1}{3}\cdot r\cdot\left(-\sqrt{\frac{4\pi}{3}}\sigma\omega R\right) \quad (r\leq R)$$

$$\alpha_{10}^{M}(r) = \frac{4\pi}{c}\cdot\frac{1}{3}\cdot\frac{1}{r^2}\cdot\left(-\sqrt{\frac{4\pi}{3}}\sigma\omega\cdot R^4\right) \quad (r>R)$$

$$\vec{A}(r\leq R) = -\frac{4\pi}{3c}\cdot\cancel{\sqrt{\frac{4\pi}{3}}}\sigma\omega Rr\cdot\vec{r}\times\left[\cancel{\sqrt{\frac{3}{4\pi}}}\cdot(-\sin\theta)\,\hat{\theta}\right]\cdot\frac{1}{r}$$

$$= \frac{4\pi}{3c}\cdot\sigma\omega\cdot Rr\cdot\sin\theta\,\hat{\varphi} \quad (r\leq R)$$

$$\vec{A}(r>R) = -\frac{4\pi}{3c}\cancel{\sqrt{\frac{4\pi}{3}}}\sigma\omega\cdot\frac{R^4}{r^2}\,\vec{r}\times\frac{1}{r}\cdot\left[\cancel{\sqrt{\frac{3}{4\pi}}}(-\sin\theta)\,\hat{\theta}\right]$$

$$= \frac{4\pi}{3c}\sigma\omega\cdot\frac{R^4}{r^2}\cdot\sin\theta\,\hat{\varphi}$$

Magnetic field is then given by

$$\vec{B} = \vec{\nabla}\times\vec{A}.$$

$$= \frac{1}{r\sin\theta}\frac{\partial(A_\varphi\sin\theta)}{\partial\theta}\hat{r} - \frac{1}{r}\frac{\partial(rA_\varphi)}{\partial r}\hat{\theta}$$

$$\vec{B}(r\le R) = \frac{4\pi}{3c}\cdot\sigma\omega\cdot 2R\cos\theta\,\hat{r} - \frac{4\pi}{3c}\cdot\sigma\omega\, 2R\sin\theta\,\hat{\theta}$$

$$\vec{B}(r>R) = \frac{4\pi}{3c}\cdot\sigma\omega\cdot\frac{2R^4}{r^3}\cos\theta\,\hat{r} + \frac{4\pi}{3c}\cdot\sigma\omega\cdot\frac{R^4}{r^3}\sin\theta\,\hat{\theta}$$

Overall, the result is

$$\vec{A}(\vec{r}) = \begin{cases} \dfrac{Q\omega}{3c}\cdot\dfrac{r}{R}\sin\theta\,\hat{\varphi} & (r\le R) \\[2ex] \dfrac{Q\omega}{3c}\cdot\dfrac{R^2}{r^2}\cdot\sin\theta\,\hat{\varphi} & (r>R). \end{cases}$$

$$\vec{B}(\vec{r}) = \begin{cases} \dfrac{Q\omega}{3c}\cdot\dfrac{2}{R}\cos\theta\,\hat{r} - \dfrac{Q\omega}{3c}\cdot\dfrac{2}{R}\sin\theta\,\hat{\theta} & (r\le R) \\[2ex] \dfrac{Q\omega}{3c}\cdot\dfrac{2R^2}{r^3}\cos\theta\,\hat{r} + \dfrac{Q\omega}{3c}\cdot\dfrac{R^2}{r^3}\sin\theta\,\hat{\theta} & (r>R) \end{cases}$$

3. (a) Since $\vec{\nabla} \times \vec{B} = \frac{4\pi}{c} \vec{J}$.

$$\vec{\nabla} \times \vec{B} = \vec{\nabla} \times \left[f(x,y,z)\, \hat{z} \right]$$

$$= \frac{\partial f}{\partial y} \hat{x} - \frac{\partial f}{\partial x} \hat{y} = \frac{4\pi}{c} \vec{J} = \vec{0}.$$

So, $\frac{\partial f}{\partial y} = 0$. $\frac{\partial f}{\partial x} = 0$. $\Rightarrow f(x,y,z) = g(z)$.

$\vec{\nabla} \cdot \vec{B} = 0 \Rightarrow \vec{\nabla} \cdot \left[g(z)\, \hat{z} \right] = 0$. $\frac{\partial g(z)}{\partial z} = 0$.

So, $f(x,y,z)$ satisfies $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = \frac{\partial f}{\partial z} = 0$.

$\Rightarrow f(x,y,z)$ should be constant.

Thus $\vec{B} = B_0 \hat{z}$.

(b) Suppose that $\vec{B} = B_x \hat{x} + B_y \hat{y} + B_z \hat{z}$.

$$\vec{\nabla} \times \vec{B} = \left(\frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z} \right) \hat{x} + \left(\frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial x} \right) \hat{y} + \left(\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} \right) \hat{z}$$

$B_x = B_x(x,y)$. $B_y = B_y(x,y)$. $B_z = B_z(x,y)$.

We have $J_z = (\nabla \times \vec{B})_z = \frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} = 0$.

That is $\frac{\partial B_y}{\partial x} = \frac{\partial B_x}{\partial y} = f_1(x,y)$

Also, $\nabla \cdot \vec{B} = 0$. That is

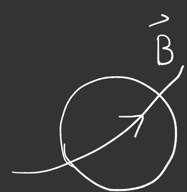
$$\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \underbrace{\frac{\partial B_z}{\partial z}}_{=0} = 0.$$

since B_z is function of x, y

So, we have $\frac{\partial B_x}{\partial y} = \frac{\partial B_y}{\partial x}$.

$$\frac{\partial B_x}{\partial x} = -\frac{\partial B_y}{\partial y}.$$

$$\Rightarrow \frac{\partial^2 B_x}{\partial x \partial y} = \frac{\partial}{\partial y}\left(-\frac{\partial B_y}{\partial y}\right) = \frac{\partial^2 B_y}{\partial x^2}.$$



$$\Rightarrow \frac{\partial^2 B_y}{\partial x^2} + \frac{\partial^2 B_y}{\partial y^2} = 0.$$

Similarly $\frac{\partial^2 B_x}{\partial x^2} + \frac{\partial^2 B_x}{\partial y^2} = 0$.

.... only when $B_x = B_y = 0$ can the criteria be satisfied. So $\vec{B}$ points in $\vec{z}$ everywhere.

(c)

- Using the conclusion in (b). We know that an arbitrary cross-section infinite, straight solenoid satisfies this kind of $\vec{J}$ distribution.
- So, $\vec{B}$ points in $\hat{z}$ direction everywhere.
- Using the conclusion in (a). We know that in/outside of the solenoid it satisfies that $\vec{J}=0$ and $\vec{B}$ points $\hat{z}$ direction.
- So, both inside and outside of the solenoid, $\vec{B}$ is uniform and point to the $\hat{z}$ direction.
- Suppose that we do not have magnetic field at infinitely far-away places. So, $\vec{B}$ outside the solenoid should be 0.

6. (a) $\vec{P} \equiv \int \vec{p}(\vec{x})\, d^3x$

$$= \int \frac{1}{4\pi c}\left(\vec{E}\times\vec{B}\right) d^3x \qquad \vec{E} = -\nabla\phi.$$

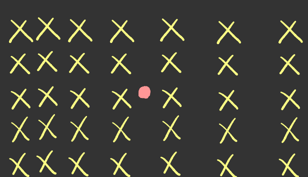
$$= \int \frac{1}{4\pi c}\left[\vec{\nabla}\times\left(-\phi\vec{B}\right) + \phi\,\vec{\nabla}\times\vec{B}\right] d^3x$$

$$= \underbrace{\int \frac{-1}{4\pi c}\,\vec{\nabla}\times\left(\phi\vec{B}\right) d^3x}_{0\,=} + \int \frac{1}{c^2}\,\phi\,\vec{J}\, d^3x$$

$$= \frac{1}{c^2}\int \phi(\vec{x})\,\vec{J}(\vec{x})\, d^3x.$$

(b) Hidden Momentum : momentum carried by a stationary electromagnetic field.

Example : A charge Q in a non-uniform $\vec{B}$ field.



• represents charge Q.

current distribution?

-0.5

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1. Show that for arbitrary solution of Maxwell's equations. the $\vec{E}$ & $\vec{B}$ satisfy

$$\Box \vec{E} = 4\pi \left[\nabla \rho + \frac{1}{c^2} \frac{\partial \vec{J}}{\partial t} \right] \qquad \Box \vec{B} = -\frac{4\pi}{c} \nabla \times \vec{J} .$$

$$\Box = \nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} .$$

Maxwell eqn =

$$\vec{E} = -\nabla \phi - \frac{1}{c} \frac{\partial \vec{A}}{\partial t} . \qquad ①$$

$$\vec{B} = \nabla \times \vec{A} . \qquad ②$$

$$\nabla \cdot \vec{E} = 4\pi \rho . \qquad ③$$

$$\nabla \times \vec{B} - \frac{1}{c} \frac{\partial \vec{E}}{\partial t} = \frac{4\pi}{c} \vec{J} . \qquad ④$$

$$\Box \vec{E} = \nabla^2 \vec{E} - \frac{1}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2} = \nabla (\nabla \cdot \vec{E}) - \nabla \times (\nabla \times \vec{E}) - \frac{1}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2} .$$

$$\xrightarrow{③} \quad = 4\pi \nabla \rho - \nabla \times (\nabla \times \vec{E}) - \frac{1}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2} . \qquad ⑤$$

With ②. ④ we have

$$\nabla \times (\nabla \times \vec{A}) - \frac{1}{c} \frac{\partial \vec{E}}{\partial t} = \frac{4\pi}{c} \vec{J} \qquad ⑥$$

Take $\frac{1}{c}\frac{\partial}{\partial t}$ of ⑥ we get

$$\nabla\times\left[\nabla\times\left(\frac{1}{c}\frac{\partial\vec{A}}{\partial t}\right)\right]-\frac{1}{c^2}\frac{\partial^2\vec{E}}{\partial t^2}=\frac{4\pi}{c}\frac{\partial\vec{J}}{\partial t}.$$

$$\xrightarrow{①}\quad -\nabla\times\underbrace{\left[\nabla\times(\nabla\phi)\right]}_{=0}-\nabla\times(\nabla\times\vec{E})-\frac{1}{c^2}\frac{\partial^2\vec{E}}{\partial t^2}=\frac{4\pi}{c}\frac{\partial\vec{J}}{\partial t}.$$

Combine with ⑤ to get

$$\Box\vec{E}=4\pi\left(\nabla\rho+\frac{1}{c}\frac{\partial\vec{J}}{\partial t}\right).$$

Take the div of ① to get

$$\nabla\times\vec{E}=\underbrace{-\nabla\times(\nabla\phi)}_{=0}-\frac{1}{c}\frac{\partial}{\partial t}(\nabla\times\vec{A}).$$

$$\xrightarrow{②}\quad =-\frac{1}{c}\frac{\partial\vec{B}}{\partial t}\qquad ⑦$$

Take the div of ④ to get

$$\nabla\times(\nabla\times\vec{B})=\nabla\underbrace{(\nabla\cdot\vec{B})}_{=0}-\nabla^2\vec{B}$$

$$=\frac{1}{c}\frac{\partial}{\partial t}(\nabla\times\vec{E})+\frac{4\pi}{c}\nabla\times\vec{J}$$

$$\xrightarrow{⑦} \quad = -\frac{1}{c^2}\frac{\partial^2 \vec{B}}{\partial t^2} + \frac{4\pi}{c}\nabla\times\vec{J}.$$

That is, $$\Box\vec{B} = -\frac{4\pi}{c}\nabla\times\vec{J}.$$

Thus, we have

$$\Box\vec{E} = 4\pi\left(\nabla\rho + \frac{1}{c^2}\frac{\partial\vec{J}}{\partial t}\right) \qquad \Box\vec{B} = -\frac{4\pi}{c}\nabla\times\vec{J}.$$

But they do not imply Maxwell's equations.

Let's find another solution to these two eqn.

−1

(How to find?)

One way is to see that these are second order DEs instead of 1st order as the Maxwell's eqns, so something linear in coordinates will work.

Another way is to realize that these are wave eqns while Maxwell's eqns are necessarily not.

2. angular momentum density of EM field:

$$\vec{l} = \vec{x} \times \vec{p} = \frac{1}{4\pi c} \vec{x} \times (\vec{E} \times \vec{B})$$

Show $\vec{L} = \int \vec{l} \, d^3\vec{x}$ is conserved for $(\rho = 0, \vec{J} = 0)$

With $\vec{E}, \vec{B}$ vanishing rapidly as $|\vec{x}| \to \infty$.

$$\vec{L} = \int_{\mathcal{V}} \frac{1}{4\pi c} \vec{x} \times (\vec{E} \times \vec{B}) \, d^3\vec{x}.$$

$\mathcal{V}$ is a large region extending to ∞.

$$\vec{x} \times (\vec{E} \times \vec{B}) = \vec{E} (\vec{x} \cdot \vec{B}) - \vec{B} (\vec{x} \cdot \vec{E})$$

$$\vec{E} = -\nabla\phi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t}.$$

$$\vec{B} = \nabla \times \vec{A}.$$

$$\nabla \cdot \vec{E} = 4\pi\rho.$$

$$\nabla \times \vec{B} - \frac{1}{c}\frac{\partial \vec{E}}{\partial t} = \frac{4\pi}{c}\vec{J}.$$

-5

3. For an arbitrary spatially bounded charge-current source, the electric dipole moment $\vec{p} = \int \rho \vec{x}\, d^3x$ satisfies

$$\frac{d\vec{p}}{dt} = \int \vec{J}\, d^3x .$$

$$\frac{d\vec{p}}{dt} = \int \frac{d\rho}{dt} \vec{x}\, d^3x + \int \rho \frac{d\vec{x}}{dt} d^3x .$$

Since $\nabla \cdot \vec{J} = -\partial\rho/\partial t$.

$$\frac{d\vec{p}}{dt} = \int -(\nabla \cdot \vec{J}) \vec{x}\, d^3x + \int \rho \frac{d\vec{x}}{dt} d^3x .$$

-5

Sorry I do not have enough time to work out all the problems now. And I will update this pset later.

HW 6 Electrodynamics yuxiang@uchicago.edu

5. Antenna of length L.

$$\vec{J}(t,z) = I_0 \sin(\pi z/L) \cos(\omega t)\, \delta(x)\, \delta(y)\, \hat{z}.$$

(a) charge density $\rho(t,z)$.

From charge-current conservation law we know

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \vec{J} = 0.$$

$$\nabla \cdot \vec{J} = \frac{\partial J_x}{\partial x} + \frac{\partial J_y}{\partial y} + \frac{\partial J_z}{\partial z} = \frac{\partial J_z}{\partial z}$$

$$\Rightarrow \frac{\partial \rho}{\partial t} = I_0 \cdot \frac{\pi}{L} \cos\left(\frac{\pi z}{L}\right) \cos\omega t\, \delta(x)\, \delta(y).$$

$$\text{So, } \rho(t,z) = \frac{\pi I_0}{\omega L} \cos\left(\frac{\pi z}{L}\right) \sin\omega t\, \delta(x)\,\delta(y) + \overset{\overset{0}{\shortparallel}}{C_0}.$$

(b) Using multipole expansion, at large distances,

$$\vec{E}(t,\vec{x}) = -\nabla\phi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t}$$

$$= \frac{1}{c^2 |\vec{x}|} \left[\left(\hat{x} \cdot \frac{d^2\vec{p}}{dt^2}\right)\hat{x} - \frac{d^2\vec{p}}{dx^2} \right]_{ret} + O\left(\frac{1}{|\vec{x}|^2}\right)$$

While $\vec{p} = \int \vec{x}\, \rho(\vec{x})\, d^3x$.

I get $\vec{p} = \iiint \overset{\hat{z}}{z}\, \rho(t,z)\, dx\, dy\, dz$

$$= \int_0^L z \cdot \frac{\pi I_0}{\omega L} \cos\left(\frac{\pi z}{L}\right) \sin\omega t \, dz \cdot \hat{z}$$

$$= \int_0^\pi \frac{I_0}{\omega} x \cos x \sin\omega t \cdot \frac{L}{\pi} dx \cdot \hat{z}$$

$$= \frac{I_0 L \hat{z}}{\omega\pi} \sin\omega t \underbrace{\int_0^\pi x \cos x \, dx}_{} = -2$$

$$= \frac{-2 I_0 L \sin\omega t}{\omega\pi} \hat{z}$$

So, $$\frac{d^2\vec{p}}{dt^2} = \frac{2 I_0 \omega L \sin\omega t}{\pi} \hat{z}$$

$$\frac{d^2\vec{p}}{dx^2} = 0. \qquad \text{at retarded time, } t = t - \frac{z}{c}.$$

So, $$\boxed{\begin{aligned} \vec{E}(t,\vec{x}) &= \frac{1}{c^2|\vec{x}|} \frac{2 I_0 \omega L \sin\omega t'}{\pi} \cos\theta \, \hat{x} \Bigg|_{t' = t - \frac{|\vec{x}|}{c}} \\ &= \frac{2 I_0 \omega L \sin\omega\left(t - \frac{|\vec{x}|}{c}\right)}{\pi c^2 |\vec{x}|} \cos\theta \, \hat{x} \end{aligned}}$$

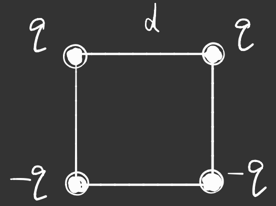
The dir doesn't look right

$$\vec{B}(t,\vec{x}) = \nabla \times \vec{A} = -\frac{1}{c^2|\vec{x}|} \hat{x} \times \frac{d^2\vec{p}}{dt^2}\Bigg|_{ret} + O\left(\frac{1}{|\vec{x}|^2}\right)$$

$$= -\frac{1}{c^2|\vec{x}|} \cdot \frac{2 I_0 \omega L \sin\omega t'}{\pi} \hat{x} \times \hat{z}$$

$$= -\frac{2 I_0 \omega L \sin\omega t'}{\pi c^2 |\vec{x}|} \hat{\varphi}$$

6. (i)



The energy flux is given by the Poynting vector,

Using multipole expansion, we get

$$\mathcal{S} = \frac{c}{4\pi} \vec{E} \times \vec{B} = \frac{1}{4\pi c^3 |\vec{x}|^2} \left[\left| \frac{d^2 \vec{p}}{dt^2} \right|^2_{ret} - \left(\hat{x} \cdot \frac{d^2 \vec{p}}{dt^2} \right)^2_{ret} \right] \hat{x}$$

Power radiated:

$$P = \frac{d\mathcal{E}}{dt} = \int \mathcal{S} \cdot \hat{x} \, dA = \frac{1}{4\pi c^3} \left| \frac{d^2 \vec{p}}{dt^2} \right|^2_{ret} \int \sin^2\theta \sin\theta \, d\theta \, d\varphi$$

$$= \frac{2}{3c^3} \left| \frac{d^2 \vec{p}}{dt^2} \right|^2_{ret}.$$

Here $\vec{p} = q \cdot d(\hat{x} + \hat{y}) + qd(-\hat{x} + \hat{y}) = 2qd\,\hat{y}$.

Considering the rotation,

$$\vec{p} = 2qd \cos\omega t \, \hat{x} + 2qd \sin\omega t \, \hat{y}$$

$$\frac{d^2 \vec{p}}{dt^2} = -2qd\omega^2 \cos\omega t \, \hat{x} - 2qd\omega^2 \sin\omega t \, \hat{y}.$$

So we have $P_1 = \frac{2}{3c^3} \cdot \left(2qd\omega^2\right)^2 = \frac{8q^2d^2\omega^4}{3c^3}$.

We also have contribution from electric quadrupole radiation.

$$Q_{ij} = \int \rho(x)\left(3x_i x_j - r^2\delta_{ij}\right) d^3x.$$

$$Q_{11} = q\cdot(3x_1^2 - r_1^2) + q(3x_2^2 - r_2^2) - q(3x_3^2 - r_3^2) - q(3x_4^2 - r_4^2)$$

$$= 3q\cdot\left(\frac{d}{2}\right)^2\left[\cos^2\omega t + \cos^2\left(\omega t + \frac{\pi}{2}\right) - \cos^2(\omega t + \pi) - \cos^2\left(\omega t + \frac{3\pi}{2}\right)\right] = 0.$$

$$Q_{21} = 3q\left(x_1y_1 + x_2y_2 - x_3y_3 - x_4y_4\right)$$

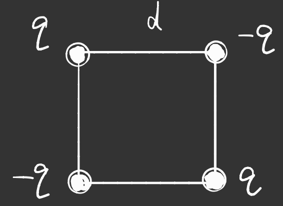
$$= 3q\cdot\left(\frac{d}{2}\right)^2\left[\cos\omega t\sin\omega t + \cos\left(\omega t+\frac{\pi}{2}\right)\sin\left(\omega t+\frac{\pi}{2}\right) - \cos(\omega t+\pi)\sin(\omega t+\pi) - \cos\left(\omega t+\frac{3\pi}{2}\right)\sin\left(\omega t+\frac{3\pi}{2}\right)\right] = 0.$$

$$Q_{22} = 3q\left(y_1^2 + y_2^2 - y_3^2 - y_4^2\right)$$

$$= 3q\cdot\left(\frac{d}{2}\right)^2\left[\sin^2\omega t + \sin^2\left(\omega t+\frac{\pi}{2}\right) - \sin^2(\omega t+\pi) - \sin^2\left(\omega t+\frac{3\pi}{2}\right)\right] = 0.$$

So, $P_1 \propto q^2 d^2 \Omega^4$. $\boxed{n_1 = 2.\ n_2 = 2.\ n_3 = 4.}$

(ii)



$\vec{P} = 0$ in this case.

Consider quadrupole contribution.

$$Q_{ij} = \int \rho(x)\left(3x_i x_j - r^2\delta_{ij}\right) d^3x.$$

$$Q_{11} = -q\cdot(3x_1^2 - r_1^2) + q(3x_2^2 - r_2^2) - q(3x_3^2 - r_3^2) + q(3x_4^2 - r_4^2)$$

$$= 3q\cdot\left(\frac{d}{2}\right)^2\left[-\cos^2\omega t + \cos^2\left(\omega t + \frac{\pi}{2}\right) - \cos^2(\omega t + \pi) + \cos^2\left(\omega t + \frac{3\pi}{2}\right)\right]$$

$$= \frac{3}{2} q d^2 \cos 2\omega t.$$

$$Q_{21} = 3q\left(-x_1 y_1 + x_2 y_2 - x_3 y_3 + x_4 y_4\right)$$

$$= 3q\cdot\left(\frac{d}{2}\right)^2\left[-\cos\omega t \sin\omega t + \cos\left(\omega t + \frac{\pi}{2}\right)\sin\left(\omega t + \frac{\pi}{2}\right) - \cos(\omega t + \pi)\sin(\omega t + \pi)\right.$$
$$\left. + \cos\left(\omega t + \frac{3\pi}{2}\right)\sin\left(\omega t + \frac{3\pi}{2}\right)\right]$$

$$= \frac{3}{2} q d^2 \sin 2\omega t.$$

$$Q_{22} = 3q\left(-y_1^2 + y_2^2 - y_3^2 + y_4^2\right)$$

$$= 3q\cdot\left(\frac{d}{2}\right)^2\left[\sin^2\omega t + \sin^2\left(\omega t + \frac{\pi}{2}\right) - \sin^2(\omega t + \pi) - \sin^2\left(\omega t + \frac{3\pi}{2}\right)\right]$$

$$= -\frac{3}{2} q d^2 \cos 2\omega t.$$

So, $\overleftrightarrow{Q} = \frac{3}{2} q d^2 \left[\cos 2\omega t \, \hat{x}\hat{x} + 2 \sin \omega t \, (\hat{x}\hat{y} + \hat{y}\hat{x}) - \cos 2\omega t \, \hat{y}\hat{y} \right]$

It give rise to radiated power proportional to

$$P \propto \left| \frac{d^3 Q_{ij}}{dt^3} \right|^2$$

So, $P \propto (q d^2 \cdot \omega^3)^2 = q^2 d^4 \omega^6$.

$$\boxed{n_1 = 2,\ n_2 = 4,\ n_3 = 6.}$$

7. A point charge on the end of a spring.

(a) $\vec{x} = \alpha \cos \omega t \, \hat{z}$ $\qquad \omega = \sqrt{\frac{k}{m}}$. $\qquad \omega \alpha = v$.

Using multipole expansion,

$$\vec{A}(t, \vec{x}) = \frac{1}{c|\vec{x}|} \frac{d\vec{p}}{dt} \Big|_{ret}$$

$$\phi(t, \vec{x}) = \frac{q}{|\vec{x}|} + \frac{1}{c|\vec{x}|} \hat{x} \cdot \frac{d\vec{p}}{dt} \Big|_{ret}$$

$$\vec{p} = q\vec{x} = q\alpha\cos\omega t\,\hat{z}.$$

$$\vec{A}(t,\vec{x}) = \frac{1}{c\cdot\alpha\cos\omega t}\cdot -q\alpha\sin\omega t\Big|_{ret}\,\hat{z}$$

$$= -q\sin\left[\omega\left(t-\frac{\alpha\cos\omega t}{c}\right)\right]\Big/ c\cdot\cos\omega t\quad\hat{z}.$$

$$\phi(t,\vec{x}) = \frac{q}{\alpha\cos\omega t} + \frac{1}{c\alpha\cos\omega t}\cdot q\omega\alpha\sin\omega t\Big|_{ret}$$

$$= \frac{q}{\alpha\cos\omega t} + \frac{q\omega}{c}\,\frac{\sin\left[\omega\left(t-\frac{\alpha\cos\omega t}{c}\right)\right]}{\cos\omega t}$$

(b) radiated power is given by:

$$P = \frac{d\varepsilon}{dt} = \int \mathcal{S}\cdot\hat{x}\,dA = \frac{2}{3c^3}\left|\frac{d^2\vec{p}}{dt^2}\right|^2_{ret}.$$

So, $$P = \frac{2}{3c^3}\left|q\alpha\omega^2\cos\omega t\right|^2_{ret}$$

$$= \frac{2q^2\alpha^2\omega^4}{3c^3}\cos^2\left[\omega\left(t-\frac{\alpha\cos\omega t}{c}\right)\right]$$

(c) The energy of oscillation :

$$E = E_{kinetic} + E_{potential} = k\alpha^2 .$$

Should obey $\frac{dE}{dt} = P$.

$$\Rightarrow \quad 2k\alpha \frac{d\alpha}{dt} = \frac{2q^2\alpha^2\omega^4}{3c^3} \cos^2\left[\omega\left(t - \frac{\alpha\cos\omega t}{c}\right)\right]$$

$$\Rightarrow \quad \frac{d\alpha}{dt} = \frac{kq^2\alpha\,\omega^4}{3c^3} \cos^2\left[\omega\left(t - \frac{\alpha\cos\omega t}{c}\right)\right]$$

$$\alpha(t=0) = \alpha_0 .$$

$$t' = t - \frac{\alpha\cos\omega t}{c} . \qquad dt' = dt + \frac{\omega\alpha\sin\omega t}{c} dt .$$

Solve to get $\alpha(t)$.

(Do not know how to solve at this moment.)

looking at it as $\frac{d\alpha}{\alpha} = (\text{something}) f(t)\, dt$

-1.

might help !!

9. e^{ikz} expanded in spherical harmonics.

This equation was applied in partial wave analysis in quantum mechanics.

$$e^{ikz} = \sum_{l=0}^{\infty} i^l \sqrt{4\pi(2l+1)}\, j_l(kr)\, Y_{l0}(\theta,\varphi) = \sum_{l=0}^{\infty} i^l (2l+1)\, j_l(kr)\, P_l(\cos\theta)$$

First show that

$$e^{ikx} = \lim_{\lambda\to\infty} \lambda e^{-ik\lambda} \frac{e^{ik|\vec{x}-\lambda\hat{z}|}}{|\vec{x}-\lambda\hat{z}|}.$$

show?

-2

Then, from eqn(5.95) we know

$$G_H(\vec{x},\vec{x}') = \frac{e^{ik(\vec{x}-\lambda\hat{z})}}{|\vec{x}-\lambda\hat{z}|} = \sum_{l,m} 4\pi ik\, j_l(kr)\, h_l^{(1)}(k\lambda)\, Y^*_{lm}(\theta',\varphi')\, Y_{lm}(\theta,\varphi)$$

While $h_l^{(1)}(\lambda k)$ as $\lambda\to\infty$ is

$$h_l^{(1)}(\lambda k) = (-i)^{l+1} \frac{e^{i\lambda k}}{\lambda k} + O\left(\frac{1}{\lambda^2 k^2}\right)$$

So,

$$\frac{e^{ik(\vec{x}-\lambda\hat{z})}}{|\vec{x}-\lambda\hat{z}|} = \sum_{l,m} \frac{4\pi}{\lambda}(-i)^l e^{i\lambda k} j_l(kr)\, Y^*_{lm}(\theta',\varphi')\, Y_{lm}(\theta,\varphi)$$

We get

$$\lambda e^{-ik\lambda} \frac{e^{ik(\vec{x}-\lambda\hat{z})}}{|\vec{x}-\lambda\hat{z}|} = \sum_{l,m} 4\pi (-i)^l j_l(kr) Y_{lm}^*(\theta,\varphi) Y_{lm}(\theta,\varphi)$$

Also, for that

$$\sum_{m=-l}^{l} Y_{lm}^*(\theta,\varphi) Y_{lm}(\hat{z}) = \frac{2l+1}{4\pi} P_l(\cos\theta).$$

$$\Rightarrow \quad \lambda e^{-ik\lambda} \frac{e^{ik(\vec{x}-\lambda\hat{z})}}{|\vec{x}-\lambda\hat{z}|} = \sum_{l} (2l+1)\cdot(-i)^l j_l(kr) P_l(\cos\theta)$$

Take the limit of $\lambda \to \infty$.

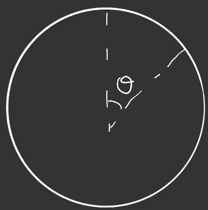
I know that

$$e^{ikz} = \sum_{l=0}^{\infty} i^l (2l+1) j_l(kr) P_l(\cos\theta)$$

So, the equation is proved.

E & M midterm yuxiang@uchicago.edu

(1)



$\vec{E}_0 = c\hat{z}$. Find $\sigma(\theta,\varphi)$.

We have Poisson eqn:

$\nabla^2\phi = -4\pi\rho$. And have the form of electric potential:

$$\phi(r) = \begin{cases} \sum_{lm} A_{lm} r^l Y_{lm}(\theta,\varphi) & (r \le R) \\ \sum_{lm} B_{lm} \frac{1}{r^{l+1}} Y_{lm}(\theta,\varphi). & (r > R) \end{cases}$$

We have $\phi(r=\infty) = -\vec{E}_0 \cdot \vec{r} = -cr\cos\theta$.

Potential inside sphere should be of form

$$\phi(\vec{r}) = a + b\vec{E}^{ext}\cdot\vec{r} = a + bcr\cos\theta.$$

Potential outside sphere should be

$$\phi(\vec{r}) = -cr\cos\theta + \sum_{lm} B_{lm}\frac{1}{r^{l+1}}Y_{lm}(\theta,\varphi).$$

Continuity of potential at $r=R$:

$$a+bcR\cos\theta = -cR\cos\theta + \frac{B_{00}}{r}Y_{00} + \frac{B_{10}}{r^2}Y_{10}$$

$\frac{\partial\phi}{\partial r}$ should equal at $r=R$

$$bc\cos\theta = -c\cos\theta - \frac{B_{00}}{R^2}Y_{00} - \frac{2B_{10}}{r^3}Y_{10}.$$

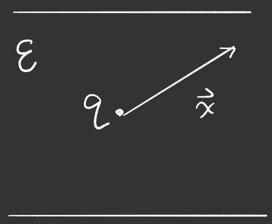
Solve to get

$$\phi(\vec{r}) = \begin{cases} 0. & (r<R) \\ -cr\cos\theta + c\frac{R^3}{r^2}\cos\theta. & (r\geq R) \end{cases}$$

So
$$\vec{E}(r=R) = \left.\frac{\partial\phi}{\partial r}\right|_{r=R}$$
$$= -3c\cos\theta\,\hat{r}$$

That is, $\sigma(\theta,\varphi) = \frac{E_n}{4\pi} = -\frac{3c}{4\pi}\cos\theta$

(2)



We have $\nabla^2 \langle \phi \rangle = -\frac{4\pi}{\varepsilon} \langle \rho_f \rangle$.

$$\langle \vec{D} \rangle = \varepsilon \langle \vec{E} \rangle .$$

We could get macroscopically averaged Maxwell eqn take exactly the same form in vacuum, except that the free charge density is renormalized by ε.

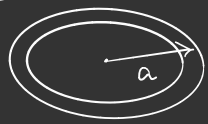
So, $\vec{E} = \frac{q}{\varepsilon r^2} \hat{r}$. $\phi = \frac{q}{\varepsilon r}$.

Since $\langle \vec{D} \rangle = \varepsilon \langle \vec{E} \rangle$.

$$\langle \vec{P} \rangle = \chi \langle \vec{E} \rangle = \frac{\varepsilon - 1}{4\pi} \langle \vec{E} \rangle$$

We get $\langle \vec{P} \rangle = \frac{\varepsilon - 1}{\varepsilon} \cdot \frac{q}{4\pi r^2} \hat{r}$.

(3)



$$\vec{J} = I \frac{1}{a} \delta(r-a)\, \delta(\theta - \tfrac{\pi}{2})\, \hat{\varphi}$$

(a) Using multipole expansion we know

$$A_i = \frac{\alpha_i}{|\vec{x}|} + \frac{\sum_j \beta_{ij} \hat{x}_j}{|\vec{x}|^2} + \cdots$$

With $\quad \alpha_i = \frac{1}{c} \int J_i(\vec{x}')\, d^3 x'$.

$$d^3 \vec{x}' = r^2 \sin\theta \, dr\, d\theta\, d\varphi$$

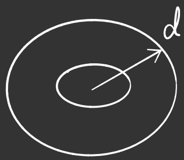
$$\alpha_\varphi = \frac{1}{c} \int I \frac{1}{a} \delta(r-a)\, \delta(\theta - \tfrac{\pi}{2}) \cdot r^2 \sin\theta\, dr\, d\theta\, d\varphi$$

$$= \frac{1}{c} \cdot \frac{I}{a} \cdot \int a^2 \sin\frac{\pi}{2} \cdot d\varphi$$

$$= \frac{2\pi a I}{c}.$$

$$\therefore \vec{A}_1 = \frac{2\pi a I}{c|\vec{x}|} \hat{\varphi}.$$

(b)



$$B_2 = \frac{4\pi}{c} \cdot \frac{I'}{2R} \quad (-\hat{z} \text{ direction})$$

$$\vec{B}_1 = \nabla \times \vec{A}_1$$

$$= \frac{2\pi a I}{c} \cdot \frac{\cos\theta}{r^2 \sin\theta} \hat{r}$$

$$E^{int} = \frac{1}{4\pi} \int \vec{B}_1 \cdot \vec{B}_2 \, d^3x$$

$$= \frac{1}{c} \int \vec{A}_1 \cdot \vec{J}_2 \, d^3x$$

$$= \frac{1}{c} \cdot \frac{2\pi a I}{c d} \cdot I' = \frac{2\pi a I I'}{c^2 d}$$