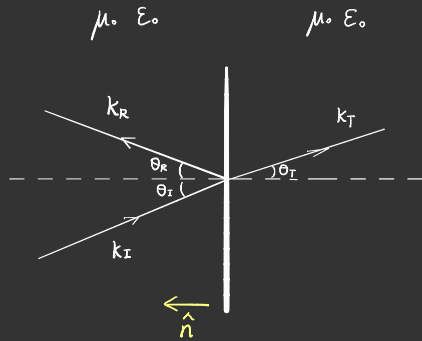
3. $\vec{C}_I$ lies in the plane determined by

$\vec{k}_I$ and the normal to the medium.



We have boundary conditions

$$(C_I - C_R)\cos\theta_I = C_T\cos\theta_T$$

$$H_I + H_R = H_T \Rightarrow \sqrt{\varepsilon_1}(C_I + C_R) = \sqrt{\varepsilon_2}\, C_T$$

Solve to get

$$\frac{C_R}{C_I} = \frac{\sqrt{\varepsilon_2}\cos\theta_I - \sqrt{\varepsilon_1}\cos\theta_T}{\sqrt{\varepsilon_2}\cos\theta_I + \sqrt{\varepsilon_1}\cos\theta_T} = \frac{\tan(\theta_I - \theta_T)}{\tan(\theta_I + \theta_T)}.$$

$$\frac{C_T}{C_I} = \frac{2\sqrt{\varepsilon_1}\cos\theta_I}{\sqrt{\varepsilon_2}\cos\theta_I + \sqrt{\varepsilon_1}\cos\theta_T} = \frac{2\cos\theta_I\sin\theta_T}{\sin(\theta_I+\theta_T)\cos(\theta_I-\theta_T)}$$

For that $n_i = c\sqrt{\varepsilon_i}$,

$$\frac{C_R}{C_I} = \frac{\frac{n_2}{n_1}\cos\theta_I - \cos\theta_T}{\frac{n_2}{n_1}\cos\theta_I + \cos\theta_T}.$$

When $\theta_I = \theta_B = \tan^{-1}\left(\frac{n_2}{n_1}\right)$. $\Rightarrow \frac{\sin\theta_I}{\cos\theta_I} = \frac{n_2}{n_1}$.

Then I get $\frac{C_R}{C_I} = \frac{\sin\theta_I - \cos\theta_T}{\sin\theta_I + \cos\theta_T}$.

While $\frac{n_2}{n_1} = \frac{\sin\theta_I}{\cos\theta_I}$, $n_1 \sin\theta_I = n_2 \sin\theta_T$.

We have $\cancel{n_2} \cos\theta_I = \cancel{n_2} \sin\theta_T$.

$$\Rightarrow \frac{C_R}{C_I} = \frac{\sin\theta_I - \cos\theta_T}{\sin\theta_I + \cos\theta_T} = 0. \qquad \vec{C_R} = 0.$$

Thus, if an "unpolarized" EM wave is incident at the Brewster angle, the reflected wave will be polarized in the direction perpendicular to the plane determined by $\vec{k}_I$ and the normal to the medium.

5. $F(k) = e^{-(k-k_0)^2/\alpha^2}$.

$$\omega(k) = \omega(k_0) + V_g(k-k_0) + \frac{1}{2}\lambda(k-k_0)^2.$$

$$\lambda = \left.\frac{d^2\omega}{dk^2}\right|_{k_0}.$$

Then amplitude of any EM field quantity ψ

will vary in spacetime as

$$\psi(t,z) = \int F(k)\, e^{-i\omega(k)t + ikz}\, dk$$

$$\omega(k) = \omega(k_0) - V_g k_0 + \frac{1}{2}\lambda k_0^2$$
$$+ V_g k - \lambda k_0 k + \frac{1}{2}\lambda k^2.$$

$$\psi(t,z) = \int e^{-(k-k_0)^2/\alpha^2}\, e^{-i\left[\omega(k_0) - V_g k_0 + \frac{1}{2}\lambda k_0^2\right]t}$$
$$e^{-i(V_g k - \lambda k_0 k)t}\, e^{-\frac{i}{2}\lambda k^2 t}\, e^{ikz}\, dk$$

$$= \int e^{-\left(\frac{1}{\alpha^2} + \frac{i}{2}\lambda t\right)k^2 + \left[\frac{2k_0}{\alpha^2} - i(V_g - \lambda k_0)t + iz\right]k - \frac{k_0^2}{\alpha^2} - i\omega(k_0)t - iV_g k_0 t - \frac{i}{2}\lambda k_0^2 t}\, dk$$

$$= \sqrt{\frac{\pi}{\frac{1}{\alpha^2} + \frac{i\lambda t}{2}}}\; e^{\left[\frac{2k_0}{\alpha^2} - i(V_g - \lambda k_0)t + iz\right]^2 / 4\left(\frac{1}{\alpha^2} + \frac{i\lambda t}{2}\right)}\; e^{-\frac{k_0^2}{\alpha^2} - i\omega(k_0)t - iV_g k_0 t - \frac{i\lambda}{2}k_0^2 t}$$

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2. $\chi(\vec{x}) = \vec{A}(\vec{x},\alpha) \exp(iS(\vec{x})/\alpha)$.

Expand $\vec{A}$ as

$$\vec{A}(\vec{x},\alpha) = \sum_{\alpha=0}^{\infty} \vec{A}^{(n)}(\vec{x})\, \alpha^n .$$

$$= A^{(0)} + A^{(1)}\alpha + A^{(2)}\alpha^2 + \cdots$$

From (7.4) we have

$$\nabla^2 \chi + \frac{\omega^2}{c^2}\chi = 0 .$$

While $\nabla^2(fg) = f\nabla^2 g + g\nabla^2 f + 2\nabla g \cdot \nabla f$.

We have

$$-\left|\nabla \frac{\vec{S}}{\alpha}\right|^2 \vec{A} + \nabla^2 \vec{A} + \frac{\omega^2}{\alpha^2 c^2}\vec{A} + i\nabla^2\left(\frac{\vec{S}}{\alpha}\right)\vec{A} + 2i\vec{\nabla}\left(\frac{\vec{S}}{\alpha}\right)\cdot\nabla\vec{A} = 0$$

Look into each power term:

α^0 term :

$$-|\nabla \vec{S}|^2 A^{(2)} + \nabla^2 A^{(0)} + \frac{\omega^2}{c^2} A^{(2)} + i\nabla^2 \vec{S} A^{(1)} + 2i\,\nabla(\vec{S})\cdot\nabla A^{(1)} = 0.$$

α^{-1} term :

$$-|\nabla \vec{S}|^2 A^{(1)} + \frac{\omega^2}{c^2} A^{(1)} + i\nabla^2 S\, A^{(0)} + 2i\nabla \vec{S}\, \nabla A^{(0)} = 0.$$

$$\Rightarrow \left(-|\nabla \vec{S}|^2 + \frac{\omega^2}{c^2}\right) A^{(1)} = 0$$

$$A^{(0)} \nabla^2 S + 2\nabla \vec{S}\cdot\nabla \vec{A}^{(0)} = 0.$$

α^{-2} term :

$$-|\nabla \vec{S}|^2 A^{(0)} + \frac{\omega^2}{c^2} A^{(0)} = 0.$$

They are (7.7) and (7.8) for $A^{(0)}$.

For $n \geq 1$, we have

α^n term :

$$-|\nabla \vec{S}|^2 A^{(n+2)} + \nabla^2 A^{(n)} + \frac{\omega^2}{c^2} A^{(n+2)} + i\nabla^2 S\, A^{(n+1)} + 2i\nabla S\, \nabla A^{(n+1)} = 0$$

$$\Rightarrow \quad \nabla^2 A^{(n)} + \left(\frac{\omega^2}{c^2} - |\nabla \vec{S}|^2\right) A^{(n+2)} = 0.$$

$$\nabla^2 \vec{S} \, A^{(n+1)} + 2 \nabla \vec{S} \, \nabla A^{(n+1)} = 0.$$

3. (a) $k \cdot \nabla \Theta = k \cdot \nabla(\nabla \cdot \vec{k})$

$$= \sum_{ij} k_i \partial_i (\partial_j k_j)$$

$$= -\sum_{ij} (\partial_i k_j)(\partial_i k_j)$$

$$= -\sum_{ij} \left(\sigma_{ij} + \frac{1}{2}\Theta[\delta_{ij} - n_i n_j]\right)^2$$

$$= -\sum_{ij} \left\{\sigma_{ij}^2 + \sigma_{ij}\Theta\left[\delta_{ij} - \frac{c^2}{\omega^2}k_i k_j\right] + \frac{1}{4}\Theta^2\left(\delta_{ij} - \frac{c^2}{\omega^2}k_i k_j\right)^2\right\}$$

$$= -\sum_{ij} \sigma_{ij}^2 - \sum_{ij}\left[\sigma_{ij}\Theta\delta_{ij} - \left(\sigma_{ij}\Theta + \frac{1}{2}\Theta^2\right)\frac{k_i k_j}{|\vec{k}|^2} - \frac{1}{4}\Theta^2\left(\frac{k_i k_j}{|\vec{k}|^2}\right)^2\right]$$

For that $\sum_{ij} \frac{k_i k_j}{|\vec{k}|^2} = 1$.

The equation becomes

$$\Rightarrow \quad = -\sum_{ij} \sigma_{ij}^2 - \frac{1}{2}\Theta^2 - \sum_{ij}\left[\sigma_{ij}\Theta\delta_{ij} - \sigma_{ij}\Theta\frac{k_i k_j}{|\vec{k}|^2} - \frac{1}{4}\Theta^2\left(\frac{k_i k_j}{|\vec{k}|^2}\right)^2\right]$$

$$= -\sum_{ij} \sigma_{ij}^2 - \frac{1}{2}\Theta^2 - \sum_{ij}\Big[\underbrace{\sigma_{ij}\Theta\delta_{ij} - \sigma_{ij}\Theta\delta_{ij}}_{\text{cancell out}} - \frac{1}{4}\Theta^2\Big(\frac{k_i k_j}{|k|^2}\Big)^2\Big]$$

$$= -\sum_{ij} \delta_{ij}^2 - \frac{1}{2}\Theta^2 + \sum_{ij}\frac{1}{4}\Theta^2\delta_{ij}^2$$

$$= -\frac{1}{2}\Theta^2 - \sum_{ij}(\sigma_{ij})^2.$$

(b) $$k\cdot\nabla\sigma_{ij} = \sum_i k_i\partial_i\Big\{\partial_i k_j - \frac{1}{2}\Theta\big[\delta_{ij} - n_i n_j\big]\Big\}$$

$$= \sum_i k_i\partial_i(\partial_i k_j) - \frac{1}{2}\sum_i k_i\partial_i\Theta\,\delta_{ij} + \underline{\frac{1}{2}\sum_i k_i\partial_i\Theta\, k_i k_j/|k|^2}.$$

We know from (7.12) that

$$\underline{\sum_i k_i\partial_i k_j = 0.}$$

So the <u>third term is 0,</u> and the equation becomes

no

$$= \sum_i k_i\partial_i\Big[(\partial_i k_j) - \frac{1}{2}\Theta\delta_{ij}\Big]$$

For that $\Theta \sigma_{ij} = \Theta \partial_i k_j - \frac{1}{2}\Theta^2 [\delta_{ij} - n_i n_j]$

We have $k \cdot \nabla \sigma_{ij} = -\Theta \sigma_{ij}$.

(2) From above we know

$$\vec{k} \cdot \nabla \Theta = -\frac{1}{2}\Theta^2 - \sum_{ij} \sigma_{ij}^2 .$$

Then $k \cdot \nabla\left(\frac{1}{\Theta}\right) = \frac{-k \cdot \nabla \Theta}{\Theta^2}$

$$= +\frac{1}{2} \cdot 1 + \frac{\sum_{ij} \sigma_{ij}^2}{\Theta^2}$$

Which is non-zero.

So if we have $\Theta < 0$ at some point,

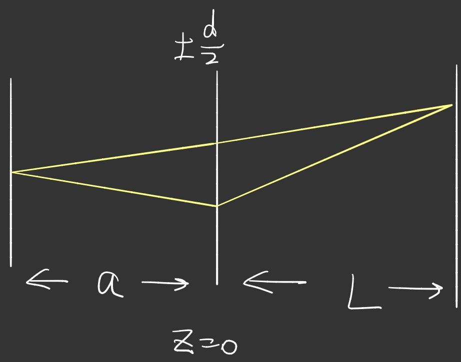
then $\frac{1}{\Theta}$ will also be < 0 but increase.

That causes a caustic $(\Theta \to -\infty)$ at some

distance along the ray.

6. (a)

From direct calcalation we know that the path length is



$$l_1 = \sqrt{a^2+\left(\frac{d}{2}\right)^2} + \sqrt{x^2+\left(y-\frac{d}{2}\right)^2+L^2}$$

$$l_2 = \sqrt{a^2+\left(\frac{d}{2}\right)^2} + \sqrt{x^2+\left(y+\frac{d}{2}\right)^2+L^2}$$

Assume that $|x|, |y| \ll L$.

We have

$$l_1 \approx \left[a^2+\left(\frac{d}{2}\right)^2\right]^{\frac{1}{2}} + L + \frac{x^2+\left(y-\frac{d}{2}\right)^2}{2L}$$

$$l_2 \approx \left[a^2+\left(\frac{d}{2}\right)^2\right]^{\frac{1}{2}} + L + \frac{x^2+\left(y+\frac{d}{2}\right)^2}{2L}.$$

(b) From above we know

$$\Delta l = \frac{yd}{2L}.$$

Also, we know that phase S is matched at the pinholes.

So, when $\frac{\omega}{c}\cdot\Delta l = 2\pi\cdot n$,

$$\frac{4\pi n c L}{\omega d}$$

$$\Rightarrow \quad y = \frac{2n\cdot 2\pi c L}{\omega d}.$$

n: integer number

intensity is maximum.

when $y = \frac{(2n+1)\cdot 2\pi c L}{\omega d}$.

intensity is minimum.

(c) (i) Source has Δy displacement

$$\text{Then} \quad l_1 \approx a + \frac{(\Delta y - \frac{d}{2})^2}{2a} + L + \frac{x^2 + (y-\frac{d}{2})^2}{2L}$$

$$l_2 \approx a + \frac{(\Delta y + \frac{d}{2})^2}{2a} + L + \frac{x^2 + (y+\frac{d}{2})^2}{2L}$$

$$\Rightarrow \quad \Delta l = \frac{d\Delta y}{2a} + \frac{yd}{2L}.$$

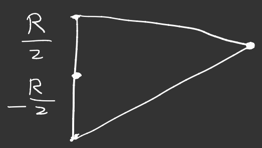
$$\left(\Delta y \approx \frac{ay}{L}\right)$$

So, when $\frac{\Delta y}{a}$ is comparable to $\frac{y}{L}$.

will Δy have obvious effect on the interference.

(ii) Source having finite size R.

This will induce additional phase difference from different points of the light source.



The superstition from these small light sources will induce

$$\Delta l' = R\frac{d}{a}.$$

$$\left(R \approx \frac{ay}{2L}\right)$$

So, if $R\frac{d}{a}$ and $\frac{yd}{2L}$ are comparable.

R will have obvious effect on interference.

(iii) $\Delta\omega$ spread of frequency.

each frequency will have its interference pattern.

$$y_1 = \frac{2n \cdot 2\pi cL}{\omega d}, \quad y_2 = \frac{(2n+1)\cdot 2\pi cL}{(\omega+\Delta\omega)d}.$$

When $y_1 = y_2$.

$$\frac{2n}{\omega} = \frac{2n+1}{\omega+\Delta\omega} \Rightarrow \Delta\omega = \frac{\omega}{2n}.$$

So, when $\Delta\omega$ is comparable to $\frac{\omega}{2n}$. (n is integer number)

it will have obvious effect on the interference pattern.

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(a) $$p^2 = 2\frac{Tr(\rho^2)}{[Tr(\rho)]^2} - 1.$$

$$Tr(\rho) = \rho_{11} + \rho_{22} = \frac{1}{\mu_0 c}\frac{1}{2T}\int_{t-T}^{t+T} E_1(t')E_1^*(t)\,dt' + \frac{1}{\mu_0 c}\frac{1}{2T}\int_{t-T}^{t+T} E_2(t')E_2^*(t)\,dt'$$

$$Tr(\rho^2) = Tr\left[\begin{pmatrix}\rho_{11} & \rho_{12}\\ \rho_{21} & \rho_{22}\end{pmatrix}\begin{pmatrix}\rho_{11} & \rho_{12}\\ \rho_{21} & \rho_{22}\end{pmatrix}\right] = \rho_{11}^2 + \rho_{22}^2 +$$

$$p^2 = \frac{2\left(\rho_{11}^2 + \rho_{22}^2 + 2\rho_{12}\rho_{21}\right)}{(\rho_{11}+\rho_{22})^2} - 1$$

$$= \frac{\rho_{11}^2 + \rho_{22}^2 + 4\rho_{12}\rho_{21} - 2\rho_{11}\rho_{22}}{(\rho_{11}+\rho_{22})^2} = \frac{(\rho_{11}-\rho_{22})^2 + 4\rho_{12}\rho_{21}}{(\rho_{11}+\rho_{22})^2} \geq 0.$$

Also, for that $\rho_{12}\rho_{21} \leq \rho_{11}\rho_{22}$. $\Rightarrow p^2 \leq 1$.

(From the conclusion that interference is largest when $\vec{C}_1 \approx \vec{C}_2 \equiv \vec{C}$.

When $p=1$. $$\int_{t-T}^{t+T} E_1(t')E_1^*(t)\,dt' \int_{t-T}^{t+T} E_2(t')E_2^*(t)\,dt'$$

$$= \int_{t-T}^{t+T} E_1(t')E_2^*(t)\,dt' \int_{t-T}^{t+T} E_2(t')E_1^*(t)\,dt'$$

Only when $E_1(t)/E_2(t) = \text{constant}$

will it obey this relation.

(b)

$$\rho = \begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix} = \frac{1}{2} I \mathbb{1} + \frac{1}{2} U \sigma_x + \frac{1}{2} V \sigma_y + \frac{1}{2} Q \sigma_z$$

$$= \frac{1}{2} \begin{pmatrix} I+Q & U-iV \\ U+iV & I-Q \end{pmatrix}$$

$$p^2 = \frac{(\rho_{11}-\rho_{22})^2 + 4\rho_{12}\rho_{21}}{(\rho_{11}+\rho_{22})^2} = \frac{Q^2+U^2+V^2}{I^2}.$$

$$\therefore p = \frac{\sqrt{U^2+V^2+Q^2}}{I}.$$

(c) For $p=1$,

(i) $U = V = 0$. This is the case when

$\rho_{12} = \rho_{21} = 0$. That is

$$\int_{t-T}^{t+T} E_1(t') E_2^*(t)\, dt' = \int_{t-T}^{t+T} E_2(t') E_1^*(t)\, dt' = 0.$$

So, either $E_1(t') = 0$ or $E_2(t') = 0$.

since t is random.

This corresponds to radiation that is linearly polarized

in either x or y direction.

(ii) $V = Q = 0$. $\rho_{11} = \rho_{22}$ $\rho_{21} = \rho_{12}$.

$$\int_{t-T}^{t+T} E_1(t') E_2^*(t)\, dt' = \int_{t-T}^{t+T} E_2(t') E_1^*(t)\, dt'$$

$$\int_{t-T}^{t+T} E_1(t') E_1^*(t)\, dt' = \int_{t-T}^{t+T} E_2(t') E_2^*(t)\, dt'$$

Only when $E_1(t) = E_2(t)$ will both equation be satisfied.

This corresponds to $45°$ linearly polarized radiation.

(iii) $U = Q = 0$.

$\rho_{21} + \rho_{12} = 0$. $\rho_{11} - \rho_{22} = 0$.

$$-\int_{t-T}^{t+T} E_1(t') E_2^*(t)\, dt' = \int_{t-T}^{t+T} E_2(t') E_1^*(t)\, dt'$$

$$\int_{t-T}^{t+T} E_1(t') E_1^*(t)\, dt' = \int_{t-T}^{t+T} E_2(t') E_2^*(t)\, dt'$$

This corresponds to the case when

$$E_1 = E\cos\omega t, \quad E_2 = E\sin\omega t.$$

Which is a circular polarized light.

9. Expand the incoming plane wave in vector spherical harmonics:

$$(\hat{x} \pm i\hat{y})\, e^{ikz} = \sum_{l=1}^{\infty} i^{l-1} \sqrt{\frac{4\pi(2l+1)}{l(l+1)}} \left[j_l(kr)\, \hat{r} \times \nabla Y_{l,\pm 1} \pm \frac{1}{k} \nabla \times j_l(kr)\, \hat{r} \times \nabla Y_{l,\pm 1} \right]$$

Consider only the case $m = +1$,

the outgoing wave could be written as:

$$\vec{A}_{scat} = e^{-i\omega t} \left[a_{l1} h_l^{(1)}(kr)\, \vec{r} \times \nabla Y_{l1} + b_{l1} \nabla \times \left(h_l^{(1)}(kr)\, \vec{r} \times \nabla Y_{l1} \right) \right]$$

For $r < R$, we have

$$\vec{A}_{int} = e^{-i\omega t} \left[a_{l1} j_l^{(1)}(k'r)\, \vec{r} \times \nabla Y_{l1} + b_{l1} \nabla \times \left(j_l^{(1)}(k'r)\, \vec{r} \times \nabla Y_{l1} \right) \right]$$

with $k' = \frac{n\omega}{c}$ $\quad n = \sqrt{\varepsilon/\varepsilon_0}$.

For a perfect conducting ball, $k' \to \infty$.

$$\Rightarrow \vec{A}_{int} = 0.$$

Match $\vec{E}_{\parallel} = 0$ at $r = R$ we have

$$-\frac{\partial}{\partial t} \vec{A}_{scat}(r = R) = 0$$

$$\Rightarrow \vec{A}_{scat} = f e^{-i\omega t}$$

$$\Rightarrow \frac{\partial f}{\partial t} = i\omega f \Big|_{r=R}.$$

that is,

$$\frac{\partial}{\partial t} a_{l_1} h_l^{(1)}(kR)\, \vec{r} \times \nabla Y_{l_1} + \frac{\partial}{\partial t} b_{l_1} \nabla \times \left(h_l^{(1)}(kR)\, \vec{r} \times \nabla Y_{l_1} \right)$$

$$= i\omega \left[a_{l_1} h_l^{(1)}(kR)\, \vec{r} \times \nabla Y_{l_1} + b_{l_1} \nabla \times \left(h_l^{(1)}(kR)\, \vec{r} \times \nabla Y_{l_1} \right) \right]$$

$$\Rightarrow a_{l_1} = 1, \quad b_{l_1} = 1.$$

$$\vec{A}_{scat} = e^{-i\omega t} \left[h_l^{(1)}(kr)\, \vec{r} \times \nabla Y_{l_1} + \nabla \times \left(h_l^{(1)}(kr)\, \vec{r} \times \nabla Y_{l_1} \right) \right]$$

10. $$\chi(\vec{x}) = -\frac{ik}{4\pi|\vec{x}|}(\cos\tilde{\theta}+\cos\theta)\, C e^{ik|\vec{x}|}\int_A e^{-ik\hat{x}\cdot\vec{x}'} e^{i(k_x x'+k_y y')}\, dx'\, dy'$$

$$= -\frac{ik}{4\pi|\vec{x}|}(\cos\tilde{\theta}+\cos\theta)\, C e^{ik|\vec{x}|} e^{-ikr_0}\int_{-a}^{a} e^{-ikx\sin\theta_1}\, dx \int_{-b}^{b} e^{-iky\sin\theta_2}\, dy$$

$$= -\frac{ik}{4\pi|\vec{x}|}(\cos\tilde{\theta}+\cos\theta)\, C e^{ik|\vec{x}|} e^{-ikr_0}\,\frac{\sin u_1}{u_1}\,\frac{\sin u_2}{u_2}.$$

of which $u_1 = \frac{2\pi a}{\lambda}\sin\theta_1$, $u_2 = \frac{2\pi b}{\lambda}\sin\theta_2$.

So, $\psi = \chi(\vec{x})\, e^{-i\omega t}$

$$= -\frac{ik}{4\pi|\vec{x}|}(\cos\tilde{\theta}+\cos\theta)\, C e^{-ikr_0} e^{i(k|\vec{x}|-\omega t)}\,\frac{\sin u_1}{u_1}\,\frac{\sin u_2}{u_2}.$$

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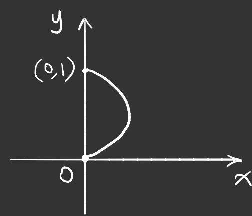
1. (a) For Euclidean geometry,



$$(\vec{x}_1, \vec{x}_2) = \vec{x}_1 \cdot \vec{x}_2 = \sum_{i,j} e_{ij} x^i x^j.$$

where $e_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

path length is given by

$$\Delta \ell = \sqrt{\sum_{ij} e_{ij} \Delta x^i \Delta x^j} = \sqrt{(\Delta x^1)^2 + (\Delta x^2)^2 + (\Delta x^3)^2}$$

$$= \sqrt{(\Delta x)^2 + (\Delta y)^2} = \sqrt{\alpha^2 (1-2y)^2 (\Delta y)^2 + (\Delta y)^2}$$

$$= \sqrt{2\alpha^2 (2y^2 - 2y + 1)}\, \Delta y.$$

$$L = \int \Delta \ell = \int_0^1 \sqrt{2\alpha^2 (2y^2 - 2y + 1)}\, dy$$

$$= \frac{\sqrt{2}}{4} \left(2 + \sqrt{2} \sinh^{-1}(1) \right) \alpha \approx 1.15 \alpha$$

(b) Elapsed proper time is given by:

$$\Delta\tau = \frac{1}{c}\sqrt{-\sum_{\mu\nu} \eta_{\mu\nu}\,\Delta x^{\mu}\,\Delta x^{\nu}}$$

$$= \frac{1}{c}\sqrt{-\left[(\Delta x^1)^2 + (\Delta x^2)^2 + (\Delta x^3)^2 - (\Delta x^0)^2\right]}$$

$$= \frac{1}{c}\sqrt{c^2\Delta t^2 - (\Delta x)^2}$$

$$= \frac{1}{c}\sqrt{c^2\Delta t^2 - \alpha^2 c^2 (1-2ct)^2 \Delta t^2}$$

$$= \sqrt{1-\alpha^2(1-2ct)^2}\;\Delta t$$

$$\tau = \int \Delta\tau = \int_0^{\frac{1}{c}} \sqrt{1-\alpha^2(1-2ct)^2}\;dt$$

$$= \int_0^{1} \sqrt{1-\alpha^2(1-2x)^2}\;dx$$

2. The matrix of Lorentz transformation is given by:

$$\Lambda^{\mu}{}_{\nu} = \begin{pmatrix} \gamma & -\gamma v/c & 0 & 0 \\ -\gamma v/c & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The inverse of $\Lambda^{\mu}{}_{\nu}$ is given by:

$$(\Lambda^{-1})^{\mu}{}_{\nu} = \begin{pmatrix} \gamma & \gamma v/c & 0 & 0 \\ \gamma v/c & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

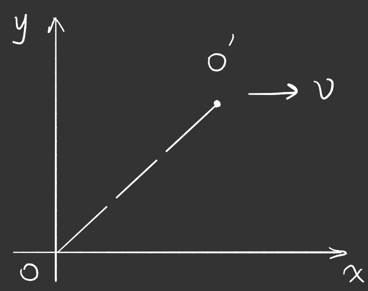
For that $\eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$.

We know that the 4-d spacetime distance is invariant under Lorentz transformation.

$$dl^2 = dx^{\mu} dx_{\mu} = \eta_{\mu\nu} dx^{\mu} dx_{\nu}.$$

So, it must obey that $\eta_{\mu\nu} dx^{\mu} dx^{\nu} = \eta_{\rho\sigma} dx'^{\rho} dx'^{\sigma}$

4. (a) We have the light source of O' at time t.



And O receives light at time t^*.

$$\Rightarrow \quad x^2+y^2+z^2 = c^2(t^*-t)^2.$$

For $v = \frac{dx}{dt}$, we have

$$x v\, dt = c^2(t^*-t)(dt^*-dt)$$

$$\Rightarrow \quad dt^* = \left(1-\frac{v}{c}\cos\theta\right)dt.$$

For that $dt_0 = \sqrt{1-\beta^2}\, dt$.

$$\Rightarrow \quad \frac{dt^*}{dt_0} = \frac{1-\frac{v}{c}\cos\theta}{\sqrt{1-\beta^2}}$$

So, $\omega' = \gamma\left(1-\frac{v}{c}\cos\theta\right)\omega$.

(b) Let k^μ be wave vector in O,

and $k^{\mu'}$ be that in O'.

$$k^\mu = (k, k\cos\theta, k\sin\theta, 0).$$

$$k^{\mu'} = (k, k\cos\theta', k\sin\theta', 0).$$

For that
$$u_x' = \frac{u_x - v}{1 - u v_x/c^2}.$$

$$u_y' = \frac{u_y\sqrt{1-\beta^2}}{1 - u v_x/c^2}.$$

$$u_z' = \frac{u_z\sqrt{1-\beta^2}}{1 - u v_x/c^2}.$$

We have
$$\cos\theta' = \frac{\cos\theta - v/c}{1 - \frac{v}{c}\cos\theta}.$$

$$\sin\theta' = \frac{\sin\theta\sqrt{1-\beta^2}}{1 - \frac{v}{c}\cos\theta}.$$

$$\Rightarrow \tan\theta' = \frac{\sin\theta}{\gamma(\cos\theta - \frac{v}{c})}.$$

HW 6 yuxiang@uchicago.edu

5. I did not work out this, but I think I will update this within 2 days.

7. (a) Apply Gauss's law, we know

$$E \cdot 2\pi r l = 4\pi \rho \cdot \pi a^2 l$$

$$\Rightarrow \vec{E} = \frac{2\rho\pi a^2}{r} \hat{e}_r .$$

$$\vec{J} = 0 . \Rightarrow \vec{B} = 0 .$$

(b) From $\rho V = \rho' V'$ we know

$$\rho' = \gamma \rho = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \rho .$$

And $\vec{J}' = \rho' v = \gamma \rho v = \dfrac{\rho v}{\sqrt{1-\frac{v^2}{c^2}}} .$

(c) $$\vec{E}' = \frac{2\rho'\pi a^2}{r} \hat{e}_r = \frac{2\rho\pi a^2}{\sqrt{1-\frac{v^2}{c^2}}} \frac{\hat{e}_r}{r} .$$

$$\vec{B}' = \frac{1}{2\pi r} \cdot \frac{4\pi}{c} \vec{J}' \pi a^2 \hat{e}_\theta = \frac{2\vec{J}'\pi a^2}{rc} \hat{e}_\theta$$

$$= \frac{2\rho v \pi a^2}{\sqrt{c^2-v^2}\, r} \hat{e}_\theta$$

(d) $E'_\perp = \gamma(\vec{E} + v\times\vec{B})_\perp$

$$= \gamma E_\perp = \gamma \frac{2\rho\pi a^2}{r} = \frac{2\rho\pi a^2}{\sqrt{1-\frac{v^2}{c^2}}}\frac{\hat{e}_r}{r}.$$

$$B'_\perp = \gamma\left(-\frac{\vec{v}\times\vec{E}}{c^2}\right)_\perp$$

$$= -\gamma\frac{vE_\perp}{c^2} = -\gamma\frac{2\rho v\pi a^2}{c^2 r}$$

$$= \frac{-2\rho v\pi a^2}{c\sqrt{c^2-v^2}\,r}.$$

And it conforms with result in (c).

(*) Prove of above transform.

$$F_{\mu\nu} = \begin{bmatrix} 0 & B_3 & -B_2 & -\frac{i}{c}E_1 \\ -B_3 & 0 & B_1 & -\frac{i}{c}E_2 \\ B_2 & -B_1 & 0 & -\frac{i}{c}E_3 \\ \frac{i}{c}E_1 & \frac{i}{c}E_2 & \frac{i}{c}E_3 & 0 \end{bmatrix}.$$

From $F'_{\mu\nu} = a_{\mu\lambda} a_{\nu\tau} F_{\lambda\tau}$

we know that

$$E_1' = E_1 . \qquad B_1' = B_1 .$$

$$E_2' = \gamma(E_2 - vB_3) . \qquad B_2' = \gamma(B_2 + \frac{v}{c^2} E_3) .$$

$$E_3' = \gamma(E_3 + vB_2) . \qquad B_3' = \gamma(B_3 - \frac{v}{c^2} E_2) .$$

$$\Rightarrow \quad E_{\parallel}' = E_{\parallel} . \qquad B_{\parallel}' = B_{\parallel} .$$

$$E_{\perp}' = \gamma(E_{\perp} + \vec{v} \times \vec{B})_{\perp} . \qquad B_{\perp}' = \gamma(B - \frac{v}{c^2} \times E)_{\perp} .$$

9. In the Lorentz frame moving with

$$\vec{v}' = -\frac{E_0}{B_0}\hat{x}.$$

$$\vec{E}' = 0. \quad (\vec{E}_{\parallel}' = \vec{E}_{\parallel}, \ \vec{E}_{\perp}' = \gamma(\vec{E}_{\perp} + \vec{v}\times\vec{B}_{\perp}))$$

$$\vec{B}' = \gamma\left(\vec{B} - \frac{\vec{v}\times\vec{E}}{c^2}\right)_{\perp} = \gamma\left(B_0 - \frac{E_0^2}{B_0 c^2}\right)\hat{y}.$$

So in this frame, the particle is moving

with $\vec{v}_p = \frac{E_0}{B_0}\vec{x}$.

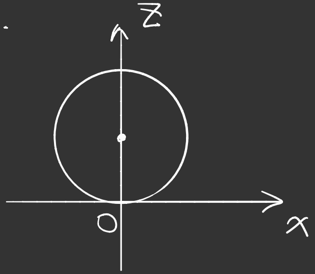
with $\vec{E}' = 0$. $\vec{B}' = \gamma\left(B_0 - \frac{E_0^2}{B_0 c^2}\right)\hat{y}$.

So, the particle will move in a circle in the

xz plane, with radius

$$R = \frac{mV_p}{q|\vec{B}'|} = \frac{mE_0}{q\gamma\left(B_0E_0 - \frac{E_0^2}{c^2}\right)}.$$

$$\omega = q\gamma\left(\frac{B_0}{m} - \frac{E_0^2}{B_0c^2 m}\right).$$



$$x_p' = R\sin(\omega t')$$

$$y_p' = R - R\cos(\omega t').$$

Transform back to the original frame.

$$x_p = \gamma(x_p' + v't')$$

$$= \gamma\left[R\sin(\omega t') - \frac{E_0}{B_0}t'\right].$$

$$y_p = R - R\cos(\omega t'). \qquad t' = \gamma\left(t + \frac{v'x_p'}{c^2}\right).$$

So, only when $v' \ll c$.

that is $c|B_0| \gg |E_0|$.

will the motion be non-relativistic.

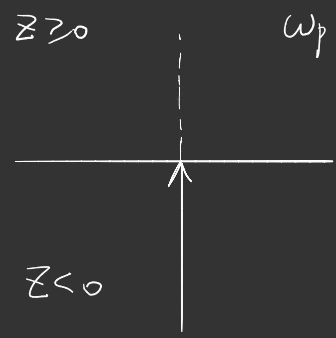
That is $x_p = R\sin\omega t - \frac{E_0}{B_0}t$

$$y_p = R - R\cos\omega t.$$

$$R = \frac{mv}{qB_0}. \qquad \omega = \frac{qB_0}{m}.$$

Midterm yuxiang@uchicago.edu

1) $\vec{A}_I = C_I e^{-i\omega t} e^{i\omega z/c} \hat{x}$.



For plasma we have

low frequency behavior:

$$n(\omega) = \frac{1+i}{\sqrt{2}} \sqrt{\frac{\sigma}{\varepsilon_0 \omega}}$$

$$\vec{k} = \frac{\omega n}{c} = (1+i)\sqrt{\frac{\mu_0 \omega \sigma}{2}}$$

Thus any EM quantity will vary as

$$\psi(t, z) = e^{-i\omega t + iz/\delta} e^{-z/\delta}.$$

where $\delta = \sqrt{\frac{2}{\mu_0 \omega \sigma}}$. While $\omega_p = \frac{Ne^2}{\varepsilon_0 m}$.

We have $\omega \gg \gamma$ and $\omega < \omega_p$.

So $\varepsilon(\omega) < 0$ and $n(\omega)$ is purely imaginary.

And $\delta = c/\sqrt{\omega_p^2 - \omega^2}$.

So, $\langle \vec{E} \rangle = \vec{E}_0 e^{-z/\delta} e^{i(z/\delta - \omega t)} e^{i\omega z/c}$.

$$= C_I e^{-z/\delta} e^{i(z/\delta - \omega t)} e^{i\omega z/c} \hat{x}.$$

$$\langle \vec{B} \rangle = \frac{1}{i\omega} \nabla \times \langle \vec{E} \rangle$$

$$= \frac{1}{c}\sqrt{\frac{\sigma}{\varepsilon_0 \omega}} C_I e^{-z/\delta} e^{i(z/\delta - \omega t)} e^{i\omega z/c} e^{\frac{\pi}{4}i} \hat{y}.$$

2) This is Fraunhofer diffraction described by (7.71)

$$\chi(\vec{x}) = \frac{-ik}{4\pi|\vec{x}|}\left(\cos\tilde{\theta} + \cos\theta\right) C e^{ik|\vec{x}|} \int_A e^{-ik\hat{x}\cdot\vec{x}'} e^{i(k_x x' + k_y y')} dx' dy'$$

$$= \frac{-ik}{4\pi|\vec{x}|}\left(\cos\tilde{\theta} + \cos\theta\right) C e^{ik|\vec{x}|} e^{-ikD}$$

$$\int_0^{2\pi}\int_0^R e^{ikr(\sin^2\theta' + \cos^2\theta')} \quad r^2 \sin\theta' d\theta' dr$$

$$= \frac{-ik}{4\pi|\vec{x}|}\left(\cos\tilde{\theta} + \cos\theta\right) C e^{ik|\vec{x}|} e^{-ikD} \int_0^{2\pi}\int_0^R e^{ikr} r^2 \sin\theta' d\theta' dr$$

For the integral inside.

When $x = y = 0$.

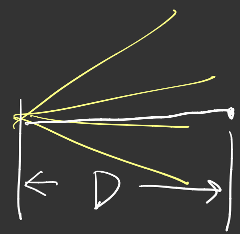
$$\vec{\chi}(x=0, y=0) = \frac{-ik}{2\pi D} C \cdot \pi R^2 = \frac{-ik}{2D} \cdot CR^2 .$$

The intensity is $I(0,0) = \dfrac{k^2 C^2 R^4}{4D^2}$.

b) For that

$$\vec{\chi}(\vec{x}) = \frac{-ik}{4\pi|\vec{x}|}\left(\cos\tilde{\theta} + \cos\theta\right) C e^{ik|\vec{x}|} \int_A e^{-ik\hat{x}\cdot\vec{x}'} e^{i(k_x x' + k_y y')} dx'dy'$$

$|\vec{x}|$ has minimum D.



So the term $\mathrm{Re}\left[\frac{e^{ik|\vec{x}|}}{|\vec{x}|}\right]$ has maximum

when $|\vec{x}| = D$. (in the center).

That is, $\mathrm{Re}(\vec{\chi})$ has maximum when $x=0$, $y=0$.

c) For the spot size, still look into

$\vec{\chi}(\vec{x})$. When $\mathrm{Re}(\vec{\chi})$ drops significantly will

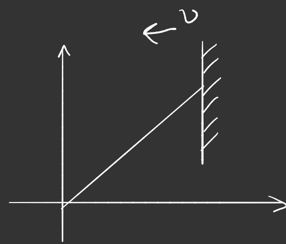
the intensity drops.

$$\mathrm{Re}\left[e^{ik\sqrt{D^2+\alpha^2}} / \sqrt{D^2+\alpha^2}\right]$$

$$= \cos k\sqrt{D^2+\alpha^2} / \sqrt{D^2+\alpha^2} .$$

When α - - - will it drop significantly.

3) First, transform to the reference frame of the mirror Σ'.



In this frame, light is moving in direction: $(k'_{ox}, k'_{oy}, 0)$.

For that
$$u_x' = \frac{u_x + v}{1 - u v_x/c^2}.$$

$$u_y' = \frac{u_y\sqrt{1-\beta^2}}{1 + u v_x/c^2}.$$

$$u_z' = \frac{u_z\sqrt{1-\beta^2}}{1 + u v_x/c^2}.$$

We have
$$\cos\theta' = \frac{\cos\theta + v/c}{1 + \frac{v}{c}\cos\theta}.$$

$$\sin\theta' = \frac{\sin\theta\sqrt{1-\beta^2}}{1 + \frac{v}{c}\cos\theta}.$$

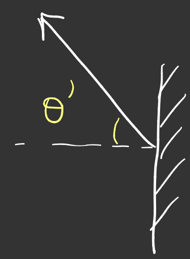
with frequency :

$$\omega' = \gamma \omega \left(1 + \frac{v}{c}\cos\theta'\right).$$

Then transfer back to the original frame

(to investigate the reflected light).

Then



$$\omega'' = \gamma \omega' \left(1 + \frac{v}{c}\cos\theta\right)$$

$$= \gamma^2 \omega \left(1 + \frac{v}{c}\cos\theta'\right)\left(1 + \frac{v}{c}\cos\theta\right)$$

$$= \gamma^2 \omega \, \frac{1 + 2\frac{v}{c}\cos\theta + \frac{v^2}{c^2}}{1 + \frac{v}{c}\cos\theta} \cdot \left(1 + \frac{v}{c}\cos\theta\right)$$

$$= \gamma^2 \omega \left(1 + 2\frac{v}{c}\cos\theta + \frac{v^2}{c^2}\right).$$

I realized that I did not take into account

the case when the mirror is parralle to x axis.

But I do not have enough time to analyze.

But I think in this case $\omega'' = \omega$.