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Problem 1: Thermodynamics

(a) We know that the energy for spin-1/2 in magnetic field is $E_\uparrow = -\mu B$, $E_\downarrow = \mu B$.

Then we have partition function for a single spin:

$$Z = e^{\mu B/kT} + e^{-\mu B/kT} = 2\cosh\left(\frac{\mu B}{kT}\right).$$

This leads to $G = -NkT\ln\left[2\cosh\left(\frac{\mu B}{kT}\right)\right]$.

For that $dG = -SdT - MdB$.

$$\Rightarrow S = -\left(\frac{\partial G}{\partial T}\right)_B = Nk\ln\left[2\cosh\left(\frac{\mu B}{kT}\right)\right] - \frac{N\mu B}{T}\tanh\left(\frac{\mu B}{kT}\right).$$

$$H = G + TS = -N\mu B\tanh\left(\frac{\mu B}{kT}\right) = -MB.$$

(b) For that $dG = -SdT - MdB$.

$$\Rightarrow \left(\frac{\partial S}{\partial B}\right)_T = \left(\frac{\partial M}{\partial T}\right)_B$$

$$= -\frac{\mu^2 BN}{kT^2}\,\mathrm{sech}^2\left(\frac{kT}{\mu B}\right)$$

While the system is thermally insulated,

$$dS = \left(\frac{\partial S}{\partial T}\right)_B dT + \left(\frac{\partial S}{\partial B}\right)_T dB = 0.$$

And
$$\left(\frac{\partial S}{\partial T}\right)_B = \frac{C_B}{T} = \frac{1}{T}\left(\frac{\partial H}{\partial T}\right)_B$$

$$= -\frac{B}{T}\left(\frac{\partial M}{\partial T}\right)_B = \frac{\mu^2 B^2 N}{kT^3}\,\mathrm{sech}^2\left(\frac{kT}{\mu B}\right).$$

We have
$$\frac{dT}{dB} = -\left(\frac{\partial S}{\partial B}\right)_T \Big/ \left(\frac{\partial S}{\partial T}\right)_B = \frac{T}{B}.$$

$$\Rightarrow \frac{T}{B} = \text{constant}.$$

That is, when B decreases, T decreases.

If the temperature remains constant,

From $$TdS = \underset{0}{\underset{\parallel}{dU}} - M\,dB$$

We have $$T\left(\frac{\partial S}{\partial B}\right)_T = -M = N\mu \tanh\left(\frac{\mu B}{kT}\right)$$

$$\Delta Q = \int_{B_i}^{B_f} M\,dB = \int_{B_i}^{B_f} -N\mu \tanh\left(\frac{\mu B}{kT}\right) dB$$

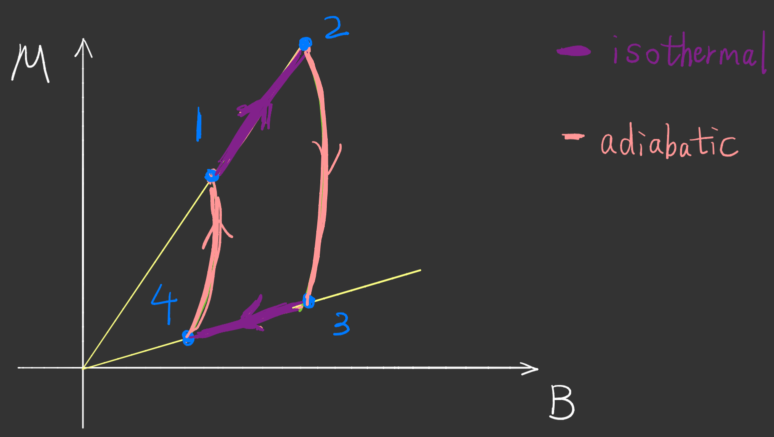
In the limit of $\frac{\mu B}{kT} \ll 1$. $\tanh\left(\frac{\mu B}{kT}\right) \approx \frac{\mu B}{kT}$.

We have

$$\Delta Q = \int_{B_i}^{B_f} -N\mu \frac{\mu B}{kT}\, dB = \frac{\mu^2 N}{2kT}\left(B_i^2 - B_f^2\right)$$

$$= \frac{-\mu^2 N}{2k_B T}\left(B_{final}^2 - B_{initial}^2\right).$$

(c)



For adiabatic process, $C_B\, dT = B\, dM$.

$$M \approx \frac{N\mu^2 B}{kT} \quad \Rightarrow \quad C_B\, dT = \frac{MkT}{N\mu^2}\, dM.$$

$$\Rightarrow \quad \frac{C_B N\mu^2}{k} \ln \frac{T_2}{T_1} = \frac{1}{2}\left(M_2^2 - M_1^2\right)$$

$$\Rightarrow \quad \frac{C_B N\mu^2}{k} \ln \frac{B_2 M_1}{B_1 M_2} = \frac{1}{2}\left(M_2^2 - M_1^2\right).$$

For isothermal process,

$$dQ = -B\, dM = -\frac{MkT}{N\mu^2}\, dM.$$

$$\Rightarrow \quad \Delta Q_{12} = -\frac{kT}{N\mu^2}\left(M_1^2 - M_2^2\right).$$

And $M = \frac{N\mu^2 B}{kT}$ is a straight line in

$M-B$ plane since T is constant.

In the graph $1 \to 2$ is in the line of $M = \frac{N\mu^2}{kT_1} B$

and $3 \to 4$ is in $M = \frac{N\mu^2}{kT_2} B$.

$1 \to 4$ is in $B \propto M e^{\frac{-2k(M_0^2 - M^2)}{C_B N\mu^2}}$.

Concerning efficiency of this cycle,

$$W = W_{12} + W_{23} + W_{34} + W_{41}.$$

$$W_{12} = \Delta Q_{12} = \frac{kT_1}{N\mu^2}(M_2^2 - M_1^2).$$

$$W_{23} = C_B(T_2 - T_1). \qquad \Delta Q_{23} = 0.$$

$$W_{34} = \frac{kT_2}{N\mu^2}(M_4^3 - M_3^2). = \Delta Q_{34}.$$

$$W_{41} = C_B(T_1 - T_2). \qquad \Delta Q_{41} = 0.$$

In this way.

$$W_{total} = \frac{kT_1}{N\mu^2}\left(M_2^2 - M_1^2\right) + \frac{kT_2}{N\mu^2}\left(M_4^2 - M_3^2\right)$$

$$\Delta Q = Q_{absorb} = \Delta Q_{34}$$

$$= \frac{kT_2}{N\mu^2}\left(M_4^2 - M_3^2\right).$$

$$\eta = \frac{W_{total}}{Q_{absorb}} = 1 + \frac{T_1\left(M_2^2 - M_1^2\right)}{T_2\left(M_4^2 - M_3^2\right)}$$

$$= 1 - \frac{T_1}{T_2}.$$

Which is maximum efficiency. (as a Carnot cycle).

Problem 2: Canonical Ensemble

(a) Partition function could be written as:

$$Z = \frac{1}{h^3}\iint e^{-\frac{\beta p^2}{2m}} e^{\frac{-\beta k x^2}{2}} \, dp\, dx .$$

For that $U = -\frac{\partial \ln Z}{\partial \beta}$.

We could further calculate

$$Z = \frac{1}{h^3}\int e^{\frac{-\beta p^2}{2m}} dp \int e^{-\frac{\beta k x^2}{2}} dx$$

$$= \frac{1}{h^3}\left(\frac{2m\pi}{\beta}\right)^{\frac{1}{2}} \cdot \left(\frac{2\pi}{k\beta}\right)^{\frac{1}{2}}$$

Then $U = -\frac{\partial \ln Z}{\partial \beta} = -\left(-\frac{1}{2\beta} - \frac{1}{2\beta}\right)$

$= k_B T.$

So, heat capacity is given by

$$C = \frac{dU}{dT} = k_B .$$

(b) $$Z = \sum_{n=0}^{\infty} e^{-\beta \varepsilon_n} = \sum_{n=0}^{\infty} e^{-\beta(n+\frac{1}{2})\hbar\omega}$$

$$= e^{-\frac{1}{2}\beta\hbar\omega} \frac{1}{1-e^{-\beta\hbar\omega}}$$

Then $$U = -\frac{\partial \ln Z}{\partial \beta}$$

$$= -\partial\left[-\frac{1}{2}\beta\hbar\omega - \ln(1-e^{-\beta\hbar\omega})\right] / \partial\beta$$

$$= \frac{1}{2}\hbar\omega + \frac{\hbar\omega\, e^{-\beta\hbar\omega}}{1-e^{-\beta\hbar\omega}}$$

$$= \frac{1}{2}\hbar\omega + \frac{\hbar\omega}{e^{\beta\hbar\omega}-1}.$$

Heat capacity is given by

$$C = \frac{dU}{dT} = \frac{dU}{d\beta}\cdot\frac{d\beta}{dT}$$

$$= \frac{-\hbar\omega\cdot\hbar\omega\, e^{\beta\hbar\omega}}{(e^{\beta\hbar\omega}-1)^2}\cdot\frac{d\beta}{dT}$$

$$= \frac{-(\hbar\omega)^2 e^{\hbar\omega/kT}}{(e^{\hbar\omega/kT}-1)^2} \cdot \frac{-1}{kT^2}$$

$$= k_B \left(\frac{\hbar\omega}{k_B T}\right)^2 \frac{e^{\hbar\omega/k_B T}}{(e^{\hbar\omega/k_B T}-1)^2}.$$

(c) Partition function could be written as:

$$Z = \frac{1}{h}\iint e^{-\frac{\beta p^2}{2m}} e^{-\beta V(x)} \, dp\, dx$$

$$= \frac{1}{h}\int e^{-\frac{\beta p^2}{2m}} dp \int e^{-\beta(\frac{k}{2}x^2 - gx^3 + fx^4)} dx$$

$$= \frac{1}{h}\left(\frac{2m\pi}{\beta}\right)^{\frac{1}{2}} \int e^{-\beta(\frac{k}{2}x^2 - gx^3 + fx^4)} dx$$

For that $e^{-\beta(\frac{k}{2}x^2 - gx^3 + fx^4)} \approx e^{-\frac{\beta}{2}kx^2}(1 + \beta g x^3 - \beta f x^4$

$+ \frac{\beta^2 g^2}{2} x^6$

We have

$$\int e^{-\beta\left(\frac{k}{2}x^2-gx^3+fx^4\right)}dx \approx \int e^{-\frac{\beta}{2}kx^2}\left(1+\beta g x^3-\beta f x^4+\frac{\beta^2 g^2}{2}x^6\right)dx$$

$$= \sqrt{\frac{2\pi}{\beta k}} - 0 - \frac{3\sqrt{\pi}}{4\left(\frac{\beta k}{2}\right)^{\frac{5}{2}}}\cdot\beta f + \frac{\beta^2 g^2}{2}\cdot\frac{15\sqrt{\pi}}{8\left(\frac{\beta k}{2}\right)^{\frac{7}{2}}}$$

$$= \sqrt{\frac{2\pi}{\beta k}} - \frac{3\sqrt{2\pi}}{(\beta k)^{\frac{5}{2}}}\cdot\beta f + \frac{15\sqrt{2\pi}}{2(\beta k)^{\frac{7}{2}}}\cdot\beta^2 g^2$$

So, $U = -\dfrac{\partial \ln Z}{\partial \beta}$

$$= -\frac{\partial}{\partial\beta}\left(-\ln h - \frac{1}{2}\ln\frac{2m\pi}{\beta} + \ln\left[\sqrt{\frac{2\pi}{\beta k}} - \frac{3\sqrt{2\pi}}{(\beta k)^{\frac{5}{2}}}\cdot\beta f + \frac{15\sqrt{2\pi}}{2(\beta k)^{\frac{7}{2}}}\cdot\beta^2 g^2\right]\right)$$

$$\approx \frac{1}{2\beta} + \frac{1}{2\beta} - \frac{\partial}{\partial\beta}\left(-3\cdot f\beta^{-1}k^{-2} + \frac{15}{2}g^2\beta^{-1}k^{-3}\right)$$

$$= \frac{1}{\beta} - 3f/k^2\beta^2 + \frac{15}{2}g^2/k^3\beta^2$$

Heat Capacity is given by

$$C = \frac{dU}{d\beta}\frac{d\beta}{dT}$$

$$= \left(-\frac{1}{\beta^2} + \frac{6f}{k^2\beta^3} - \frac{15g^2}{2k^3\beta^3}\right)\cdot\left(\frac{1}{-k_B T^2}\right)$$

$$= \left(k_B^2 T^2 - \frac{6k_B^3 T^3 f}{k^2} + \frac{15g^2 k_B^3 T^3}{2k^3}\right)\cdot\frac{1}{k_B T^2}$$

$$= k_B - \frac{6k_B^2 Tf}{k^2} + \frac{15g^2 k_B^2 T}{2k^3}$$

$$= k_B + k_B^2\left(\frac{15g^2}{k^3} - 6\frac{f}{k^2}\right)T.$$

For the average position,

$$\langle x \rangle = \int x\, e^{-\beta\left(\frac{kx^2}{2} - gx^3 + fx^4\right)} dx \Big/ \int e^{-\beta\left(\frac{kx^2}{2} - gx^3 + fx^4\right)} dx$$

$$= \frac{\int x\, e^{-\frac{\beta}{2}kx^2}\left(1 + \beta g x^3 - \beta f x^4 - \frac{\beta g^2}{2} x^6\right) dx}{\sqrt{\frac{2\pi}{\beta k}} - \frac{3\sqrt{2\pi}}{(\beta k)^{\frac{5}{2}}} \cdot \beta f + \frac{15\sqrt{2\pi}}{2(\beta k)^{\frac{7}{2}}} \cdot \beta^2 g^2}$$

$$= \frac{0 + \beta g \cdot \frac{3\sqrt{2\pi}}{(\beta k)^{\frac{5}{2}}} - 0 + 0}{\sqrt{\frac{2\pi}{\beta k}} - \frac{3\sqrt{2\pi}}{(\beta k)^{\frac{5}{2}}} \cdot \beta f + \frac{15\sqrt{2\pi}}{2(\beta k)^{\frac{7}{2}}} \cdot \beta^2 g^2}$$

$$\approx \beta g \cdot \frac{3}{(\beta k)^2} = \frac{3g}{\beta k^2} = \frac{3g k_B T}{k^2}.$$

(d) Partition function could be written as

$$Z = \sum_{n=0}^{\infty} e^{-\beta \varepsilon_n}$$

$$= \sum_{n=0}^{\infty} e^{-\beta(n+\frac{1}{2})\hbar\omega} \, e^{\beta\lambda(n+\frac{1}{2})^2\hbar\omega}$$

$$= \sum_{n=0}^{\infty} e^{\beta\left[\lambda n^2 + (\lambda-1)n + \frac{\lambda-2}{4}\right]}$$

Let $f(n) = \lambda n^2 + (\lambda-1)n + \frac{\lambda-2}{4}$.

Then $$\sum_n [f(n)]^m e^{\alpha f(n)} = \frac{d^m}{d\alpha^m} \sum_n e^{\alpha f(n)}$$

$$\Rightarrow \sum_n f(n) e^{\alpha f(n)} = \frac{d}{d\alpha} \sum_n e^{\alpha f(n)}$$

$$\sum_n \left[\lambda n^2 + (\lambda-1)n + \frac{\lambda-2}{4}\right] e^{\alpha f(n)} = \frac{d}{d\alpha} \sum_n e^{\alpha f(n)}.$$

Cannot find a general solution.

So, use approximation.

$$Z = \sum_{n=0}^{\infty} e^{-\beta(n+\frac{1}{2})\hbar\omega} \, e^{\beta\lambda(n+\frac{1}{2})^2\hbar\omega}$$

$$\approx \sum_{n=0}^{\infty} e^{-\beta(n+\frac{1}{2})\hbar\omega} \left[1 + \beta\lambda(n+\tfrac{1}{2})^2\hbar\omega\right]$$

Then it becomes easier. Let $x = \beta\hbar\omega$.

$$Z = \sum_{n=0}^{\infty} e^{-x(n+\frac{1}{2})} \left[1 + x\lambda(n+1)^2\right]$$

$$= e^{-\frac{x}{2}} \sum_{n=0}^{\infty} e^{-xn} + e^{-\frac{x}{2}} \sum_{n=0}^{\infty} x\lambda(n+1)^2 e^{-xn}$$

$$= (1+x\lambda) e^{-\frac{x}{2}} \sum_{n=0}^{\infty} e^{-xn} + 2\lambda e^{-\frac{x}{2}} \sum_{n=0}^{\infty} xn e^{-xn}$$

$$+ e^{-\frac{x}{2}} \cdot \lambda \sum_{n=0}^{\infty} xn^2 e^{-xn}$$

For that $\displaystyle \frac{1}{e-1} = \sum_{n=0}^{\infty} e^{-n} \quad \left(\frac{1}{x-1} = \sum_{n=0}^{\infty} x^{-n} \right)$

$$\Rightarrow \quad \frac{-e}{(e-1)^2} = \sum_{n=0}^{\infty} -n e^{-n} \qquad \left(\frac{x}{(x-1)^2} = \sum_n n x^{-n} \right)$$

$$\Rightarrow \quad \frac{e(1+e)}{(e-1)^3} = \sum_{n=0}^{\infty} n^2 x^{-n} \qquad \left(\frac{x(1+x)}{(x-1)^3} = \sum_n n^2 x^{-n} \right)$$

So, $Z = (1+x\lambda) e^{-\frac{x}{2}} \cdot \frac{1}{e-1} + 2\lambda e^{-\frac{x}{2}} \cdot \frac{e}{(e-1)^2}$

$$+ \lambda e^{-\frac{x}{2}} \cdot \frac{1}{x} \cdot \frac{e(1+e)}{(e-1)^3}$$

$$C = \frac{dU}{dT} = \frac{\partial(\ln Z)}{\partial \beta} \frac{\partial \beta}{\partial T}$$

$$= k_B \left[\left(1 - \frac{1}{12}\gamma^2 + \frac{1}{240}\gamma^4\right) + 4\lambda\left(\frac{1}{\gamma} + \frac{1}{80}\gamma^3\right) \right].$$

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Problem 1: Thermodynamics

$$\left(\frac{\partial C_v}{\partial V}\right)_T = \left[\frac{\partial}{\partial V}\cdot T\left(\frac{\partial S}{\partial T}\right)_V\right]_T = T\left[\frac{\partial}{\partial V}\left(\frac{\partial S}{\partial T}\right)_V\right]_T$$

$$= T\left[\frac{\partial}{\partial T}\left(\frac{\partial S}{\partial V}\right)_T\right]_V = T\left[\frac{\partial}{\partial T}\left(\frac{\partial p}{\partial T}\right)_V\right]_V$$

$$= T\left(\frac{\partial^2 p}{\partial T^2}\right)_V.$$

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$$\left(\frac{\partial C_p}{\partial p}\right)_T = \left[\frac{\partial}{\partial p}\cdot T\left(\frac{\partial S}{\partial T}\right)_p\right]_T = T\left[\frac{\partial}{\partial p}\left(\frac{\partial S}{\partial T}\right)_p\right]_T$$

$$= T\left[\frac{\partial}{\partial T}\left(\frac{\partial S}{\partial p}\right)_T\right]_p = -T\left[\frac{\partial}{\partial T}\left(\frac{\partial V}{\partial T}\right)_p\right].$$

$$= -T\left(\frac{\partial^2 V}{\partial T^2}\right)_p.$$

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Problem 2 : Canonical Ensemble

1. $$\varepsilon = -g\mu_B J_i B = -g\mu_B m B.$$

$$Z_1 = \sum_{m=-J}^{J} e^{-\beta\varepsilon} = \frac{e^{J\varepsilon_0} - e^{-(J+1)\varepsilon_0}}{1-e^{-\varepsilon_0}}.$$

$Z = Z_1^N$

$$= \frac{\sinh\left(\frac{2J+1}{2}\varepsilon_0\right)}{\sinh\left(\frac{1}{2}\varepsilon_0\right)}. \qquad \varepsilon_0 = \beta g \mu_B B.$$

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2. $$M = \frac{N}{\beta}\frac{\partial \ln Z_1}{\partial B} = \frac{N}{\beta}\frac{\partial \ln Z_1}{\partial \varepsilon_0}\frac{\partial \varepsilon_0}{\partial B}$$

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$$= N g \mu_B \left[\frac{2J+1}{2}\coth\left(\frac{2J+1}{2}\varepsilon_0\right) - \frac{1}{2}\coth\left(\frac{\varepsilon_0}{2}\right)\right].$$

3. $$\chi = \left(\frac{\partial M}{\partial B}\right)_{\cancel{T}=0}^{B=0} = \left(\frac{\partial M}{\partial \varepsilon_0}\right)_{T=0}\left(\frac{\partial \varepsilon_0}{\partial B}\right)_{T=0}$$

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$$= \frac{N g^2 \mu_B^2}{k_B T}\left[-\left(\frac{2J+1}{2}\right)^2 \operatorname{csch}^2\left(\frac{2J+1}{2}\varepsilon_0\right) + \frac{1}{8}\operatorname{csch}^2\left(\frac{\varepsilon_0}{2}\right)\right]$$

Problem 3 : Fermions

The grand canonical partition function :

$$Z_{\vec{p}} = \sum_{n_{\vec{p}}} e^{-(\alpha+\beta\varepsilon_{\vec{p}})n_{\vec{p}}} = 1 + e^{-(\alpha+\beta\varepsilon_{\vec{p}})}.$$

The grand potential :

$$\Phi = -k_B T \sum_{n_{\vec{p}}} \ln\left[1+\exp[\beta(\mu-\varepsilon_{\vec{p}})]\right]$$

$$= k_B T \sum_{n_{\vec{p}}} \ln\left(1-\langle n_{\vec{p}}\rangle\right).$$

$$\Uparrow$$

For that $\langle n_{\vec{p}}\rangle = 1 - \dfrac{1}{1+\exp[\beta(\mu-\varepsilon_{\vec{p}})]}$.

So, the entrophy is given by

$$S = -\left(\frac{\partial\Phi}{\partial T}\right)_{\mu,V} = -k_B \sum_{n_{\vec{p}}} \ln\left(1-\langle n_{\vec{p}}\rangle\right) - k_B T \sum_{n_{\vec{p}}} \frac{\partial \ln\left(1-\langle n_{\vec{p}}\rangle\right)}{\partial T}$$

$$= -k_B \sum_{n_{\vec{p}}} \ln\left(1-\langle n_{\vec{p}}\rangle\right) + k_B T \sum_{n_{\vec{p}}} \frac{1}{1-\langle n_p\rangle} \frac{\partial \langle n_{\vec{p}}\rangle}{\partial T}$$

$$= -k_B \sum_{n_{\vec{p}}} \ln\left(1-\langle n_{\vec{p}}\rangle\right) + k_B T \cdot \sum_{n_{\vec{p}}} \frac{\mu-\varepsilon_{\vec{p}}}{k_B T^2} \frac{\langle n_{\vec{p}}\rangle}{\langle n_{\vec{p}}\rangle - 1}$$

$$= -k_B \sum_{n_{\vec{p}}} \ln\left(1-\langle n_{\vec{p}}\rangle\right) + \sum_{n_{\vec{p}}} k_B \left[\ln\langle n_p\rangle - \ln\left(1-\langle n_p\rangle\right)\right] \langle n_{\vec{p}}\rangle$$

$$= k_B \sum_{n_{\vec{p}}} \left[-\langle n_{\vec{p}}\rangle \ln\langle n_{\vec{p}}\rangle - \left(1-\langle n_{\vec{p}}\rangle\right)\ln\left(1-\langle n_{\vec{p}}\rangle\right)\right]$$

$$= g k_B V \int \frac{d\vec{p}}{(2\hbar\pi)^3} \left[-\langle n_{\vec{p}}\rangle \ln\langle n_{\vec{p}}\rangle - \left(1-\langle n_{\vec{p}}\rangle\right)\ln\left(1-\langle n_{\vec{p}}\rangle\right)\right].$$

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Problem 4 : Bosons

For that $E = TS + \mu N - pV$.

At $T = T_c$, $\mu = 0$.

And Bose gas is a kind of 3-d, non-relativistic ideal gas, which has relation $pV = \frac{2}{3}E$.

So, $$S = \frac{5E}{3T}.$$

From the expression for energy in the grand canonical ensemble, $$E = \frac{3}{2}pV = \frac{3}{2}V\frac{g}{\lambda^3}k_B T\,\zeta\left(\frac{5}{2}\right). \quad (7.62)$$

while density of excited states is given by

$$n = \frac{g}{\lambda^3}\zeta\left(\frac{3}{2}\right). \qquad N = nV.$$

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So, $$\frac{S}{N} = \frac{\frac{5}{3}\frac{E}{T}}{nV} = \frac{\frac{5}{2}\frac{Vg}{\lambda^3}k_B\,\zeta\left(\frac{5}{2}\right)}{\frac{Vg}{\lambda^3}\zeta\left(\frac{3}{2}\right)} = k_B\frac{5}{2}\frac{\zeta\left(\frac{5}{2}\right)}{\zeta\left(\frac{3}{2}\right)}.$$