QFT I Pset 1 yuxiang@uchicago.edu

1. **Read Srednicki Sec. 1** – Unifying quantum mechanics and special relativity.

2. **Srednicki Problem 1.2** – QFT for many-body quantum mechanics.

1.2) With the hamiltonian of eq. (1.32), show that the state defined in eq. (1.33) obeys the abstract Schrödinger equation, eq. (1.1), if and only if the wave function obeys eq. (1.30). Your demonstration should apply both to the case of bosons, where the particle creation and annihilation operators obey the commutation relations of eq. (1.31), and to fermions, where the particle creation and annihilation operators obey the anticommutation relations of eq. (1.38).

$$|\psi,t\rangle = \int d^3x_1 \cdots d^3x_n \, \psi(\vec{x}_1,\dots\vec{x}_n;t)\, a^\dagger(x_1)\dots a^\dagger(\vec{x}_n)\,|0\rangle$$

$$H = \sum_{j=1}^{n}\left(-\frac{\hbar^2}{2m}\nabla_j^2 + U(\vec{x}_j)\right) + \sum_{j=1}^{n}\sum_{k=1}^{j-1} V(\vec{x}_j-\vec{x}_k)$$

$$= \int d^3x\, a^\dagger(\vec{x})\left(-\frac{\hbar^2}{2m}\nabla^2 + U(x)\right)a(\vec{x}) + \frac{1}{2}\int d^3x\, d^3y\, V(\vec{x}-\vec{y})\, a^\dagger(\vec{x})a^\dagger(\vec{y})a(\vec{y})a(\vec{x})$$

$$H|\psi,t\rangle_{①} = \int d^3x\, a^\dagger(\vec{x})\left(-\frac{\hbar^2}{2m}\nabla^2 + U(\vec{x})\right)\underline{a(\vec{x})}\int d^3x_1\cdots d^3x_n\,\psi(\vec{x}_1,\dots\vec{x}_n,t)\,\underline{a^\dagger(\vec{x}_1)\cdots a^\dagger(\vec{x}_n)}\,|0\rangle$$

$$= \int d^3x\, a^\dagger(\vec{x})\left(-\frac{\hbar^2}{2m}\nabla^2 + U(\vec{x})\right)\int d^3x_1\cdots d^3x_n\,\psi(\vec{x}_1,\dots\vec{x}_n,t)\,\underline{\left[\pm a^\dagger(\vec{x}_1)a(\vec{x}) + \delta(\vec{x}-\vec{x}_1)\right]a^\dagger(\vec{x}_2)\cdots a^\dagger(\vec{x}_n)}\,|0\rangle$$

$$= \pm a^\dagger(\vec{x}_1)a(\vec{x})a^\dagger(\vec{x}_2)\cdots a^\dagger(\vec{x}_n) + \delta(\vec{x}-\vec{x}_1)a^\dagger(\vec{x}_2)\cdots a^\dagger(\vec{x}_n)$$

$$= (\pm1)^2\, a^\dagger(\vec{x}_1)a^\dagger(\vec{x}_2)a(\vec{x})\cdots a^\dagger(\vec{x}_n) + (\pm1)\,\delta(\vec{x}-\vec{x}_2)\,a^\dagger(\vec{x}_1)a^\dagger(\vec{x}_3)\cdots a^\dagger(\vec{x}_n)$$

$$= \cdots = \sum_{i=1}^{n}(-1)^{i-1}\,\delta(\vec{x}-\vec{x}_i)\,a^\dagger(\vec{x}_1)\cdots a^\dagger(\vec{x}_{i-1})\,a^\dagger(\vec{x}_{i+1})\cdots a^\dagger(\vec{x}_n)$$

$$= \int d^3x\, a^\dagger(\vec{x})\left(-\frac{\hbar^2}{2m}\nabla^2 + U(\vec{x})\right)\int d^3x_1\cdots d^3x_n\,\psi(\vec{x}_1,\dots\vec{x}_n,t)\sum_{i=1}^{n}(-1)^{i-1}\delta(\vec{x}-\vec{x}_i)\,a^\dagger(\vec{x}_1)\cdots a^\dagger(\vec{x}_{i-1})\,a^\dagger(\vec{x}_{i+1})\cdots a^\dagger(\vec{x}_n)\,|0\rangle$$

$$= \left(-\frac{\hbar^2}{2m}\nabla_i^2 + U(\vec{x}_i)\right)\int d^3x_1\cdots d^3x_n\,\psi(\vec{x}_1,\dots\vec{x}_n,t)\sum_{i=1}^{n}(-1)^{i-1}a^\dagger(\vec{x}_1)\cdots a^\dagger(\vec{x}_{i-1})\,a^\dagger(\vec{x}_{i+1})\cdots a^\dagger(\vec{x}_n)\,|0\rangle$$

3. **Classical Klein-Gordon theory:** We derived in class the Klein Gordon equation for a scalar field

$$(\partial_t^2 - c^2\partial_x^2 + \omega_o^2)\phi(t,x) = 0\,. \tag{1}$$

(a) Look for plane wave solutions to this equation $\phi(t,x) = \phi_0 e^{-i\omega t + ikx}$, and find a condition on the frequency $\omega = \omega(k)$ for this to be a solution [**Hint:** there are two such solutions $\omega_\pm(k)$]. This describes propagating with a wavevector k and a frequency $\omega(k)$. Write the most general form of a solution to Eq. (1).

(b) Let's try to throw a Gaussian wavepacket of width ℓ to the right, with velocity v. Specifically, we want to consider the initial conditions $\phi(0,x) = e^{-x^2/(2\ell^2)}$, and $\partial_t\phi(0,x) = -v\partial_x\phi(0,x) = \frac{xv}{\ell^2}e^{-x^2/(2\ell)^2}$. What is the time-dependent profile $\phi(t,x)$ of the wavepacket? Is it moving to the right?

(c) Compute the 'group velocity' $d\omega(k)/dk$, and show that it is bounded by c.

(a) $(-\omega^2 + c^2k^2 + \omega_0^2)\,\phi_0 = 0$.

$\omega_\pm(k) = \pm\sqrt{\omega_0^2 + c^2k^2}$

$\phi_\pm(t,x) = \phi_0\, e^{\pm i\sqrt{\omega_0^2+c^2k^2}\,t + ikx}$

k has a continuous spectrum, so $\int dk$ or $\int \frac{dk}{2\pi}$.

(b) $\Psi(x,0) = \int \tilde{\Psi}(k)\,\phi_k(x)\,dk$.

$\tilde{\Psi}(k) = \int e^{-\frac{x^2}{2\ell^2}} e^{-ikx}\,dx \sim e^{-\frac{\ell^2k^2}{2}}$.

$\Psi(t,x) = \int e^{-i\sqrt{\omega_0^2+c^2k^2}\,t}\, e^{-\frac{\ell^2k^2}{2}}\, e^{ikx}\,dk$

X Oops, dimension changes

$\sim \int e^{-\frac{\ell^2k^2}{2}} e^{ixk} e^{-ictk\left(1+\frac{\omega_0^2}{2c^2k^2}\right)} dk = \int e^{-\frac{\ell^2k^2}{2} + i(x-ct)k}\, e^{-\frac{i\omega_0^2 t}{2ck}}\, dk$

$= e^{-\frac{i\omega_0^2 t}{2ck}} \int e^{-\frac{\ell^2k^2}{2} + i(x-ct)k}\,dk - \int \frac{i\omega_0^2 t}{2ck^2}\, e^{-\frac{i\omega_0^2 t}{2ck}}\, e^{-\frac{\ell^2k^2}{2} + i(x-ct)k}\,dk$

$$\sim e^{-\frac{i\omega_0^2 t}{2ck}} e^{-\frac{(x-ct)^2}{2\ell^2}}$$

Yes, moving to the right.

(c) $\dfrac{d\omega(k)}{dk} = \dfrac{ck}{\sqrt{\frac{\omega_0^2}{c^2}+k^2}} \leq c.$ ($\omega_0=0$ equlity holds)

4. **Quantum Klein-Gordon theory:** We found in class that the 1+1 dimensional Klein-Gordon action could be diagonalized by working in momentum space

$$S = \sum_{k=\frac{2\pi}{L}\{\ldots,-2,-1,0,1,2,\ldots\}} \int dt \, \frac{1}{2}|\dot{\phi}_k|^2 - \frac{1}{2}m^2c^4|\phi_k|^2 - \frac{1}{2}c^2k^2|\phi_k|^2 . \tag{2}$$

We have changed notation $\omega_o \to mc^2$, because this will be the energy of a single particle state with zero momentum (as you will show in point (a) below). The spectrum is that of decoupled SHOs labeled by k

$$\begin{aligned} |\{n_k\}\rangle &\equiv |\cdots, n_{-\frac{2\pi}{L}2}, n_{-\frac{2\pi}{L}}, n_0, n_{\frac{2\pi}{L}}, n_{\frac{2\pi}{L}2}, n_{\frac{2\pi}{L}3}, \ldots\rangle \\ H|\{n_k\}\rangle &= |\{n_k\}\rangle E_{\{n_k\}} = |\{n_k\}\rangle \sum_k E_{n_k} \end{aligned} \tag{3}$$

with $E_{n_k} = \left(n_k + \frac{1}{2}\right)\sqrt{m^2c^4 + (ck)^2}$. Consider the ground state $|\mathrm{GS}\rangle \equiv |0,0,0,\ldots\rangle$. Let us measure the energy of excited states relative to the ground state:

$$\delta E_{\{n_k\}} = E_{\{n_k\}} - E_{\mathrm{GS}} = \sum_k n_k \sqrt{m^2c^4 + (ck)^2} . \tag{4}$$

Finally, define the total *occupation number* and *momentum* of a state as

$$N_{\{n_k\}} \equiv \sum_k n_k , \qquad P_{\{n_k\}} \equiv \sum_k n_k \, k . \tag{5}$$

(a) What is the energy of the single particle state with zero momentum, $|\ldots, 0, 0, n_0 = 1, 0, 0, \ldots\rangle$?

(b) What is the energy and momentum of the state

$$|\ldots, 0, 0, n_{k_1} = 3, 0, 0, \ldots, 0, 0, n_{k_2} = 5, 0, 0, \ldots\rangle \tag{6}$$

i.e. the state where the SHO labeled by k_1 has occupation number 3, the SHO labeled by k_2 has occupation number 5, and all other modes have occupation number 0?

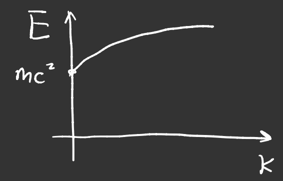
(c) What is the lowest energy state with momentum K? [**Hint:** consider states with total occupation number $N_{\{n_k\}} = 1$]. Plot its energy as a function of K.

(d) What are the second and third lowest energy states with momentum K? You can assume $K = \frac{2\pi}{L}l$ with l an even integer. What is their total occupation number $N_{\{n_k\}}$? Add their energies as functions of K to your plot from point (b). [**Hint:** you should find in the thermodynamic limit $l, L \to \infty$ with $K = \frac{2\pi}{L}l$ held fixed that the spacing in energy from the second to the third state goes to zero, whereas the spacing between first and second state is constant]

(a) $\delta E = E - E_{GS} = \left(1+\frac{1}{2}-\frac{1}{2}\right)\sqrt{m^2c^4+c^2k^2} = \sqrt{m^2c^4+c^2k^2}$ X

(b) $\delta E = n_{k_1}\sqrt{m^2c^4+c^2k_1^2} + n_{k_2}\sqrt{m^2c^4+c^2k_2^2}$

$P = n_{k_1}k_1 + n_{k_2}k_2 = 3k_1 + 5k_2$.



(c) $\delta E = n_k\sqrt{m^2c^4+c^2k^2} = \sqrt{m^2c^4+c^2k^2}$

with $N_{\{n_k\}} = 1$.

(d) $k = \frac{2\pi}{L}\ell$. $\delta E = n_1\sqrt{m^2c^4+c^2k_1^2} + n_2\sqrt{m^2c^4+c^2k_2^2}$

$k_1 = \frac{\pi}{L}(\ell+1)$, $k_2 = \frac{\pi}{L}(\ell-1)$.

$$\delta E = \frac{\ell+1}{2}\sqrt{m^2c^4+c^2\frac{\pi^2}{L^2}(\ell+1)^2} + \frac{\ell-1}{2}\sqrt{m^2c^4+c^2\frac{\pi^2}{L^2}(\ell-1)^2}.$$

$N_{\{n_k\}} = \ell$.

2nd lowest or 3rd lowest state ?

5. **Reading ahead:** (if you have time!) Srednicki Sections 2 and 3.

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1. Srednicki Problem 2.8 (a) and 2.9 (a-d)

2.8) a) Let $\Lambda = 1 + \delta\omega$ in eq. (2.26), and show that

$$[\varphi(x), M^{\mu\nu}] = \mathcal{L}^{\mu\nu}\varphi(x)\ , \qquad (2.29)$$

where

$$\mathcal{L}^{\mu\nu} \equiv \tfrac{\hbar}{i}(x^\mu\partial^\nu - x^\nu\partial^\mu)\ . \qquad (2.30)$$

$$U(\Lambda)^{-1}\varphi(x)U(\Lambda) = \varphi(\Lambda^{-1}x). \qquad \Lambda = 1+\delta\omega.$$

$$U(\Lambda)^{-1}\varphi(x)U(\Lambda) = U^{-1}(1+\delta\omega)\,\varphi(x)\,U(1+\delta\omega)$$

$$= \left(I - \tfrac{i}{2\hbar}\delta\omega_{\mu\nu}M^{\mu\nu}\right)\varphi(x)\left(I + \tfrac{i}{2\hbar}\delta\omega_{\mu\nu}M^{\mu\nu}\right)$$

$$= \left(I - \tfrac{i}{2\hbar}\delta\omega_{\mu\nu}M^{\mu\nu}\right)\left(\varphi(x) + \tfrac{i}{2\hbar}\varphi(x)\,\delta\omega_{\mu\nu}M^{\mu\nu}\right)$$

$$= \varphi(x) - \tfrac{i}{2\hbar}\delta\omega_{\mu\nu}M^{\mu\nu}\varphi(x) + \tfrac{i}{2\hbar}\varphi(x)\,\delta\omega_{\mu\nu}M^{\mu\nu} - \tfrac{1}{4\hbar^2}\delta\omega_{\mu\nu}M^{\mu\nu}\varphi(x)\,\delta\omega_{\rho\sigma}M^{\rho\sigma}$$

$$\approx \varphi(x) - \tfrac{i}{2\hbar}\delta\omega_{\mu\nu}\left[M^{\mu\nu}, \varphi(x)\right]$$

$$\varphi(\Lambda^{-1}x) = \varphi(\Lambda_\nu{}^\rho x^\nu) = \varphi((\delta_\nu{}^\mu + \delta\omega_\nu{}^\mu)x^\nu) = \varphi(x^\mu + \delta\omega_\nu{}^\mu x^\nu)$$

$$\approx \varphi(x^\mu) + \partial_\mu\varphi\,\delta\omega_\nu{}^\mu x^\nu$$

$$= \varphi(x) - \tfrac{1}{2}\delta\omega_\nu{}^\mu(x^\nu\partial_\mu\varphi - \partial_\nu\varphi\, x^\mu)$$

We have $\quad \frac{i}{\hbar}\left[M^{\mu\nu}, \varphi(x)\right] = (x^\nu\partial_\mu - \partial_\nu x^\mu)\varphi.$

$$\Rightarrow \quad \left[\varphi(x), M^{\mu\nu}\right] = \frac{\hbar}{i}\underbrace{(x^\mu\partial_\nu - x^\nu\partial_\mu)}_{\mathcal{L}^{\mu\nu}}\varphi(x).$$

2.9) Let us write

$$\Lambda^\rho{}_\tau = \delta^\rho{}_\tau + \tfrac{i}{2\hbar}\delta\omega_{\mu\nu}(S_{\rm V}^{\mu\nu})^\rho{}_\tau \; , \tag{2.32}$$

where

$$(S_{\rm V}^{\mu\nu})^\rho{}_\tau \equiv \tfrac{\hbar}{i}(g^{\mu\rho}\delta^\nu{}_\tau - g^{\nu\rho}\delta^\mu{}_\tau) \tag{2.33}$$

are matrices which constitute the *vector representation* of the Lorentz generators.

a) Let $\Lambda = 1 + \delta\omega$ in eq. (2.27), and show that

$$[\partial^\rho\varphi(x), M^{\mu\nu}] = \mathcal{L}^{\mu\nu}\partial^\rho\varphi(x) + (S_{\rm V}^{\mu\nu})^\rho{}_\tau\partial^\tau\varphi(x) \; . \tag{2.34}$$

b) Show that the matrices $S_{\rm V}^{\mu\nu}$ must have the same commutation relations as the operators $M^{\mu\nu}$. Hint: see the previous problem.

c) For a rotation by an angle θ about the z axis, we have

$$\Lambda^\mu{}_\nu = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} . \tag{2.35}$$

Show that

$$\Lambda = \exp(-i\theta S_{\rm V}^{12}/\hbar) \; . \tag{2.36}$$

d) For a boost by *rapidity* η in the z direction, we have

$$\Lambda^\mu{}_\nu = \begin{pmatrix} \cosh\eta & 0 & 0 & \sinh\eta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh\eta & 0 & 0 & \cosh\eta \end{pmatrix} . \tag{2.37}$$

Show that

$$\Lambda = \exp(+i\eta S_{\rm V}^{30}/\hbar) \; . \tag{2.38}$$

2. **Lorentz-invariant delta function:** Prove that the inner product of single-particle states

$$\langle \mathbf{k}|\mathbf{k}'\rangle = \langle 0|a_{\mathbf{k}}a^{\dagger}_{\mathbf{k}'}|0\rangle = 2\varepsilon_{\mathbf{k}}(2\pi)^d\delta^d(\mathbf{k}-\mathbf{k}') \tag{1}$$

is Lorentz-invariant. Use the fact that a Lorentz transformation Λ acts on single-particle states as

$$|\mathbf{k}\rangle \to |\tilde{\mathbf{k}}\rangle\,, \qquad \text{with} \quad \tilde{\mathbf{k}}^i = \Lambda^i{}_\mu k^\mu\,, \quad \text{and} \quad k^\mu = (\varepsilon_{\mathbf{k}}, \mathbf{k})\,. \tag{2}$$

In other words, show that $\langle \mathbf{k}|\mathbf{k}'\rangle = \langle \tilde{\mathbf{k}}|\tilde{\mathbf{k}}'\rangle$.

$$|\tilde{k}\rangle = |\Lambda \vec{k}\rangle. \qquad \tilde{k}^i = \Lambda^i{}_\mu k^\mu. \qquad k^\mu = (\varepsilon_{\vec{k}}, \vec{k})$$

$$\langle \tilde{k}|\tilde{k}'\rangle = 2\,\varepsilon_{\Lambda\vec{k}}(2\pi)^d\delta^d(\tilde{k}-\tilde{k}')$$

$$= 2\,\varepsilon_{\Lambda\vec{k}}(2\pi)^d\delta^d(\Lambda\vec{k}-\Lambda\vec{k}')$$

$$= 2\,\varepsilon_{\vec{k}}(2\pi)^d\delta^d(\vec{k}-\vec{k}').$$

$$= \langle \vec{k}|\vec{k}'\rangle.$$

$$k'^\mu k'_\mu = g_{\mu\nu}k'^\mu k'^\nu$$

$$= g_{ij}(\Lambda^i{}_\mu k^\mu)(\Lambda^j{}_\nu k^\nu)$$

$$= g_{\mu\nu}k^\mu k^\nu = k^\mu k_\mu.$$

3. **Total momentum operator:** In the SHO, the operator $a^\dagger a$ counts the occupation number. Build the operator $\hat{N}_{\mathbf{k}}$ that measures the number of quanta with momentum $\mathbf{k}$. Build the operator $\hat{N}$ that measures the total number of quanta. Finally, build the operator $\hat{\mathbf{P}}$ that measures the total momentum of the state (**Hint:** measure each quanta weighted by their momentum). Express this operator in terms of $a_{\mathbf{k}}$, $a^\dagger_{\mathbf{k}'}$, and then $\phi_{\mathbf{k}}$, $\pi_{\mathbf{k}}$. Finally, show that it terms of $\phi(x)$, $\pi(x)$, it takes the form $\hat{P}_i = -\int d^3x\, \hat{\phi}(x)\partial_i\hat{\pi}(x)$. Compute the commutator of this operator with $\hat{\phi}(x)$. Finally, compute $e^{i\mathbf{y}\cdot\hat{\mathbf{P}}}\phi(\mathbf{x})e^{-i\mathbf{y}\cdot\hat{\mathbf{P}}}$ (**Hint:** you can simplify this last point by aligning your axes with the $\mathbf{y}$ vector, so that $\mathbf{y} = (y^1, 0, 0)$).

$$\hat{N}_{\vec{k}} = a^\dagger_{\vec{k}} a_{\vec{k}} . \qquad \hat{N} = \sum_{\vec{k}} a^\dagger_{\vec{k}} a_{\vec{k}} .$$

$$\hat{P} = \sum_{\vec{k}} \vec{k}\, a^\dagger_{\vec{k}} a_{\vec{k}} .$$

$$\phi_{\vec{k}} = \int d^3x\, \hat{\phi}(\vec{x})\, e^{i\vec{k}\cdot\vec{x}} = \int d^3x \left(\int \frac{d^3k}{\sqrt{(2\pi)^3 2\mathcal{E}_k}} \left(a_{\vec{k}} e^{-i\vec{k}\cdot\vec{x}} + a^\dagger_{\vec{k}} e^{i\vec{k}\cdot\vec{x}} \right) \right) e^{i\vec{k}'\cdot\vec{x}}$$

$$= \frac{1}{\sqrt{2\mathcal{E}_{\vec{k}}}} a_{\vec{k}} + \frac{1}{\sqrt{2\mathcal{E}_{\vec{k}}}} a^\dagger_{-\vec{k}} ,$$

$$\partial_0 \phi(\vec{x}) = \int \frac{d^3k}{\sqrt{(2\pi)^3}\, 2\mathcal{E}_{\vec{k}}} (-i\mathcal{E}_k)\left(a_{\vec{k}} e^{-ik\cdot x} - a^\dagger_{\vec{k}} e^{ik\cdot x} \right)$$

$$\pi_{\vec{k}} = \int d^3x\, \partial_0 \phi(\vec{x})\, e^{ik\cdot x} = \frac{-i}{\sqrt{2}} \sqrt{\mathcal{E}_k}\, (a_{\vec{k}} - a^\dagger_{\vec{k}})$$

$$a_{\vec{k}} = \sqrt{2\mathcal{E}_{\vec{k}}}\, \phi_{\vec{k}} + \frac{\sqrt{2}i}{\sqrt{\mathcal{E}_k}} \pi_{\vec{k}} . \qquad a^\dagger_{\vec{k}} = \sqrt{2\mathcal{E}_{\vec{k}}}\, \phi_{\vec{k}} - \frac{\sqrt{2}i}{\sqrt{\mathcal{E}_{\vec{k}}}} \pi_{\vec{k}} .$$

$$\hat{P} = \sum_{\vec{k}} \vec{k} \left(\sqrt{2\mathcal{E}_{-\vec{k}}}\, \phi_{-\vec{k}} - \frac{\sqrt{2}i}{\sqrt{\mathcal{E}_{-\vec{k}}}} \pi_{-\vec{k}} \right) \left(\sqrt{2\mathcal{E}_{\vec{k}}}\, \phi_{\vec{k}} + \frac{\sqrt{2}i}{\sqrt{\mathcal{E}_{\vec{k}}}} \pi_{\vec{k}} \right)$$

$$\hat{P}_i = -\int d^3x\, \hat{\phi}(x)\, \partial_i \hat{\pi}(x) .$$

$$[P_i, \phi(x)] = \int d^3x \left[\pi(x)\,\partial_i\phi(x), \phi(x')\right]$$

$$= -\int d^3x \; i\,\delta^3(\vec{x}-\vec{x}')\,\partial_i\phi(x) = -i\,\partial_i\phi(x)$$

$$e^{i\vec{y}\cdot\vec{P}}\,\phi(x)\,e^{-i\vec{y}\cdot\vec{P}} =$$

4. **More on Lorentz and Poincaré algebras (optional): (i)** Starting from Eq. (2.17) in Srednicki, check that the generators $J_i^{\pm} \equiv J_i + iK_i$ form two $su(2)$ subalgebras: $[J_i^+, J_j^+] = i\epsilon_{ijk}J_k^+$ and $[J_i^-, J_j^-] = i\epsilon_{ijk}J_k^-$. You know from QM that representataions of $su(2)$ are labeled by a spin $s = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$; fields in 3+1d relativistic QFT (such as the standard model!) are therefore labeled by irreducible representations of $su(2) \oplus su(2)$, i.e. with two spins (s_1, s_2). The scalar representation (0,0) is the simplest (the standard model Higgs boson transforms in this representation).

(ii) In class we obtained the Lorentz algebra (Eq. (2.17) in Srednicki). Spacetime translations $x^\mu \to x^\mu + a^\mu$ are also a symmetry of nature. Their generators are $\hat{P}_\mu = (\hat{H}, \hat{\mathbf{P}})$, where $\hat{H}$ is the Hamiltonian. Starting from Eq. (2.15) in Srednicki, obtain the commutator of $\hat{P}_\mu$ with Lorentz generators Eq. (2.19). This, together with the Lorentz algebra (2.16), forms the Poincaré algebra.

5. **Classical field theory and effective field theory: (i)** construct the most general Lorentz-invariant action for a scalar field ϕ that is quadratic in ϕ, and contains at most 4 derivatives. Use integration by parts (you may assume that $\phi(x)$ vanishes at infinity $x \to \infty$) to reduce the number of terms in the action. What are the mass dimensions (units) of the coefficients in the action? **(ii)** obtain the equation of motion. **(iii)** generalize to include cubic terms $O(\phi^3)$, with up to four derivatives.

(i) $\mathcal{L}(\phi, \partial_\mu\phi) = \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - V(\phi)$. $\quad \frac{1}{2}m^2\phi^2$

$$S = \int \mathcal{L}(\phi, \partial_\mu\phi)\, dx^4$$

$$= \mathcal{L}(\phi, \partial_\mu\phi)\, x^4 \Big|^{=0} - \int d\mathcal{L} \cdot x^4$$

$$= -\int \left[\frac{\partial\mathcal{L}}{\partial\phi} d\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} d(\partial_\mu\phi)\right] x^4$$

$$= -\int \left[\frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right] \boxed{d\phi\, x^4}$$

mass dimension :

(ii) $\frac{\partial\mathcal{L}}{\partial\phi} = \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right)$, $\quad (\delta S = 0)$

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \frac{1}{2}m^2\phi^2$$

$$\frac{\partial}{\partial\phi}\left(\frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi\right) - m^2\phi = \frac{1}{2}\partial_\mu(\partial^\mu\phi)$$

$$\Rightarrow \quad \partial_\mu\partial^\mu\phi + m^2\phi = 0.$$

(iii) $\mathcal{L}' = \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{3!}\phi^3$.

$$S = \int \mathcal{L}' dx^4 = \int \left(\frac{1}{2} \partial_\mu \phi \, \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 - \frac{\lambda}{3!} \phi^3 \right) dx^4$$

$$\delta S = \int \left(\partial_\mu \phi \, \partial^\mu (\delta \phi) - m^2 \phi \, \delta \phi - \frac{\lambda}{2} \phi^2 \delta \phi \right) dx^4$$

$$= \int \left(- \partial_\mu \partial^\mu \phi \, \delta \phi - m^2 \phi \, \delta \phi - \frac{\lambda}{2} \phi^2 \delta \phi \right) dx^4$$

$$= - \int \left(\partial_\mu \partial^\mu \phi + m^2 \phi + \frac{\lambda}{2} \phi^2 \right) \delta \phi \, dx^4 .$$

$$\Rightarrow \quad \partial_\mu \partial^\mu \phi + m^2 \phi + \frac{\lambda}{2} \phi^2 = 0 .$$

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1. **Transformation properties of Wightman function:** **(i)** Show that the Wightman function $G_W(x) \equiv \langle 0|\hat{\phi}(t,\mathbf{x})\hat{\phi}(0,0)|0\rangle$ of a Lorentz-invariant theory is unchanged if evaluated at the Lorentz transformed coordinate $\Lambda^{-1}x$, i.e.: $G_W(x) = G_W(\Lambda^{-1}x)$. Show then that its Fourier transform satisfies a similar relation $G_W(p) = G_W(\Lambda^{-1}p)$. **(ii)** Show that the result $G_W(p) = \frac{2\pi}{2\varepsilon_k}\delta(\omega - \varepsilon_k)$ found in class can be made manifestly invariant under Lorentz transformations (strictly, it will only be invariant under proper orthochronous Lorentz transformations, and not invariant under time-reversal). [**Hint:** you might use the fact that for a function f with zeros $f(x_i) = 0$, then $\delta(f(x)) = \sum_i \frac{1}{|f'(x_i)|}\delta(x - x_i)$].

(i) $G_W(\Lambda^{-1}x) = \langle 0|\hat{\phi}(t,\Lambda^{-1}\vec{x})\hat{\phi}(0,0)|0\rangle = \langle 0|\hat{\phi}(t,\vec{x})\hat{\phi}(0,0)|0\rangle = G_W(x)$.

$G_W(\Lambda^{-1}p) = \int dt\, d^dx\, e^{-i\Lambda^{-1}\vec{k}\cdot\vec{x}+i\omega t}\, G_W(x) = ?$

(ii) $G_W(p) = \int dt\, d^dx\, e^{-i\vec{k}\cdot\vec{x}+i\omega t}\, G_W(x)$

$$= \int dt\, \frac{e^{i(\omega-\varepsilon_{\vec{k}})t}}{2\varepsilon_{\vec{k}}} = \frac{2\pi}{2\varepsilon_k}\delta(\omega-\omega_k)$$

$$G_W(\Lambda p) = \frac{2\pi}{2\varepsilon_k}\delta(\Lambda(\omega-\omega_k)) = \frac{2\pi}{2\varepsilon_k}\sum_k \frac{1}{|(\Lambda\omega-\omega_k)'|}\delta(\omega-\omega_k)$$

$$= \frac{2\pi}{2\varepsilon_k}\delta(\sinh\eta\, \omega_x + \cosh\eta\, \varepsilon_{\vec{p}}) = \frac{2\pi}{2\varepsilon_k}\delta(\omega-\omega_k)\left(\sinh\eta + \cosh\eta \frac{d\varepsilon_p}{d q_x}\Big|_{q_x=\omega}\right)^{-1}$$

$$= \frac{2\pi}{2\varepsilon_{k'}}\delta(\omega-\omega_k). \quad \Rightarrow \text{Lorentz invariant.}$$

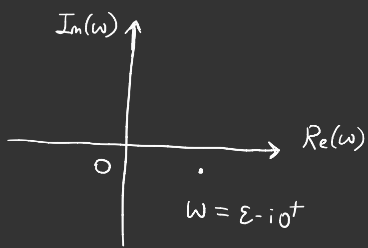
See sol'n

2. **Retarded Green's function:** The *retarded* Green's function is defined as $G^R(t,x) \equiv i\theta(t)\langle 0|[\hat{\phi}(t,\mathbf{x}),\hat{\phi}(0,0)]|0\rangle$, and plays an important role in linear response (the factor of i is a convention). Compute its Fourier transform, following similar steps to the derivation of the Feynman correlator in class. Pay special close attention to the required $i0^+$ one needs to introduce for convergence of the integral. Plot the poles of this correlation function in the complex ω plane. You should find that, except for the $i0^+$'s, the retarded Greens function is proportional to the Feynman one. Finally, show that for $\omega > 0$, $\operatorname{Im} G^R(\omega,\mathbf{k}) = \frac{1}{2}G_W(\omega,\mathbf{k})$ [**Hint**: you may want to use $\operatorname{Im}\frac{1}{x \pm i0^+} = \mp\pi\delta(x)$].

$$G^R(p) = \int dt\, d^d\vec{x}\; e^{-ip_\mu x^\mu}\, G^R(x)$$

$$= \int_0^\infty i\,dt\, \frac{e^{i(\omega - \varepsilon_k + i0^+)}}{2\varepsilon_k} = -\frac{1}{2\varepsilon_k}\,\frac{1}{\omega - \varepsilon_k + i0^+}$$

$$\operatorname{Im} G^R(\omega,\vec{k}) = \frac{\pi}{2\varepsilon_k}\,\delta(\omega - \varepsilon_k) = \frac{1}{2}G_W(\omega,\vec{k})$$



3. **Higher point functions:** **(i)** Show that the three-point function $\langle 0|\hat{\phi}(x_1)\hat{\phi}(x_2)\hat{\phi}(x_3)|0\rangle$ vanishes for the free scalar.

Next you will show that the four-point function 'factorizes' into products of two-point functions:

$$\langle 0|\hat{\phi}(x_1)\hat{\phi}(x_2)\hat{\phi}(x_3)\hat{\phi}(x_4)|0\rangle = G(x_{12})G(x_{34}) + G(x_{13})G(x_{24}) + G(x_{14})G(x_{23})\,, \quad (1)$$

where $x_{ij} = x_i - x_j$ and $G(x) = \langle 0|\hat{\phi}(x)\hat{\phi}(0)|0\rangle$ is the regular (Wightman) two-point function. You will do this in several steps: **(ii)** Define $\hat{\phi}_-$, $\hat{\phi}_+$ as the parts of the operator $\hat{\phi}(x)$ that have the lowering ($a_{\mathbf{k}}$) and raising ($a^\dagger_{-\mathbf{k}}$) operator, respectively:

$$\begin{aligned}\hat{\phi}(x) &= \int \frac{d^d\mathbf{k}}{(2\pi)^d}\frac{1}{2\varepsilon_{\mathbf{k}}} e^{i\mathbf{k}\cdot\mathbf{x}}\left(e^{-i\varepsilon_{\mathbf{k}}t}a_{\mathbf{k}} + e^{i\varepsilon_{\mathbf{k}}t}a^\dagger_{-\mathbf{k}}\right)\\ &\equiv \hat{\phi}_- + \hat{\phi}_+\,.\end{aligned}$$

It will be useful to work with these operators in this exercise. Show that their commutator is just a number (not an operator), equal to the two-point function:

$$[\hat{\phi}_-(x_1), \hat{\phi}_+(x_2)] = G(x_{12})\,.$$

(iii) Define the *normal ordering* of an operator $\mathcal{O}$, denoted by $:\mathcal{O}:$ as the same operator, but with all a's on the right and $a^\dagger$'s on the left, e.g.

$$:a_1(a_2 + a_2^\dagger)a_3a_4^\dagger: \;=\; a_2^\dagger a_4^\dagger a_1 a_3 + a_4^\dagger a_1 a_2 a_3\,.$$

This is a useful definition because $\langle 0|:\mathcal{O}:|0\rangle = 0$ for any $\mathcal{O}$. Show that

$$\hat{\phi}(x_1)\hat{\phi}(x_2) \;=\; :\hat{\phi}(x_1)\hat{\phi}(x_2): \,+\, G(x_{12})\,.$$

(iv) Show that [**Hint:** make use of $\hat{\phi}_\pm$]

$$\hat{\phi}(x_1) :\hat{\phi}(x_2)\hat{\phi}(x_3): \;=\; :\hat{\phi}(x_1)\hat{\phi}(x_2)\hat{\phi}(x_3): \,+\, G(x_{12})\hat{\phi}(x_3) + G(x_{13})\phi(x_2)\,.$$

and therefore

$$\hat{\phi}(x_1)\hat{\phi}(x_2)\hat{\phi}(x_3) \;=\; :\hat{\phi}(x_1)\hat{\phi}(x_2)\hat{\phi}(x_3): \,+\, G(x_{12})\hat{\phi}(x_3) + G(x_{13})\phi(x_2) + G(x_{23})\phi(x_1)\,.$$

(v) Derive a similar expression for $\hat{\phi}(x_1)\hat{\phi}(x_2)\hat{\phi}(x_3)\hat{\phi}(x_4)$, and show that it implies (1).

(i) $\mathcal{L}$ not change under $\phi \to -\phi$.

$\langle 0 | \phi_1 \phi_2 \phi_3 | 0 \rangle = - \langle 0 | \phi_1 \phi_2 \phi_3 | 0 \rangle$ $\Rightarrow$ $\langle 0 | \phi_1 \phi_2 \phi_3 | 0 \rangle$ vanishes.

(ii) $\hat{\phi}(x) = \int \frac{d^d \vec{k}}{(2\pi)^d} \frac{1}{2\varepsilon_k} e^{i\vec{k}\cdot\vec{x}} \left(e^{-i\varepsilon_k t} a_k + e^{i\varepsilon_k t} a^\dagger_{-k} \right)$

$$= \underbrace{\int \frac{d^d \vec{k}}{(2\pi)^d} \frac{1}{2\varepsilon_k} e^{i\vec{k}\cdot\vec{x} - i\varepsilon_k t} a_k}_{\hat{\phi}_-} + \underbrace{\int \frac{d^d \vec{k}}{(2\pi)^d} \frac{1}{2\varepsilon_k} e^{i\vec{k}\cdot\vec{x} + i\varepsilon_k t} a^\dagger_{-k}}_{\hat{\phi}_+}$$

$$[\hat{\phi}_-(x_1), \hat{\phi}_+(x_2)] = \iint \frac{d^d \vec{k}}{(2\pi)^d} \frac{d^d \vec{k}'}{(2\pi)^d} \frac{1}{4\varepsilon_k \varepsilon_{k'}} \left[e^{ikx_1 - i\varepsilon_k t} a_k , e^{ik'x_2 + i\varepsilon_k t} a^\dagger_{-k} \right]$$

$$= \int \frac{d^d \vec{k}}{(2\pi)^d} \frac{(2\pi)^d}{(2\pi)^d} \frac{1}{2\varepsilon_k} e^{ik(x_1 - x_2)} = \langle 0 | \phi(x_1) \phi(x_2) | 0 \rangle$$

Yep, but show more steps next time.

(iii) $\phi(x_1)\phi(x_2) = \left(\phi_-(x_1) + \phi_+(x_1)\right)\left(\phi_-(x_2) + \phi_+(x_2)\right)$

$$= \phi_-(x_1)\phi_-(x_2) + \phi_+(x_1)\phi_-(x_2) + \phi_-(x_1)\phi_+(x_2) + \phi_+(x_1)\phi_+(x_2)$$

$$:\phi(x_1)\phi(x_2): = \phi_-(x_1)\phi_-(x_2) + \phi_-(x_2)\phi_+(x_1) + \phi_-(x_1)\phi_+(x_2) + \phi_+(x_1)\phi_+(x_2)$$

$$\langle 0 | \phi(x_1)\phi(x_2) | 0 \rangle = \langle 0 | \phi_-(x_1)\phi_+(x_2) | 0 \rangle = [\phi_-(x_1), \phi_+(x_2)].$$

$$\Rightarrow \quad \phi(x_1)\phi(x_2) = :\phi(x_1)\phi(x_2): + \langle 0 | \phi(x_1)\phi(x_2) | 0 \rangle.$$

(iv) $\phi_1 :\phi_2\phi_3: = :\phi_1\phi_2\phi_3: + \phi_1\phi_2\phi_3 - :\phi_1\phi_2:\phi_3 + \phi_1\phi_3\phi_2 - :\phi_1\phi_3:\phi_2$

?

4. **Dimensional analysis and correlation functions: (i)** Consider the free *massless* relativistic scalar:

$$S = -\int d^{d+1}x\, \frac{1}{2}(\partial_\mu \phi)^2 .$$

Given that we have set the mass $m = 0$, there is no scale in this theory. In quantum and relativistic units (where $c = \hbar = 1$), time and spatial derivatives have dimensions of energy $\partial_t \sim \partial_x \sim E$. Show that for the action to be dimensionless, the dimensions of ϕ is $\phi \sim E^{(d-1)/2}$. Using Lorentz invariance and dimensional analysis, can you guess what the Wightman function $G_W(x) = \langle 0|\phi(t,\mathbf{x})\phi|0\rangle$ is equal to in this theory (up to a factor)? How does your result fall off for $t = 0$, $\mathbf{x} \to \infty$? **(ii)** Let us contrast this to a theory with a finite mass gap $m \neq 0$. We found in class that the Wightman function was equal to

$$G_W(t,\mathbf{x}) = \int \frac{d^d\mathbf{k}}{(2\pi)^d}\frac{e^{i\mathbf{k}\cdot x - i\varepsilon_{\mathbf{k}} t}}{2\varepsilon_{\mathbf{k}}},$$

with $\varepsilon_{\mathbf{k}} = \sqrt{\mathbf{k}^2 + m^2}$. Because this integral is difficult to compute, we will focus on $d = 1$ spatial dimension, and set $t = 0$. Using the tool of your choice (Mathematica, tables of integrals, the internet, or integration skills), show that

$$G_W(0,x) = \frac{1}{2\pi} K_0(m|x|)\,, \tag{2}$$

where K_n is a modified Bessel function. How does this behave at long distances $m|x| \gg 1$? You should find that it decays much faster than in the $m = 0$ case: gapped systems are *short-range* correlated.

So what is $[\phi]$?

(i) $\partial_\mu \phi \sim E^{\frac{d+1}{2}}$. $S \sim \int d^{d+1}x\, E^{d+1}$. dimensionless.

$G_W(x) =$?

(ii) $G_W(t,x) = \int \frac{d\vec{k}}{2\pi} \frac{e^{ikx - i\varepsilon_k t}}{2\varepsilon_k}$.

$$G_w(0,x) = \int \frac{dk}{2\pi} \frac{e^{ikx}}{2\sqrt{k^2+m^2}}$$

$$= \frac{1}{2\pi} K_0(m|x|).$$

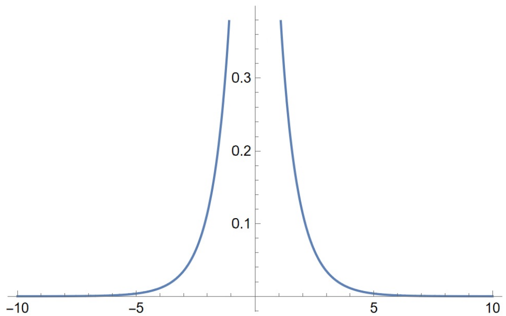
In[2]:= $\displaystyle\int_{-\infty}^{\infty} \frac{e^{i k x}}{2\sqrt{k^2+m^2}}\, dk$

Out[2]= $\displaystyle\sqrt{\frac{1}{m^2}}\,\sqrt{m^2}\ \text{BesselK}\left[0, \frac{\text{Abs}[x]}{\sqrt{\frac{1}{m^2}}}\right]$ if $x \in \mathbb{R}$ && $\text{Re}[m^2] > 0$

In[6]:= m = 1;

$\displaystyle\text{Plot}\left[\left\{\text{BesselK}\left[0, \frac{\text{Abs}[x]}{\sqrt{\frac{1}{m^2}}}\right]\right\}, \{x, -10, 10\}\right]$

Out[7]=



QFT Pset 4 yuxiang@uchicago.edu

1. **Gaussian integrals: (i)** Consider the integral

$$I_n = \int_{-\infty}^{\infty} dx\, x^n e^{-Mx^2}, \qquad n \in \mathbb{N}. \tag{1}$$

Argue using symmetry that $I_n = 0$ if n is odd. Next, show for $n = 2m$ even, $I_{2m} = (-1)^m \partial_M^m I_0$. Finally, compute (or look up) I_0, and obtain an expression for I_{2m}. Show that $(I_4/I_0) = 3(I_2/I_0)^2$, and speculate about a possible connection with problem 3 in Problem Set 3.

(ii) Next we consider Gaussian integrals with multiple variables $x_i \in \mathbb{R}$

$$I(M,\alpha) = \int dx_1 \cdots dx_N\, e^{-\frac{1}{2}M_{ij}x_i x_j + \alpha_i x_i}, \tag{2}$$

where $i, j = 1, 2, \ldots, N$, and repeated indices are summed. The exponent can also be written $-\frac{1}{2}x^T M x + \alpha^T x$ in matrix notation. The integral is over $\mathbb{R}^N$. First show, using a suitable change of variable, that $I(M,\alpha) = e^{\frac{1}{2}\alpha^T M^{-1}\alpha} I(M,0)$. Then, compute $I(M,0)$ by using the orthogonal matrix O that diagonalizes M.

(iii) As a toy version of path integrals in QM/QFT, we will consider the regular integral $I(M,0)$ with 'action' $\frac{1}{2}x^T M x$. We define the expectation value of the 'operator' $\hat{x}_i$, or more generally of a function of all the $\{\hat{x}_i\}$, as

$$\langle f(\{\hat{x}_i\})\rangle \equiv \frac{1}{I(M,0)} \int dx_1 \cdots dx_N\, f(\{x_i\}) e^{-\frac{1}{2}M_{ij}x_i x_j}. \tag{3}$$

(for example, we will consider $f(\{\hat{x}_i\}) = \hat{x}_i\hat{x}_j$, and $f(\{\hat{x}_i\}) = \hat{x}_i\hat{x}_j\hat{x}_k\hat{x}_l$, below). After checking that $\langle 1\rangle = 1$, show that $\langle \hat{x}_i\hat{x}_j\rangle = (M^{-1})_{ij}$ [**Hint**: you may want to first show that $\langle \hat{x}_i\hat{x}_j\rangle = \frac{1}{I(M,0)}\left(\partial_{\alpha_i}\partial_{\alpha_j} I(M,\alpha)\right)|_{\alpha=0}$, where α is set to zero after taking the derivatives]. Generalize to show that $\langle f(\{\hat{x}_i\})\rangle = \frac{1}{I(M,0)}\left(f(\{\partial_{\alpha_i}\})I(M,\alpha)\right)|_{\alpha=0}$.

(i) $I_n = \int_{-\infty}^{+\infty} dx\, x^n e^{-Mx^2}$

For odd n, $I_n(-x) = \int_{-\infty}^{+\infty} -dx\,(-x)^n e^{-Mx^2} = I_n(x)$.

$$\int_{-\infty}^{+\infty} dx\, x^n e^{-Mx^2} = \int_{-\infty}^{0} dx\, x^n e^{-Mx^2} + \int_{0}^{+\infty} dx\, x^n e^{-Mx^2}$$

$$= \int_{+\infty}^{0} d(-x)\, (-x)^n e^{-Mx^2} + \int_{0}^{+\infty} dx\, x^n e^{-Mx^2}$$

$$= -\int_{0}^{+\infty} dx\, x^n e^{-Mx^2} + \int_{0}^{+\infty} dx\, x^n e^{-Mx^2} = 0.$$

For $n = 2m$.

$$\partial_M I_0 = \int_{-\infty}^{+\infty} dx\, \partial_M e^{-Mx^2} = \int_{-\infty}^{+\infty} dx\, -x^2 e^{-Mx^2}.$$

$$\partial_M^m I_0 = \int_{-\infty}^{+\infty} dx\, (-x^2)^m e^{-Mx^2} = (-1)^m \int_{-\infty}^{+\infty} dx\, x^{2m} e^{-Mx^2}$$

$$I_{2m} = \int_{-\infty}^{+\infty} dx\, x^{2m} e^{-Mx^2} = (-1)^m \partial_M^m I_0.$$

$$I_0 = \int_{-\infty}^{+\infty} dx\, e^{-Mx^2} = \sqrt{\frac{\pi}{M}} = \sqrt{\pi}\, M^{-\frac{1}{2}}. \qquad \partial_M I_0 = -\tfrac{1}{2}\sqrt{\pi}\, M^{-\frac{3}{2}}.$$

$$I_{2m} = (-1)^m \cdot \left(-\tfrac{1}{2}\right)\cdot\left(-\tfrac{3}{2}\right)\cdots\left(-\tfrac{2m-1}{2}\right)\sqrt{\pi}\, M^{-\frac{2m+1}{2}} = \frac{(2m-1)!!}{2^m}\sqrt{\pi}\, M^{-(m+\frac{1}{2})}$$

$$\frac{I_4}{I_0} = \frac{\frac{3}{4} M^{-\frac{5}{2}}}{M^{-\frac{1}{2}}} = \frac{3}{4} M^{-2}. \qquad \frac{I_2}{I_0} = \frac{\frac{1}{2} M^{-\frac{3}{2}}}{M^{-\frac{1}{2}}} = \frac{1}{2} M^{-1}.$$

$$\frac{I_4}{I_0} = 3\left(\frac{I_2}{I_0}\right)^2.$$

Connection: 4 point function "factorize" into sum of products of 2 point functions.

$$\langle 0|\phi_1\phi_2\phi_3\phi_4|0\rangle = G(x_{12})\,G(x_{34}) + G(x_{13})\,G(x_{24}) + G(x_{14})\,G(x_{23}).$$

(ii) $$I(M,\alpha) = \int dx_1 \ldots dx_N\, e^{-\frac{1}{2}M_{ij}x_i x_j + \alpha_i x_i} = \int dx_1 \ldots dx_N\, e^{-\frac{1}{2}x^T M x + \alpha^T x}$$

$$-\frac{1}{2}x^T M x + \alpha^T x = -\frac{1}{2}\left(x^T M + 2\alpha^T\right)x$$ make this $x'^T x'$.

$$= -\frac{1}{2}\left(x^T + 2\alpha^T M^{-1}\right) M x .$$

$x \to x + rM^{-1}\alpha$.
$$= -\frac{1}{2}\left(x^T - (r-2)\alpha^T M^{-1}\right) M \left(x - rM^{-1}\alpha\right)$$

$$= -\frac{1}{2}\left[x^T M x - (r-2)\alpha^T x - r x^T \alpha + r(r-2)\alpha^T M^{-1}\alpha\right]$$

set $r=1$.
$$= -\frac{1}{2}x^T M x + \frac{1}{2}\alpha^T M^{-1}\alpha .$$

$$I(M,\alpha) = \int dx_1 \dots dx_N \, e^{-\frac{1}{2}x^T M x + \alpha^T x}$$

$$= e^{-\frac{1}{2}\alpha^T M^{-1}\alpha} \int dx_1 \dots dx_N \, e^{-\frac{1}{2}x^T M x} \, e^{\frac{1}{2}\alpha^T M^{-1}\alpha}$$

$$\xrightarrow{x' \to x + M^{-1}\alpha} = e^{-\frac{1}{2}\alpha^T M^{-1}\alpha} \int dx_1' \dots dx_N' \, e^{-\frac{1}{2}x'^T M x'} = e^{-\frac{1}{2}\alpha^T M^{-1}\alpha} \, I(M,0) .$$

compute
$$I(M,0) = \int dx_1 \dots dx_N \, e^{-\frac{1}{2}x^T M x}$$
$O^{-1} M O = D$.

$$= \int dx_1 \dots dx_N \, e^{-\frac{1}{2}x^T O D O^{-1} x}$$

$$\xrightarrow{x' \to O^{-1} x} = \int dx_1' \dots dx_N' \, e^{-\frac{1}{2}x'^T D x'} = \int dx_1' \dots dx_N' \, e^{-\frac{1}{2}(m_1 + \dots m_N) x'^2}$$

$$= \prod_{i=1}^{N} \sqrt{\frac{2\pi}{m_i}} .$$

(ii)
$$\langle 1 \rangle = \frac{1}{I(M,0)} \int dx_1 \dots dx_N \, e^{-\frac{1}{2}M_{ij}x_i x_j} = 1 .$$

$$\frac{1}{I(M,0)}\left(\partial_{\alpha_i}\partial_{\alpha_j} I(M,\alpha)\right)\Big|_{\alpha=0} = \frac{1}{I(M,0)} \int dx_1 \dots dx_N \, x_i x_j \, e^{-\frac{1}{2}M_{kl}x_k x_l + \alpha_k x_k}\Big|_{\alpha=0}$$

$$= \frac{1}{I(M,0)} \int dx_1 \dots dx_N \, x_i x_j \, e^{-\frac{1}{2}M_{kl}x_k x_l} = \langle x_i x_j \rangle .$$

As we already showed, $I(M,\alpha) = e^{\frac{1}{2}\alpha^T M^{-1}\alpha} I(M,0)$.

$$\langle x_i x_j \rangle = \partial_{\alpha_i}\partial_{\alpha_j} e^{\frac{1}{2}\sum \alpha_k^T M_{k\ell}^{-1}\alpha_\ell}\Big|_{\alpha=0} = ?$$

not quite! include exp term.

$$\frac{\partial}{\partial\alpha_j}\left(\sum_{k\ell}\alpha_k M_{k\ell}^{-1}\alpha_\ell\right) = \sum_\ell M_{j\ell}^{-1}\alpha_\ell + \sum_k \alpha_k M_{kj}^{-1} = 2\sum_\ell M_{j\ell}^{-1}\alpha_\ell.$$

$$\frac{\partial}{\partial\alpha_i}(\quad) = 2M_{ij}^{-1}.$$

$$\Rightarrow \langle x_i x_j \rangle = M_{ij}^{-1} e^{\frac{1}{2}\alpha^T M^{-1}\alpha}\Big|_{\alpha=0} = M_{ij}^{-1}.$$

Correction:

$$\partial_{\alpha_j} e^{\frac{1}{2}\sum \alpha_k^T M_{k\ell}^{-1}\alpha_\ell} = \frac{1}{2}\sum_\ell M_{j\ell}^{-1}\alpha_\ell\, e^{\frac{1}{2}\alpha^T M^{-1}\alpha} + \frac{1}{2}\sum_k \alpha_k M_{kj}^{-1} e^{\frac{1}{2}\alpha^T M^{-1}\alpha}$$

$$= \sum_\ell M_{j\ell}^{-1}\alpha_\ell\, e^{\frac{1}{2}\alpha^T M^{-1}\alpha}$$

$$\partial_{\alpha_i}\left(\partial_{\alpha_j} e^{\frac{1}{2}\sum \alpha_k^T M_{k\ell}^{-1}\alpha_\ell}\right) = M_{ji}^{-1} e^{\frac{1}{2}\alpha^T M^{-1}\alpha} + \sum_\ell M_{j\ell}^{-1}\alpha_\ell \sum_k M_{ik}^{-1}\alpha_k\, e^{\frac{1}{2}\alpha^T M^{-1}\alpha}$$

$$\xrightarrow{\text{set } \alpha=0} \langle x_i x_j \rangle = M_{ji}^{-1} = (M^{-1})_{ij}. \qquad (\text{symmetric } M)$$

$$\frac{1}{I(M,0)}\left(f(\{\partial\alpha_i\})\, I(M,\alpha)\right)\Big|_{\alpha=0} = \frac{1}{I(M,0)} f(\partial\alpha_i)\left(\int dx_1\cdots dx_N\, e^{-\frac{1}{2}M_{ij}x_i x_j + \alpha_i x_i}\right)\Big|_{\alpha=0}$$

(due to linearity?)

$$= \frac{1}{I(M,0)}\int dx_1\cdots dx_N\, f(\partial\alpha_i)\, e^{-\frac{1}{2}M_{ij}x_i x_j + \alpha_i x_i}\Big|_{\alpha=0}$$

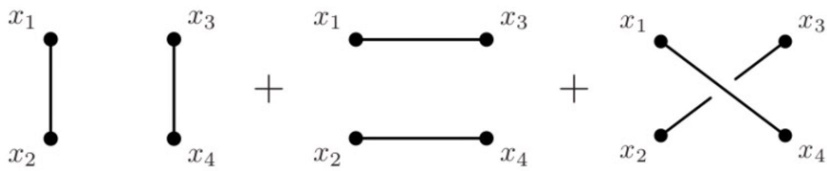
$$= \langle f(\{x_i\})\rangle \qquad \times$$

$$= f(\{\partial\alpha_i\})\, e^{\frac{1}{2}\alpha^T M^{-1}\alpha}\Big|_{\alpha=0}$$

Finally, show that

$$\langle \hat{x}_i \hat{x}_j \hat{x}_k \hat{x}_l \rangle = \langle \hat{x}_i \hat{x}_j \rangle \langle \hat{x}_k \hat{x}_l \rangle + \langle \hat{x}_i \hat{x}_k \rangle \langle \hat{x}_j \hat{x}_l \rangle + \langle \hat{x}_i \hat{x}_l \rangle \langle \hat{x}_j \hat{x}_k \rangle \,. \tag{4}$$

This correlator is, in a sense, 'disconnected' – it factors into two-point functions, as illustrated below:



In fact, one usually defines the *connected* 4-point function $\langle \hat{x}_i \hat{x}_j \hat{x}_k \hat{x}_l \rangle_c$ as the left-hand side of (4) minus the right-hand side of (4). Clearly, the connected 4-point vanishes for the Gaussian theory, as you have just shown, but for more complicated theories it will be non-vanishing. For a more general theory

$$I(\alpha) = \int dx_1 \cdots dx_N \, e^{-S(\{x\}) + \alpha_i x_i} \,, \tag{5}$$

the logarithm of the path integral can be used to generate connected correlators. Show that for the 4-point function (still using a definition analogous to (3)), i.e.:

$$\langle \hat{x}_i \hat{x}_j \hat{x}_k \hat{x}_l \rangle_c = \Big(\partial_{\alpha_i} \partial_{\alpha_j} \partial_{\alpha_k} \partial_{\alpha_l} \log I(\alpha) \Big) |_{\alpha=0} \,. \tag{6}$$

(iv) In QM and QFT, we will typically have to deal with Gaussian integrals with imaginary coefficients. Using complex analysis, show that

$$\int_{-\infty}^{\infty} dx \, x^n \, e^{iMx^2} = e^{\frac{i\pi}{4}(n+1)\mathrm{sgn}M} I_n \,, \tag{7}$$

where I_n is the integral from Eq. (1). "Wick's theorem", which you proved in point (iii), therefore also works for these complex integrals. [**Hint:** show by contour integration that the integral along the real line $\int_{-\Lambda}^{\Lambda} dx \, x^n \, e^{iMx^2}$ is equal to the integral along the line rotated by 45 degrees $\int_{-\Lambda e^{i\pi/4}}^{\Lambda e^{i\pi/4}} dx \, x^n \, e^{iMx^2}$, plus a contribution from an arc that vanishes as $\Lambda \to \infty$]

$$\langle x_i x_j x_k x_l \rangle = \partial_{\alpha_i}\partial_{\alpha_j}\partial_{\alpha_k}\partial_{\alpha_l}\, e^{\frac{1}{2}\alpha^T M^{-1}\alpha}\Big|_{\alpha=0}$$

$$\begin{matrix} i & j \\ k & l \\ m & n \end{matrix}$$

$$= \partial_{\alpha_i}\partial_{\alpha_j}\left(M^{-1}_{lk}\, e^{\frac{1}{2}\alpha^T M^{-1}\alpha} + \sum_n M^{-1}_{ln}\alpha_n \sum_m M^{-1}_{km}\alpha_m\, e^{\frac{1}{2}\alpha^T M^{-1}\alpha} \right)\Big|_{\alpha=0}$$

$$= M^{-1}_{lk}\,\partial_{\alpha_i}\partial_{\alpha_j}\, e^{\frac{1}{2}\alpha^T M^{-1}\alpha} + \partial_{\alpha_i}\left(\sum_n M^{-1}_{ln}\alpha_n \cdot M^{-1}_{kj}\, e^{\frac{1}{2}\alpha^T M^{-1}\alpha} + M^{-1}_{lj} \sum_m M^{-1}_{km}\alpha_m\, e^{\frac{1}{2}\alpha^T M^{-1}\alpha} \right)$$

other term will be 0 after setting $\alpha = 0$

$$= M^{-1}_{lk} M^{-1}_{ij} + M^{-1}_{li} M^{-1}_{kj} + M^{-1}_{lj} M^{-1}_{ki}$$

$$= \langle x_i x_j \rangle \langle x_k x_l \rangle + \langle x_i x_l \rangle \langle x_j x_k \rangle + \langle x_i x_k \rangle \langle x_j x_l \rangle$$

For connected 4-point function,

$$I(\alpha) = \int dx_1 \cdots dx_N\, e^{-S(\{x\}) + \alpha_i x_i} \qquad \partial_{\alpha_l} \log I = \frac{1}{I}\frac{\partial I}{\partial \alpha_l}$$

$$\partial_{\alpha_i}\partial_{\alpha_j}\partial_{\alpha_k}\partial_{\alpha_l} \log I(\alpha)\Big|_{\alpha=0} = \partial_{\alpha_i}\partial_{\alpha_j}\partial_{\alpha_k}\left(\frac{1}{I(\alpha)} \int dx_1 \dots dx_N\, x_l\, e^{-S(\{x\}) + \alpha_i x_i} \right)\Big|_{\alpha=0}$$

$$= \partial_{\alpha_i}\partial_{\alpha_j}\left(\frac{1}{I(\alpha)} \int dx_1 \dots dx_N\, x_l x_k\, e^{-S(\{x\}) + \alpha_i x_i} - \frac{1}{I^2(\alpha)} \int dx_1 \dots dx_N\, x_l\, e^{-S(\{x\}) + \alpha_i x_i} \int dx_1 \dots dx_N\, x_k\, e^{-S(\{x\}) + \alpha_i x_i} \right)\Big|_{\alpha=0}$$

$$= \partial_{\alpha_i}\left(\frac{1}{I(\alpha)} \int dx_1 \cdots dx_N\, x_l x_k x_j\, e^{-S(\{x\}) + \alpha_i x_i} - \frac{1}{I^2(\alpha)} \int dx_1 \dots dx_N\, x_l x_k\, e^{-S(\{x\}) + \alpha_i x_i} \int dx_1 \dots dx_N\, x_j\, e^{-S(\{x\}) + \alpha_i x_i} \right.$$

$$- \frac{1}{I^2(\alpha)} \int dx_1 \dots dx_N\, x_j x_l\, e^{-S(\{x\}) + \alpha_i x_i} \int dx_1 \dots dx_N\, x_k\, e^{-S(\{x\}) + \alpha_i x_i}$$

$$- \frac{1}{I^2(\alpha)} \int dx_1 \dots dx_N\, x_l\, e^{-S(\{x\}) + \alpha_i x_i} \int dx_1 \dots dx_N\, x_j x_k\, e^{-S(\{x\}) + \alpha_i x_i}$$

$$\left. + \frac{2}{I^3(\alpha)} \int dx_1 \dots dx_N\, x_l\, e^{-S(\{x\}) + \alpha_i x_i} \int dx_1 \dots dx_N\, x_k\, e^{-S(\{x\}) + \alpha_i x_i} \int dx_1 \dots dx_N\, x_j\, e^{-S(\{x\}) + \alpha_i x_i} \right)\Big|_{\alpha=0}$$

$$= \frac{1}{I(\alpha)} \int dx_1 \cdots dx_N \, x_l x_k x_j x_i \, e^{-S(\{x\}) + \alpha_i x_i} - \frac{1}{I^2(\alpha)} \int dx_1 \cdots dx_N \, x_l x_k x_j \, e^{-S(\{x\}) + \alpha_i x_i} \int dx_1 \cdots dx_N \, x_i \, e^{-S(\{x\}) + \alpha_i x_i}$$

$$- \frac{1}{I^2(\alpha)} \int dx_1 \cdots dx_N \, x_l x_k \, e^{-S(\{x\}) + \alpha_i x_i} \int dx_1 \cdots dx_N \, x_i x_j \, e^{-S(\{x\}) + \alpha_i x_i}$$

$$- \frac{1}{I^2(\alpha)} \int dx_1 \cdots dx_N \, x_l x_k x_i \, e^{-S(\{x\}) + \alpha_i x_i} \int dx_1 \cdots dx_N \, x_j \, e^{-S(\{x\}) + \alpha_i x_i}$$

$$+ \frac{2}{I^3(\alpha)} \int dx_1 \cdots dx_N \, x_l x_k \, e^{-S(\{x\}) + \alpha_i x_i} \int dx_1 \cdots dx_N \, x_j \, e^{-S(\{x\}) + \alpha_i x_i} \int dx_1 \cdots dx_N \, x_i \, e^{-S(\{x\}) + \alpha_i x_i}$$

$$- \frac{1}{I^2(\alpha)} \int dx_1 \cdots dx_N \, x_j x_l \, e^{-S(\{x\}) + \alpha_i x_i} \int dx_1 \cdots dx_N \, x_i x_k \, e^{-S(\{x\}) + \alpha_i x_i}$$

$$- \frac{1}{I^2(\alpha)} \int dx_1 \cdots dx_N \, x_j x_l x_i \, e^{-S(\{x\}) + \alpha_i x_i} \int dx_1 \cdots dx_N \, x_k \, e^{-S(\{x\}) + \alpha_i x_i}$$

$$+ \frac{2}{I^3(\alpha)} \int dx_1 \cdots dx_N \, x_j x_l \, e^{-S(\{x\}) + \alpha_i x_i} \int dx_1 \cdots dx_N \, x_k \, e^{-S(\{x\}) + \alpha_i x_i} \int dx_1 \cdots dx_N \, x_i \, e^{-S(\{x\}) + \alpha_i x_i}$$

$$- \frac{1}{I^2(\alpha)} \int dx_1 \cdots dx_N \, x_i x_l \, e^{-S(\{x\}) + \alpha_i x_i} \int dx_1 \cdots dx_N \, x_j x_k \, e^{-S(\{x\}) + \alpha_i x_i}$$

$$- \frac{1}{I^2(\alpha)} \int dx_1 \cdots dx_N \, x_l \, e^{-S(\{x\}) + \alpha_i x_i} \int dx_1 \cdots dx_N \, x_i x_j x_k \, e^{-S(\{x\}) + \alpha_i x_i}$$

$$+ \frac{2}{I^3(\alpha)} \int dx_1 \cdots dx_N \, x_l \, e^{-S(\{x\}) + \alpha_i x_i} \int dx_1 \cdots dx_N \, x_j x_k \, e^{-S(\{x\}) + \alpha_i x_i} \int dx_1 \cdots dx_N \, x_i \, e^{-S(\{x\}) + \alpha_i x_i}$$

$$+ \frac{2}{I^3(\alpha)} \int dx_1 \cdots dx_N \, x_i x_l \, e^{-S(\{x\}) + \alpha_i x_i} \int dx_1 \cdots dx_N \, x_k \, e^{-S(\{x\}) + \alpha_i x_i} \int dx_1 \cdots dx_N \, x_j \, e^{-S(\{x\}) + \alpha_i x_i}$$

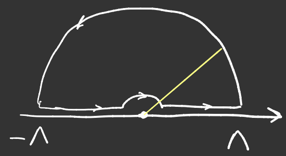
$$+ \frac{2}{I^3(\alpha)} \int dx_1 \cdots dx_N \, x_l \, e^{-S(\{x\}) + \alpha_i x_i} \int dx_1 \cdots dx_N \, x_i x_k \, e^{-S(\{x\}) + \alpha_i x_i} \int dx_1 \cdots dx_N \, x_j \, e^{-S(\{x\}) + \alpha_i x_i}$$

$$+ \frac{2}{I^3(\alpha)} \int dx_1 \cdots dx_N \, x_l \, e^{-S(\{x\}) + \alpha_i x_i} \int dx_1 \cdots dx_N \, x_k \, e^{-S(\{x\}) + \alpha_i x_i} \int dx_1 \cdots dx_N \, x_i x_j \, e^{-S(\{x\}) + \alpha_i x_i}$$

$$- \frac{6}{I^4(\alpha)} \int dx_1 \cdots dx_N \, x_l \, e^{-S(\{x\}) + \alpha_i x_i} \int dx_1 \cdots dx_N \, x_k \, e^{-S(\{x\}) + \alpha_i x_i} \int dx_1 \cdots dx_N \, x_j \, e^{-S(\{x\}) + \alpha_i x_i} \int dx_1 \cdots dx_N \, x_i \, e^{-S(\{x\}) + \alpha_i x_i}$$

other terms cancell out ?

(iv) $\int_{-\infty}^{+\infty} dx\, x^n e^{iMx^2}$



run out of time, will make up later

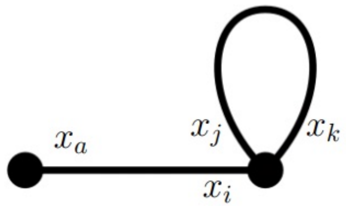
2. **Non-Gaussian integrals:** In the path integral formalism, interactions in QFT will enter as non-Gaussian terms in the integrals (e.g., $S_{\text{int}} = \int \lambda\phi^3 + \lambda'\phi^4$). As a toy version of an interacting QFT, we will consider the integral

$$\int dx_1 \cdots dx_N \, e^{-\frac{1}{2}M_{ij}x_ix_j - \frac{\lambda_{ijk}}{3!}x_ix_jx_k} \,. \tag{8}$$

We will assume the 'interaction' λ is small, and work perturbatively in λ. In practice, this means expanding the exponential—in this problem we will only expand to linear order in λ. **(i)** Considering 'expectation values' as in problem 1, let us start with the 1-point function in the interacting theory:

$$\langle \hat{x}_a \rangle_\lambda = \langle \hat{x}_a \rangle_0 + O(\lambda) \,, \tag{9}$$

where $\langle \cdot \rangle_\lambda$ denotes expectation values in the interacting theory, and $\langle \cdot \rangle_0$ in the theory with $\lambda = 0$. Show that $\langle \hat{x}_a \rangle_0 = 0$ by symmetry, and then find the $O(\lambda)$ term (you can express your answer in terms of 'free' 2-point functions, $\langle \hat{x}_i \hat{x}_j \rangle_0$). Your answer might be illustrated by the diagram:



(ii) Next, compute the 3-point function in the interacting theory

$$\langle \hat{x}_a \hat{x}_b \hat{x}_c \rangle \,, \tag{10}$$

to leading order in λ. Try to draw diagrams representing each term you find. Finally, compute the *connected* correlator

$$\langle \hat{x}_a \hat{x}_b \hat{x}_c \rangle_{\text{con}} = \langle \hat{x}_a \hat{x}_b \hat{x}_c \rangle - \langle \hat{x}_a \rangle \langle \hat{x}_b \hat{x}_c \rangle - \langle \hat{x}_b \rangle \langle \hat{x}_c \hat{x}_a \rangle - \langle \hat{x}_c \rangle \langle \hat{x}_a \hat{x}_b \rangle \tag{11}$$

to leading order in λ. Show that only the fully connected diagram survives.

(i)

$$\langle x_a \rangle_0 = \frac{\int dx_1 \cdots dx_N \; x_a \, e^{-\frac{1}{2}M_{ij}x_i x_j}}{\int dx_1 \cdots dx_N \; e^{-\frac{1}{2}M_{ij}x_i x_j}} = \frac{\left(\int \cdots\right)\int dx_a \; x_a \, e^{-\frac{1}{2}M_{aa}x_a^2}}{\int dx_1 \cdots dx_N \; e^{-\frac{1}{2}M_{ij}x_i x_j}}$$

As we proved $I_n = 0$ for odd n. $\langle x_a \rangle_0 = 0$.

$$\langle x_a \rangle_\lambda = \frac{\int dx_1 \cdots dx_N \; x_a \, e^{-\frac{1}{2}M_{ij}x_i x_j - \frac{\lambda_{ijk}}{3!} x_i x_j x_k}}{\int dx_1 \cdots dx_N \; e^{-\frac{1}{2}M_{ij}x_i x_j - \frac{\lambda_{ijk}}{3!} x_i x_j x_k}}$$

3. **Srednicki Problem 7.3**

7.3) a) Use the Heisenberg equation of motion, $\dot{A} = i[H, A]$, to find explicit expressions for $\dot{Q}$ and $\dot{P}$. Solve these to get the Heisenberg-picture operators $Q(t)$ and $P(t)$ in terms of the Schrödinger picture operators Q and P.

b) Write the Schrödinger picture operators Q and P in terms of the creation and annihilation operators a and $a^\dagger$, where $H = \hbar\omega(a^\dagger a + \frac{1}{2})$. Then, using your result from part (a), write the Heisenberg-picture operators $Q(t)$ and $P(t)$ in terms of a and $a^\dagger$.

c) Using your result from part (b), and $a|0\rangle = \langle 0|a^\dagger = 0$, verify eqs. (7.16) and (7.17).

(a) $\dot{Q} = i[H, Q] = i\left[\frac{P^2}{2m} + \frac{m\omega^2}{2}Q^2, Q\right]$ $\qquad [Q,P] = i$.

$= \frac{i}{2m}[P^2, Q] + \frac{im\omega^2}{2}[Q^2, Q]$

$= \frac{i}{2m}\left([P,Q]P + P[P,Q]\right) = \frac{P}{m}$.

$\dot{P} = i[H, P] = i\left[\frac{P^2}{2m} + \frac{m\omega^2}{2}Q^2, P\right]$

$= \frac{im\omega^2}{2}[Q^2, P] = -m\omega^2 Q$.

$\Rightarrow \ddot{P} = -m\omega^2\dot{Q} = -\omega^2 P$. $\quad P(t) = Ae^{i\omega t} + Be^{-i\omega t}$

$Q(t) = -\frac{1}{m\omega^2}\left(i\omega A e^{i\omega t} - i\omega B e^{-i\omega t}\right)$

$= \frac{-i}{m\omega}Ae^{i\omega t} + \frac{i}{m\omega}Be^{-i\omega t}$

A.B dependent on initial condition.

$P(0) = A + B$. $\quad Q(0) = \frac{-i}{m\omega}A + \frac{i}{m\omega}B$.

$$\Rightarrow \quad A = \frac{1}{2}\left(P(0) + im\omega\, Q(0)\right).$$

$$B = \frac{1}{2}\left(P(0) - im\omega\, Q(0)\right).$$

(b) $H = \hbar\omega\left(a^{+}a + \frac{1}{2}\right)$. $\quad H = \frac{P^2}{2m} + \frac{m\omega^2}{2} Q^2$.

$$Q = \sqrt{\frac{\hbar}{2m\omega}}\,(a + a^{+}) \qquad P = i\sqrt{\frac{m\hbar\omega}{2}}\,(-a + a^{+}) \qquad \to P(0),\ Q(0).$$

$$\Rightarrow \quad P(t) = \frac{1}{2}\left(P(0) + im\omega Q(0)\right) e^{i\omega t} + \frac{1}{2}\left(P(0) - im\omega Q(0)\right) e^{-i\omega t}.$$

$$= \frac{1}{2}\left(i\sqrt{\frac{m\hbar\omega}{2}}\, 2a^{+}\right) e^{i\omega t} + \frac{1}{2}\left(i\sqrt{\frac{m\hbar\omega}{2}} \cdot -2a\right) e^{-i\omega t}$$

$$= i\sqrt{\frac{m\hbar\omega}{2}}\left(a^{+}e^{i\omega t} - a e^{-i\omega t}\right).$$

$$Q(t) = \frac{-i}{m\omega}\left(i\sqrt{\frac{m\hbar\omega}{2}}\, a^{+}\right) e^{i\omega t} + \frac{i}{m\omega}\left(-i\sqrt{\frac{m\hbar\omega}{2}}\, a\right) e^{-i\omega t}$$

$$= \sqrt{\frac{\hbar}{2m\omega}}\left(a^{+}e^{i\omega t} + a e^{-i\omega t}\right).$$

(c)

$$\begin{aligned}
\langle 0|\mathrm{T}Q(t_1)Q(t_2)|0\rangle &= \frac{1}{i}\frac{\delta}{\delta f(t_1)}\frac{1}{i}\frac{\delta}{\delta f(t_2)}\langle 0|0\rangle_f\Big|_{f=0} \\
&= \frac{1}{i}\frac{\delta}{\delta f(t_1)}\left[\int_{-\infty}^{+\infty} dt'\, G(t_2 - t')f(t')\right]\langle 0|0\rangle_f\Big|_{f=0} \\
&= \left[\tfrac{1}{i}G(t_2 - t_1) + (\text{term with } f\text{'s})\right]\langle 0|0\rangle_f\Big|_{f=0} \\
&= \tfrac{1}{i}G(t_2 - t_1)\,. \qquad (7.16)
\end{aligned}$$

$$\begin{aligned}
\langle 0|\mathrm{T}Q(t_1)Q(t_2)Q(t_3)Q(t_4)|0\rangle = \frac{1}{i^2}\Big[&G(t_1{-}t_2)G(t_3{-}t_4) \\
+\,&G(t_1{-}t_3)G(t_2{-}t_4) \\
+\,&G(t_1{-}t_4)G(t_2{-}t_3)\Big]. \qquad (7.17)
\end{aligned}$$

$$\langle 0|TQ_1Q_2|0\rangle = \frac{\hbar}{2m\omega}\langle 0|T\left(a^\dagger e^{i\omega t_1}+a e^{-i\omega t_1}\right)\left(a^\dagger e^{i\omega t_2}+a e^{-i\omega t_2}\right)|0\rangle \qquad (t_2>t_1)$$

$$= \frac{\hbar}{2m\omega}\langle 0|T\left(a^\dagger e^{i\omega t_1}a^\dagger e^{i\omega t_2} + \underline{a e^{-i\omega t_1}a^\dagger e^{i\omega t_2}} + \underline{a^\dagger e^{i\omega t_1}a e^{-i\omega t_2}} + a e^{-i\omega t_1}a e^{-i\omega t_2}\right)|0\rangle$$

$$= \langle 0| \cdots \quad a^\dagger e^{i\omega t_2} a e^{-i\omega t_1} + e^{i\omega(t_2-t_1)} + \underline{a e^{-i\omega t_2}a^\dagger e^{i\omega t_1}} - e^{i\omega(t_1-t_2)} \cdots |0\rangle$$

$$= \frac{\hbar}{2m\omega}\langle 0|e^{i\omega(t_1-t_2)}|0\rangle = \frac{\hbar}{2m\omega}e^{-i\omega(t_2-t_1)}.$$

will be more convenient to have $t_1>t_2$ to get $e^{i\omega(t_2-t_1)}$

$$G(t_2-t_1) = \frac{i}{2\omega}e^{-i\omega|t_2-t_1|}.$$

for $t_2>t_1$, we have

$$\langle 0|TQ_1Q_2|0\rangle = \frac{\hbar}{2m\omega}e^{-i\omega(t_2-t_1)} = \frac{\hbar}{m}\frac{G(t_2-t_1)}{i}.$$

set $t_1>t_2>t_3>t_4$. to be convenient.

$$\langle 0|TQ_1Q_2Q_3Q_4|0\rangle = \left(\frac{\hbar}{2m\omega}\right)^2\langle 0|T\left(a^\dagger e^{i\omega t_1}+a e^{-i\omega t_1}\right)\left(a^\dagger e^{i\omega t_2}+a e^{-i\omega t_2}\right)\left(a^\dagger e^{i\omega t_3}+a e^{-i\omega t_3}\right)\left(a^\dagger e^{i\omega t_4}+a e^{-i\omega t_4}\right)|0\rangle$$

(neglect $\langle 0|a^\dagger$ and $a|0\rangle$ terms)

$$= \left(\frac{\hbar}{2m\omega}\right)^2\langle 0|\left(aa^\dagger e^{i\omega(t_2-t_1)} + aa e^{-i\omega(t_1+t_2)}\right)\left(aa^\dagger e^{i\omega(t_4-t_3)} + a^\dagger a^\dagger e^{i\omega(t_3+t_4)}\right)|0\rangle$$

$\langle 0\rvert$	$aa^\dagger aa^\dagger$	$aaaa^\dagger$	$aa^\dagger a^\dagger a^\dagger$	$aaa^\dagger a^\dagger$	$\lvert 0\rangle$
	1	0	0	2	

$$= \left(\frac{\hbar}{2m\omega}\right)^2\left(e^{i\omega(t_2-t_1)}e^{i\omega(t_4-t_3)} + 2e^{-i\omega(t_1+t_2)}e^{i\omega(t_3+t_4)}\right)$$

$$= \left(\frac{\hbar}{2m\omega}\right)^2\left(e^{-i\omega(t_1-t_2)}e^{-i\omega(t_3-t_4)} + e^{-i\omega(t_1-t_3)}e^{-i\omega(t_2-t_4)} + e^{-i\omega(t_1-t_4)}e^{-i\omega(t_2-t_3)}\right)$$

$$= \left(\frac{\hbar}{m}\right)^2\frac{1}{i^2}\left[G(t_1-t_2)G(t_3-t_4) + G(t_1-t_3)G(t_2-t_4) + G(t_1-t_4)G(t_2-t_3)\right]$$

4. **Topological aspects from the P.I.: Dirac charge quantization (optional)**

Consider the action for a non-relativistic particle of mass m and charge q in a EM potential $A_\mu(t,x)$

$$S = \int dt \frac{1}{2} m \dot{x}_i(t)^2 - q \left[A_0(t, x(t)) + A_i(t, x(t)) \dot{x}^i(t) \right] . \tag{12}$$

(i) Compute the classical equation of motion $\delta S / \delta x(t) = 0$ and show that it leads to the Lorentz force (in particular, show that it is gauge invariant $A_\mu \to A_\mu + \partial_\mu \lambda(t,x)$).

(ii) Let us now fix $A_i(x)$ to correspond to the field generated by a magnetic monopole at the origin. That is, we choose $A_0 = 0$ and $\mathbf{A}(\mathbf{x})$ such that

$$\mathbf{B} = \nabla \times \mathbf{A}(\mathbf{x}) = q_B \frac{\mathbf{r}}{r^3} . \tag{13}$$

(the form of $\mathbf{A}(\mathbf{x})$ will not be important). Compute the magnetic flux through a sphere of radius R. **(iii)** Consider a non-intersecting trajectory for the particle constrained to a sphere of radius R, with periodic boundary conditions in time $\lim_{t\to-\infty} x^i(t) = \lim_{t\to\infty} x^i(t)$. This defines a curve $\mathcal{C} \in S^2$. The contribution of this trajectory to the action is

$$S = -q \int dt \dot{\mathbf{x}} \cdot \mathbf{A} = -q \oint_{\mathcal{C}} d\mathbf{x} \cdot \mathbf{A} . \tag{14}$$

Using Stokes' theorem, show that this can be expressed as the integral of the magnetic flux through a section of the sphere D_1 with boundary $\partial D_1 = \mathcal{C}$. Note that there are two possible such sections D_1, D_2, such that $D_1 \cup D_2 = S^2$, and that the value of the action depends on which one you choose! By requiring that both contributions to the path integral e^{iS} agree, derive a quantization condition on the charge q.

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1. **Noether currents in classical mechanics:** Consider the action of a system of particles $\vec{q}_a(t)$ of masses m_a:

$$S = \int dt \sum_a \frac{1}{2} m_a (\dot{\vec{q}}_a)^2 - \sum_{a \neq b} V(|\vec{q}_a - \vec{q}_b|)\,. \tag{1}$$

From a field theory perspective, this can be thought of as a field theory in 0+1 dimensions, because the degrees of freedom $\vec{q}_a$ only depend on time t. The Noether procedure also works in this context. Find the Noether 'currents' associated with the following infinitesimal transformations (parametrized by $c, \vec{\alpha}, \theta \ll 1$):

- Time translation: $\vec{q}_a \to \vec{q}_a{}'(t) = \vec{q}_a(t) + c\partial_t \vec{q}_a(t)$;
- Coordinate translations: $\vec{q}_a \to \vec{q}_a{}'(t) = \vec{q}_a(t) + \vec{\alpha}$;
- Coordinate rotations. For this we can specialize for simplicity to vectors $\vec{q}_a$ in $\mathbb{R}^2$, so that coordinate rotations of infinitesimal angle θ act as $(q_a)_i \to (q_a{}')_i = (q_a)_i + \theta\epsilon_{ij}(q_a)_j$, where ϵ_{ij} is the matrix $\epsilon = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$.

(Notice that the second and third are *internal* symmetries in this context, because they do not act on time, but just on the 'fields' $\vec{q}_a$)

Do the currents look familiar?

$$S = \int dt \left(\sum_a \frac{1}{2} m_a \dot{q}_a^2 - \sum_{a\neq b} V(|q_a - q_b|) \right) \qquad \delta \vec{q}_a = c(t)\, \partial_t \vec{q}_a$$

$$\delta S = \int dt \left(\sum_a m_a \dot{q}_a \underbrace{\delta \dot{q}_a}_{\frac{d}{dt}(\delta q_a)} - \sum_{a\neq b} \frac{\partial V(|q_a - q_b|)}{\partial |q_a - q_b|} \frac{\partial |q_a - q_b|}{\partial q_a} \delta q_a \right)$$

$$= \underbrace{\sum_a m_a \dot{q}_a \delta q_a \Big|_{t_1}^{t_2}}_{\text{vanish for equal-time variation ?}} - \int \sum_a m_a \ddot{q}_a \delta q_a - \int \sum_{a\neq b} \frac{\partial V(|q_a - q_b|)}{\partial |q_a - q_b|} \frac{q_a - q_b}{|q_a - q_b|} \delta q_a$$

$$= -\int \left(\sum_a m_a \ddot{q}_a - \sum_{a\neq b} \frac{\partial V(|q_a - q_b|)}{\partial |q_a - q_b|} \frac{q_a - q_b}{|q_a - q_b|} \right) \delta q_a$$

- $\vec{q}_a \longrightarrow \vec{q}_a{}'(t) = \vec{q}_a(t) + c\,\partial_t \vec{q}_a(t)$

$c\,\partial_t q_a$

$$\delta S = -\int\left(\sum_a m_a\left(\ddot{q}_a + c\,\dddot{q}_a\right) - \sum_{a\neq b} \frac{\partial V(|q_a - q_b + c(\dot{q}_a - \dot{q}_b)|)}{\partial |q_a - q_b|} \frac{q_a - q_b + c(\dot{q}_a - \dot{q}_b)}{|q_a - q_b|}\right)\underset{\sim}{\delta q_a}$$

$$=$$

Let's try something different. start from

from a book

$$S = \int \mathcal{L}(\phi, \partial_\mu \phi)\, d^d x. \qquad \phi(x) \to \phi'(x).$$

$$\delta S = \int\left[\frac{\partial \mathcal{L}}{\partial \phi}\underbrace{\delta\phi}_{\bar{\delta}\phi + (\partial_\mu\phi)\delta x^\mu} + \frac{\partial \mathcal{L}}{\partial_\mu \phi}\underbrace{\delta(\partial_\mu\phi)}_{\bar{\delta}(\partial_\mu\phi) + \partial_\nu(\partial_\mu\phi)\delta x^\nu} + \mathcal{L}\,\partial_\mu(\delta x^\mu)\right] d^d x.$$

$$= \int\left[\underbrace{\frac{\partial \mathcal{L}}{\partial \phi}\bar{\delta}\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi)}\bar{\delta}(\partial_\mu\phi)}_{\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi)}\bar{\delta}\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi)}\bar{\delta}(\partial_\mu\phi)} + \underbrace{\frac{\partial \mathcal{L}}{\partial \phi}\partial_\mu\phi\,\delta x^\mu + \frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi)}\partial_\nu(\partial_\mu\phi)\delta x^\nu}_{\partial_\nu \mathcal{L}\,\delta x^\nu}\right] d^d x$$

$$= \int\left[\partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi)}\bar{\delta}\phi\right) + \partial_\nu(\mathcal{L}\,\delta x^\nu)\right] d^d x.$$

$$= \partial_\mu\left[\underbrace{\frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi)}\bar{\delta}\phi + \mathcal{L}\,\delta x^\mu}_{J^\mu}\right]$$

- $\vec{q}_a \longrightarrow \vec{q}_a{}'(t) = \vec{q}_a(t) + c\,\partial_t \vec{q}_a(t).$ $\qquad \bar{\delta} q_a = q_a'(t) - q_a(t) =$

$$J^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_t q_a)}\bar{\delta} q_a + \mathcal{L}\, a^\mu$$

We should expect to get

energy , momentum , angular-momentum

as noether current. (consered quantity)

but here is 0+1 d field theory , what we get?

2. **Complex scalar**: consider the action for a complex scalar field $\Phi(x) \in \mathbb{C}$:

$$S = \int d^{d+1}x \, \partial_\mu \Phi \partial^\mu \Phi^* - m^2 |\Phi|^2 \,. \tag{2}$$

(i) Write $\Phi = \phi_1 + i\phi_2$ with $\phi_{1,2} \in \mathbb{R}$ and obtain the equations of motion $\delta S / \delta \phi_{1,2} = 0$. Show that one can equivalently obtain this equation from $\delta S / \delta \Phi = 0$ by treating Φ, Φ^* as independent.

(ii) By coupling the field to sources $S \to S + \int d^{d+1}x \, J^* \Phi + J \Phi^*$, compute from the path integral the Feynman correlators in momentum space, i.e. the Fourier transforms of $\langle 0 | T\{\Phi(x)\Phi(y)\} | 0 \rangle$, $\langle 0 | T\{\Phi(x)\Phi^*(y)\} | 0 \rangle$, and $\langle 0 | T\{\Phi^*(x)\Phi^*(y)\} | 0 \rangle$.

[**Hint:** if you work in momentum space, be careful that Φ_p^* is independent from Φ_{-p} for complex fields!]

(iii) Show that the action is invariant under the transformation $\Phi \to e^{i\alpha}\Phi$. This is called a $U(1)$ symmetry (because $e^{i\alpha}$ is a unitary 1 by 1 matrix: $(e^{i\alpha})^\dagger = (e^{i\alpha})^{-1}$). How does the symmetry act on the (ϕ_1, ϕ_2) fields? Show that the Noether current associated with this symmetry is:

$$j^\mu = i\Phi^* \partial^\mu \Phi - i\partial^\mu \Phi^* \Phi \,. \tag{3}$$

Define the charge operator as $\hat{Q} = \int d^d x \, \hat{j}^0$. Check, using the Hamiltonian formalism, that $[\hat{Q}, \hat{\Phi}(x)] = -\hat{\Phi}(x)$ [**Hint**: you may want to express $\hat{Q}$ in terms of ϕ_1, ϕ_2 and their conjugate momenta π_1, π_2, and then use canonical commutation relations. Alternatively, you can find the momenta conjugate to $\hat{\Phi}$ and $\hat{\Phi}^*$].

(i) $\mathcal{L} = \partial_\mu \Phi \partial^\mu \Phi^* - m^2 |\Phi|^2$. $\quad \Phi = \phi_1 + i\phi_2$.

$$\mathcal{L} = (\partial_\mu \phi_1 + i\partial_\mu \phi_2)(\partial^\mu \phi_1 - i\partial^\mu \phi_2) - m^2|\phi_1|^2 + m^2|\phi_2|^2$$

$$= \partial_\mu \phi_1 \partial^\mu \phi_1 - \partial_\mu \phi_2 \partial^\mu \phi_2 - m^2|\phi_1|^2 + m^2|\phi_2|^2.$$

eqn: $\quad \partial^\mu \dfrac{\partial \mathcal{L}}{\partial(\partial^\mu \phi_i)} - \dfrac{\partial \mathcal{L}}{\partial \phi_i} = 0. \quad$ (equivalent to $\delta S / \delta \phi_i = 0$).

$$\Rightarrow \quad \partial^\mu \partial_\mu \phi_1 + m^2 \phi_1 = 0.$$

$$\partial^\mu \partial_\mu \phi_2 + m^2 \phi_2 = 0.$$

$$\mathcal{L} = \partial_\mu \Phi \partial^\mu \Phi^* - m^2 |\Phi|^2 = \partial_\mu \Phi \partial^\mu \Phi^* - m^2 \Phi^* \Phi$$

$$\partial^\mu \frac{\partial \mathcal{L}}{\partial(\partial^\mu \Phi)} - \frac{\partial \mathcal{L}}{\partial \Phi} = 0. \quad \Rightarrow \quad \partial^\mu \partial_\mu \Phi^* + m^2 \Phi^* = 0.$$

$$\partial^\mu \frac{\partial \mathcal{L}}{\partial(\partial^\mu \Phi^*)} - \frac{\partial \mathcal{L}}{\partial \Phi^*} = 0. \quad \Rightarrow \quad \partial^\mu \partial_\mu \Phi + m^2 \Phi = 0.$$

$$\Rightarrow \quad \partial^\mu \partial_\mu (\phi_1 + i\phi_2) + m^2 (\phi_1 + i\phi_2) = 0.$$

$$\partial^\mu \partial_\mu \phi_1 + m^2 \phi_1 = 0. \qquad \partial^\mu \partial_\mu \phi_2 + m^2 \phi_2 = 0.$$

(ii) $$S = \int d^{d+1}x \left(\partial_\mu \Phi \partial^\mu \Phi - m^2 |\Phi|^2 + J^* \Phi + J \Phi^* \right).$$

?

(iii) $S = \int d^{d+1}x \left(\partial_\mu \Phi \partial^\mu \Phi - m^2 |\Phi|^2 \right)$.

$\Phi \longrightarrow e^{i\alpha} \Phi$. $\quad \mathcal{L} = \partial_\mu e^{i\alpha} \Phi \, \partial^\mu e^{i\alpha} \Phi - m^2 |\Phi|^2$.

$$J^\mu = \frac{\partial \mathcal{L}}{\partial(\partial^\mu \phi)} \delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial^\mu \phi^+)} \delta\phi^+$$

$$\phi \to e^{i\alpha} \phi \simeq \phi + \underbrace{i\alpha\phi}_{=\delta\phi}$$

(iv) Generalizing, show that $[\hat{Q}, (\hat{\Phi})^q] = -q(\hat{\Phi})^q$, where $q \in \mathbb{N}$. The Hermitian operator $\hat{Q}$ allows us to construct a unitary operator $\hat{U}(\alpha) = e^{i\alpha\hat{Q}}$ realizing the symmetry transformations on the Hilbert space of states (similar to what we had done with Lorentz transformations). Show that these act on the quantum field operator as

$$\hat{U}(\alpha)^{-1}\hat{\Phi}\hat{U}(\alpha) = e^{i\alpha}\hat{\Phi}\,. \tag{4}$$

Using this and the fact that the vacuum is invariant under the symmetry $\hat{U}(\alpha)|0\rangle = |0\rangle$, show that two of the correlators from point (ii) *had* to vanish by symmetry.

(v) Compute the connected correlation function $\langle j^\mu(x)\Phi^*(y)\Phi(z)\rangle$ using Wick contractions. Because we are focusing on the connected correlator, you can drop the UV divergent contraction that would come from contracting both fields in j^μ. Check that your answer satisfies the Ward identity. (You can choose to work in position or momentum space).

(vi) Add a term $(\Phi)^q$ to the action $q \in \mathbb{Z}$, and show using the new equation of motion that the current j^μ that you found previously is no longer classically conserved. Then, show that the action is still invariant under a $\mathbb{Z}_q$ symmetry, that acts as $\Phi \to e^{i2\pi n/q}\Phi$, with $n \in \{0, 1, \ldots, q-1\}$.

Apologies, I was running out of time to submit

any more job ... feel free to take off

more points

3. Srednicki problem 22.3

22.3) a) With $T^{\mu\nu}$ given by eq. (22.31), compute the equal-time ($x^0 = y^0$) commutators $[T^{00}(x), T^{00}(y)]$, $[T^{0i}(x), T^{00}(y)]$, and $[T^{0i}(x), T^{0j}(y)]$.

b) Use your results to verify eqs. (2.17), (2.19), and (2.20).

$$T^{\mu\nu} = \partial^\mu \varphi_a \partial^\nu \varphi_a + g^{\mu\nu} \mathcal{L}.$$

Verify

$$[J_i, J_j] = i\hbar \varepsilon_{ijk} J_k.$$

$$[J_i, K_j] = i\hbar \varepsilon_{ijk} K_k.$$

$$[K_i, K_j] = -i\hbar \varepsilon_{ijk} J_k.$$

$$[J_i, H] = 0$$

$$[J_i, P_j] = i\hbar \varepsilon_{ijk} P_k.$$

$$[K_i, H] = i\hbar P_i.$$

$$[K_i, P_j] = i\hbar \delta_{ij} H.$$

$$[P_i, P_j] = 0.$$

$$[P_i, H] = 0.$$

express everything in tensor form and do algebra.

$$P^j = \int d^3x \, T^{0j}(x)$$

$$J^i = \varepsilon^{ijk} \int d^3x \, x^j T^{0k}.$$

QFT I Pset 6 yuxiang@uchicago.edu

1. **Noether's charge as generator of transformations** Consider an infintesimal internal transformation parametrized by $\alpha^i \ll 1$, acting on fields ϕ_a as

$$\phi_a \to \phi'_a = \phi_a + \alpha^i (\Delta\phi)_{ia} \,. \tag{1}$$

The index i labels the transformation (there could be several), and a labels the fields. Consider the Noether charge Q_i associated with this transformation. We have shown in class using the Ward identity and the path integral formalism that the operator satisfies the equal time commutator

$$[Q_i, \phi_a] = i(\Delta\phi)_{ia} \,. \tag{2}$$

Show this using instead the Hamiltonian formalism, using the definition of the Noether current and momentum conjugate to ϕ_a

Noether current $j_i^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \Delta\phi_{ia}$, with Noether charge Q_i

integrated over space $Q_i = \int d^3x \, j_i^0$.

Momentum conjugate $[\phi_a(\vec{x}), \pi_b(\vec{y})] = i\delta_{ab}\delta^3(\vec{x}-\vec{y})$.

Since $\pi_a^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)}$, $Q_i = \int d^3x \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi_a)} \Delta\phi_{ia} = \int d^3x \, \pi_a^0 \Delta\phi_{ia}$.

Let $\Delta\phi_{ia} = (T^i)_{ab}\phi_b$.

$-i\delta_{aa'}\delta^3(\vec{x}-\vec{y})$

$$[Q_i, \phi_a] = \left[\int d^3x \, \pi_{a'}^0 (T^i)_{a'b}\phi_b, \phi_a\right] = \int d^3x \, (T^i)_{a'b}\phi_b [\pi_{a'}^0, \phi_a]$$

$$= i(T^i)_{ab}\phi_b = i(\Delta\phi)_{ia}.$$

2. $SU(2)$ **symmetry: (i)** $SU(2)$ is the group of 2x2 special ($\det U = 1$) unitary ($U^\dagger = U^{-1}$) complex matrices. Writing $U = e^{iM}$, what conditions must M satisfy? Show that the most general such matrix can be written $M = \sum_{i=1}^{3} \alpha_i \sigma_i$, where σ_i are the Pauli matrices and $\alpha_i \in \mathbb{R}$. The matrices $J_i^{\mathbf{2}} = \frac{1}{2}\sigma_i$ form the two-dimensional representation of the $SU(2)$ algebra: $[J_i^{\mathbf{2}}, J_j^{\mathbf{2}}] = i\epsilon_{ijk} J_k^{\mathbf{2}}$. Find the 3x3 matrices $J_i^{\mathbf{3}}$ satisfying the same commutation relations [**Hint**: these are the generators of rotations in $\mathbb{R}^3$].

(ii) Consider the following two theories:

$$\mathcal{L}_1 = -\sum_{a=1,2} \partial_\mu \Phi_a^* \partial^\mu \Phi_a + M^2 \Phi_a^* \Phi_a \,, \tag{3}$$

$$\mathcal{L}_2 = -\sum_{i=1,2,3} \partial_\mu \chi_i \partial^\mu \chi_i + m^2 (\chi_i)^2 \,, \tag{4}$$

where Φ_1, Φ_2 are two complex fields, and χ_1, χ_2, χ_3 are three real fields. Show that $\mathcal{L}_1$ has an $SU(2)$ symmetry that acts on the field as $\Phi_a \to [e^{i\alpha^i J_i^{\mathbf{2}}}]_{ab} \Phi_b$, and find its associated Noether currents j_μ^i. Show that $\mathcal{L}_2$ has an $SU(2)$ symmetry that acts on the field as $\chi_i \to [e^{i\alpha^k J_k^{\mathbf{3}}}]_{ij} \chi_j$, and find its associated Noether currents j_μ^i.

(iii) Find a interaction term for S_1 (i.e. a term that is not quadratic in Φ) that preserves the $SU(2)$ symmetry. Do the same for S_2. Finally, show that the following interaction, which couples both theories, also preserves $SU(2)$:

$$\mathcal{L}_{\text{int}} = \lambda\, \chi_i \Phi_a^* \sigma_{ab}^i \Phi_b \,, \tag{5}$$

where repeated indices are summed, $\lambda \in \mathbb{R}$ is a real parameter, and σ^i are again the Pauli matrices. What is the Noether current of the theory $\mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_{\text{int}}$?

(i) $U = e^{iM} \in SU(2)$. For Hermite M, $e^{iM}(e^{iM})^\dagger = e^{iM} e^{-iM} = 1$.

Then satisfies $UU^\dagger = 1$. and $\det(e^{iM}) = e^{i\,\mathrm{Tr}M} = 1$. $\Rightarrow \mathrm{Tr}(M) = 0$.

With $M^\dagger = M$, $\mathrm{Tr}(M) = 0$, we have

$$M = \sum_{i=1}^{3} \alpha_i \sigma_i .$$

Find $[J_i^3, J_j^3] = i\varepsilon_{ijk} J_k^3$.

Construct : $J_i^3 = J_i^2 \otimes I + I \otimes J_i^2$. $\times$

generator of rotation in $\mathbb{R}^3$: $J_x^3 = \frac{1}{\sqrt{2}}\begin{pmatrix} & 1 & \\ 1 & & 1 \\ & 1 & \end{pmatrix}$. $J_y^3 = \frac{1}{\sqrt{2}}\begin{pmatrix} & -i & \\ i & & -i \\ & i & \end{pmatrix}$. $J_z^3 = \begin{pmatrix} 1 & & \\ & 0 & \\ & & -1 \end{pmatrix}$.

$$[J_x, J_y] = \frac{1}{2}\begin{pmatrix} i & & i \\ & 0 & \\ -i & & -i \end{pmatrix} - \frac{1}{2}\begin{pmatrix} -i & & i \\ & & \\ -i & & i \end{pmatrix} = \begin{pmatrix} i & & \\ & 0 & \\ & & -i \end{pmatrix} = i\varepsilon_{xyz} J_z^3.$$

$[J_y, J_z]$, $[J_z, J_x]$ also satisfies.

(ii) $$\mathcal{L}_1 = -\sum_{a=1,2} \partial_\mu \Phi_a^* \partial^\mu \Phi_a + M^2 \Phi_a^* \Phi_a$$

$$\Phi_a \longrightarrow \left[e^{i\alpha^i J_i^2}\right]_{ab} \Phi_b.$$

$$\mathcal{L}_1 \longrightarrow -\sum_{a=1,2} \partial_\mu\left(\Phi_b^* \left[e^{-i\alpha^i J_i^2}\right]_{ba}\right) \partial^\mu\left(\left[e^{i\alpha^i J_i^2}\right]_{ac} \Phi_c\right) + M^2 \sum_{\alpha=1,2} \Phi_b^* \underbrace{\left[e^{-i\alpha^i J_i^2}\right]_{ba} \left[e^{i\alpha^i J_i^2}\right]_{ac}}_{\delta_{bc}} \Phi_c$$

$$\partial_\mu \Phi_b^* \underbrace{\left[e^{-i\alpha^i J_i^2}\right]_{ba} \left[e^{i\alpha^i J_i^2}\right]_{ac}}_{\delta_{bc}} \partial^\mu \Phi_c$$

$$= -\sum_{a=1,2} \partial_\mu \Phi_b^* \partial^\mu \Phi_b + M^2 \Phi_b^* \Phi_b$$

Thus $\mathcal{L}_1$ does not change under $SU(2)$. noether current :

$$j_i^\mu = \sum_{a=1,2} \frac{\partial \mathcal{L}_1}{\partial(\partial_\mu \Phi_a)} \delta\Phi_a + \frac{\partial \mathcal{L}_1}{\partial(\partial_\mu \Phi_a^*)} \delta\Phi_a^*$$

$\delta\Phi_a = i\alpha^i (J_i^2)_{ab} \Phi_b$.

$\delta\Phi_a^* = -i\alpha^i \Phi_b^* (J_i^2)_{ba}$.

$$= \sum_{a=1,2} (-\partial^\mu \Phi_a^*)\, i\alpha^i (J_i^2)_{ab} \Phi_b + (-\partial^\mu \Phi_a)(-i\alpha^i \Phi_b^* (J_i^2)_{ba})$$

$$= \sum_{a=1,2} i\alpha^i \left(\partial^\mu \Phi_a \Phi_b^* (J_i^2)_{ba} - \partial^\mu \Phi_a^* (J_i^2)_{ab} \Phi_b\right).$$

$$\mathcal{L}_2 = -\sum_{i=1,2,3} \partial_\mu \chi_i \partial^\mu \chi_i + m^2 (\chi_i)^2$$

$$\chi_i \to [e^{i\alpha^k J_k^3}]_{ij} \chi_j$$

$$\mathcal{L}_2 \to -\sum_{i=1,2,3} \left([e^{i\alpha^k J_k^3}]_{ij} \partial_\mu \chi_j\right)\left([e^{i\alpha^k J_k^3}]_{im} \partial^\mu \chi_m\right) + m^2 (\chi_i)^2$$

$$= -\sum_{i,j,m} \underbrace{[e^{i\alpha^k J_k^3}]_{ij} [e^{i\alpha^k J_k^3}]_{im}}_{\delta_{jm}} \partial_\mu \chi_j \partial^\mu \chi_m + m^2 \left(\underline{[e^{i\alpha^k J_k^3}]_{ij} \chi_j}\right)^2$$

$$\sum_{ijk} [e^{i\alpha^k J_k^3}]_{ij} [e^{i\alpha^k J_k^3}]_{im} \chi_j \chi_m$$

$$= -\sum_j \partial_\mu \chi_j \partial^\mu \chi_j + m^2 \chi_j^2 .$$

noether current: $\delta \chi_i = i\alpha^k (J_k^3)_{ij} \chi_j$

$$j_k^\mu = \sum_i \frac{\partial \mathcal{L}_2}{\partial(\partial_\mu \chi_i)} \delta \chi_i = \sum_i (-\partial^\mu \chi_i)(i (J_k^3)_{ij} \chi_j)$$

$\times 2$ ✓

$$= i \sum_{ij} (J_k^3)_{ij} \chi_j \partial^\mu \chi_i .$$

(iii) Set $\mathcal{L}_{1int} = -\lambda (\underline{\Phi_a^* \Phi_a})^2$. SU(2) invariant

$\mathcal{L}_{2int} = -\lambda' (\underline{\chi_i \chi_i})^2$. SU(2) invariant

Show $\mathcal{L}_{int} = \lambda \chi_i \Phi_a^* \sigma_{ab}^i \Phi_b$ preserves SU(2).

$\delta \chi_i = i\alpha^k (J_k^3)_{ij} \chi_j$. $\quad \delta(\Phi_a^* \sigma_{ab}^i \Phi_b) = (\delta \Phi_a^*) \sigma_{ab}^i \Phi_b + \Phi_a^* \sigma_{ab}^i (\delta \Phi_b)$

$$= -i\alpha^k \Phi_d^* \frac{\sigma_{da}^k}{2} \Phi_b + i\alpha^k \Phi_a^* \sigma_{ab}^i \frac{\sigma_{bc}^k}{2} \Phi_c .$$

preserve SU(2) ?

Hint: Try BCH identity.

Noether current

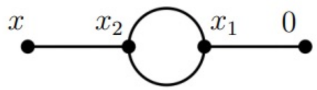
$$j_i^\mu = \sum_a \frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi_a)}\delta\Phi_a + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi_a^*)}\delta\Phi_a^* + \sum_i \frac{\partial\mathcal{L}}{\partial(\partial_\mu\chi_i)}\delta\chi_i$$

$$= \sum_a (-\partial^\mu\Phi_a^*)\, i\alpha^k \frac{\sigma^k_{ab}}{2}\Phi_b - \partial^\mu\Phi_a\left(-i\alpha^k\Phi_b^*\frac{\sigma^k_{ba}}{2}\right) + \sum_i(-\partial^\mu\chi_i)\, i\alpha^k\varepsilon_{ijk}\chi_j$$

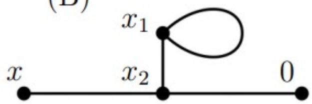
$$= \sum_{ab} i\alpha^k\frac{\sigma^k_{ab}}{2}\left(\Phi_b\partial^\mu\Phi_a^* - \Phi_a^*\partial^\mu\Phi_b\right) + \sum_{ij} i\alpha^k\varepsilon_{ijk}\chi_j\partial^\mu\chi_i$$

3. **More on ϕ^3 perturbation theory:** When studying the 1-loop correction to the two-point function in the theory $\mathcal{L} = -\frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}m^2\phi^2 - \frac{1}{3!}\lambda\phi^3$ in class, we found that there are only two diagrams contributing at order λ^2 to the connected two-point function:

(A)



(B)



We studied diagram (A) in class. The goal of this exercise is to compute diagram (B).

(i) Remind yourself of how we derived in class the form of the $O(\lambda^2)$ correction to the connected two-point function:

$$\langle\phi(x)\phi(0)\rangle_{\rm c} = \langle\phi(x)\phi(0)\rangle_0 + \frac{i^2}{2}\,\langle\phi(x)S_{\rm int}S_{\rm int}\phi(0)\rangle_0\big|_{\rm conn.} + O(\lambda^4) \tag{6}$$

where 'conn.' means that one only performs Wick contractions resulting in fully connected diagrams. Show that all such contractions lead to the diagrams above (with sometimes x_1 and x_2 swapped). Summing up the ones that produce the (B) diagram, you should find

$$\frac{i^2}{2}\,\langle\phi(x)S_{\rm int}S_{\rm int}\phi(0)\rangle_0\big|_{\rm conn.} = \frac{i^2}{2}\lambda^2\int_{x_1}\int_{x_2} G(x-x_2)G(x_2)G(x_2-x_1)G(0)\,, \tag{7}$$

where we used the shorthand $\int_x \equiv \int d^Dx$, and $G(x) = \langle\phi(x)\phi(0)\rangle_0$ is the free two-point function (also called 'propagator'). Notice that the $(1/3!)^2$ canceled with symmetry factors.

?

(ii) Next, Fourier transform to show that $\int \frac{d^D p}{(2\pi)^D}$ [Eq. (7)] $e^{-ix^\mu p_\mu}$ is equal to

$$\frac{i^2}{2}\lambda^2[\widetilde{G}(p)]^2\widetilde{G}(0)\int \frac{d^D p'}{(2\pi)^D}\widetilde{G}(p') \tag{8}$$

where $\widetilde{G}(p) \equiv \int \frac{d^D p}{(2\pi)^D} G(x) = \frac{-i}{p^2+m^2-i\epsilon}$. Confirm that the form of this expression satisfies our 'momentum space Feynman rules'.

(iii) Finally, we need to perform the integral over p'. Let us focus on $D = 4$ spacetime dimensions for concreteness (although you can keep D general if you wish). Wick rotate $p'_0 = ip'_{0E}$ so that the integral becomes

$$\int \frac{d^4 p'}{(2\pi)^4}\widetilde{G}(p') = \int \frac{d^4 p'_E}{(2\pi)^4}\frac{1}{p'^2_E + m^2}, \tag{9}$$

with now regular 'Euclidean' norm: $p_E^2 = p_{E,0}^2 + p_{E,1}^2 + p_{E,2}^2 + p_{E,3}^2$. Working in spherical coordinates (and using the fact that the area of a 3-sphere of radius 1 is $2\pi^2$) reduce this to a single integral of the radial momentum:

$$\int \frac{d^4 p'}{(2\pi)^4}\widetilde{G}(p') = \frac{1}{8\pi^2}\int_0^\infty dp_E \frac{p_E^3}{p_E^2 + m^2}. \tag{10}$$

This integral diverges for large momentum p_E—it is 'UV divergent'. To make sense of it, let us replace it with an integral that has a sharp momentum cutoff $\int_0^\infty dp_E \to \int_0^\Lambda dp_E$. Assuming $\Lambda \gg m$, compute the leading order integral at large Λ (your answer should be $O(\Lambda^2)$). In summary, we have

$$\widetilde{G}(p) = \frac{-i}{p^2+m^2} + \delta\widetilde{G}(p)\Big|_{(A)} + \delta\widetilde{G}(p)\Big|_{(B)} + O(\lambda^4), \tag{11}$$

where the contribution $\delta\widetilde{G}(p)|_{(A)}$ from diagram (A) was found in class; spell out your answer for $\delta\widetilde{G}(p)|_{(B)}$ and then show that the UV divergence can be absorbed by a 'mass counterterm' $m^2 \to m^2 + \delta m^2$, providing an expression for δm^2.

?

4. **Wavefunction renormalization:** When computing diagram (A) in class, we found that in $D \geq 4$ spacetime dimensions there is a UV divergence coming from the integral

$$\int \frac{d^D q}{(2\pi)^D} \frac{1}{(q^2+\Delta)^2} \tag{12}$$

which can be absorbed by a mass counterterm like above (this is called 'mass renormalization'). In this exercise, show that in $D \geq 6$ there is another (subleading) UV divergence, which however can be absorbed by adding the counterterm $-\frac{1}{2}\delta Z(\partial_\mu \phi)^2$ to $\mathcal{L}$. For historical reasons this is called 'wavefunction renormalization'. [**Hint:** you don't need to compute the loop integral; to get the leading and subleading UV divergences you can expand the integrand for large q (the integral is actually evaluated in Eq. (14.27) of Srednicki in case you would like a consistency check for your answer)]

$$\int \frac{d^D q}{(2\pi)^D} \frac{1}{(q^2+\Delta)^2}, \quad \text{expand for } q \to \infty.$$

$$\to \int \frac{d^D q}{(2\pi)^D} \frac{1}{q^4}. \qquad d^D q = \Omega_D q^{D-1} dq.$$

$$\to \frac{\Omega_D}{(2\pi)^D} \int q^{D-5} dq \xrightarrow{q \to \infty} \frac{\Omega_D}{(2\pi)^D} \left[\frac{q^{D-4}}{D-4}\right]^{\Lambda}.$$

① For $D \geq 4$, UV divergence.

② For $D = 6$. $\left[\frac{q^{D-4}}{D-4}\right]^{\Lambda} \sim \Lambda^2$. subdivergence.

✗ (Λ set as upper-bound of integral)

Should do in this way:

$$\int_0^{\Lambda} \frac{d^D q}{(2\pi)^D} \frac{1}{(q^2+\Delta)^2} \qquad d^D q = \Omega_D q^{D-1} dq.$$

$$= \Omega_D \int_0^{\Lambda} dq \, \frac{q^{D-1}}{(q^2+\Delta)^2}$$

$$d\sqrt{u} = \frac{du}{2\sqrt{u}}$$

$$\xrightarrow{u=q^2} \quad = \Omega_D \int_0^{\Lambda^2} \frac{du}{2} \, \frac{u^{\frac{D-2}{2}}}{(u+\Delta)^2}$$

$$\sim \int_0^{\Lambda^2} - u^{\frac{D-2}{2}} \, d\left(\frac{1}{u+\Delta}\right) = - \left. \frac{u^{\frac{D-2}{2}}}{u+\Delta} \right|_0^{\Lambda^2} + \frac{D-2}{2} \int_0^{\Lambda^2} \frac{u^{\frac{D-4}{2}}}{u+\Delta} du$$

$$= \frac{D-2}{2} \int_0^{\Lambda^2} \frac{u^{\frac{D-4}{2}}}{u+\Delta} du - \frac{\Lambda^{D-2}}{\Lambda^2+\Delta}$$

$$\sim \frac{D-2}{2} \int_0^{\Lambda^2} u^{\frac{D-6}{2}} du - \frac{\Lambda^{D-2}}{\Lambda^2+\Delta}$$

$$= \frac{D-2}{2(D-4)} \Lambda^{D-4} - \frac{\Lambda^{D-2}}{\Lambda^2+\Delta}$$

$$\xrightarrow{D \geq 6} \quad \sim \underline{\Lambda^2} \longrightarrow$$

$\mathcal{O}(\Lambda^2)$ term is absorbed by mass renormalization.

by adding $-\frac{1}{2}\delta z (\partial_\mu \phi)^2$ to $\mathcal{L}$.

$$\mathcal{L}' \to -\frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2}\underset{\sim}{\delta z}(\partial_\mu \phi)^2$$

Λ^2

then we have divergence cancelled.

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1. **Unstable particle:** Consider a theory of two interacting real fields χ, $\phi \in \mathbb{R}$:

$$S = -\int d^D x \, \frac{1}{2}(\partial\chi)^2 + \frac{1}{2}m^2\chi^2 + \frac{1}{2}(\partial\phi)^2 + \frac{1}{2}M^2\phi^2 + \frac{\lambda}{2}\phi\chi^2 \,. \tag{1}$$

We will assume that ϕ is the heavier particle, i.e. $M > m$.

(i) What are the symmetries of this action? Show that a term χ^3 can be ruled out if one imposes one of these symmetries. Can one similarly rule out ϕ^3? Provide the energy dimensions of the fields, their masses, and their coupling λ.

(ii) We will focus on the connected two-point function of the heavy field

$$G_\phi(p) = \int d^D x \, e^{-ip_\mu x^\mu} \langle 0|\mathcal{T}\phi(x)\phi(0)|0\rangle_c \,. \tag{2}$$

What is it when $\lambda = 0$? Show that there is no $O(\lambda)$ correction, and then show that at $O(\lambda^2)$ there is a single 1-loop diagram, similar to diagram (A) in Problem Set 6. Express your result as a self-energy $\Pi_\phi(p)$ as in class, and show that it takes the form

$$\Pi_\phi(p) = \frac{i}{2}\lambda^2 \int \frac{d^D p'}{(2\pi)^D} G_{\chi,0}(p') G_{\chi,0}(p-p') + O(\lambda^4) \,, \tag{3}$$

and give an expression for $G_{\chi,0}$.

(iii) Following the steps in class (Feynman parameters and Wick rotation), reduce the integrals in (3) to an integral over a Feynman parameter, and over a radial (Euclidean) momentum p'_E.

(i) $\chi \to -\chi$. $\mathbb{Z}_2$ symmetry.

χ^3 term: $\int d^D x \frac{g}{3!}\chi^3 \xrightarrow{\chi\to-\chi} \cdots -\chi^3$. break $\mathbb{Z}_2$ and ruled out.

ϕ^3 term: $\int d^D x \frac{g}{3!}\phi^3 \xrightarrow{\phi\to-\phi}$ does not break $\mathbb{Z}_2$. cannot rule out.

Dimension: $(\underset{1}{\partial}\chi)^2$, $[\chi] \sim \frac{D-1}{2}$. $(\underset{1}{\partial}\phi)^2$. $[\phi] \sim \frac{D-1}{2}$.

$[m], [M] \sim 1$. $\lambda\phi\chi^2 \sim [\lambda]\frac{3D-3}{2} \sim D \Rightarrow [\lambda] \sim 3-D$.

(ii) $\lambda = 0$. $\quad G_\phi(p) = \int d^D x \, e^{-ip_\mu x^\mu} \langle 0 | T\phi(x)\phi(0) | 0 \rangle$

$$= \frac{1}{p^2 + M^2 - i\varepsilon}.$$

$\frac{\lambda}{2}\phi\chi^2$. $\quad O(\lambda)$ vanish.

$O(\lambda^2)$.

$$\Pi_\phi(p^2) = \lambda^2 \int \frac{d^D q}{(2\pi)^D} \frac{1}{(q^2+m^2)((q-p)^2+m^2)}$$

$$G_{int}(p) = G(p) + \delta G(p)$$

$$= \frac{-i}{p^2+m^2-\Pi(p)} \approx \frac{-i}{p^2+m^2} \frac{1}{1 - \frac{\Pi}{p^2+m^2}} \approx \frac{-i}{p^2+m^2} + i\left(\frac{-i}{p^2+m^2}\right)^2 \Pi = G(p) + i\,[G(p)]^2\,\Pi.$$

$$\Pi_\phi(p) = \frac{i}{2}\lambda^2 \int \frac{d^D p}{(2\pi)^D} G(p')\, G(p-p')$$

$$= \frac{i}{2}\lambda^2 \int \frac{d^D p'}{(2\pi)^D} \underbrace{\frac{-i}{p'^2+m^2-i\varepsilon}}_{G_{\chi,0}(p)} \frac{-i}{(p-p')^2+m^2-i\varepsilon} = \lambda^2 \int \frac{d^D q}{(2\pi)^D} \frac{1}{(q^2+m^2)((q-p)^2+m^2)}.$$

Feynman parameter:

$$\lambda^2 \int \frac{d^D q}{(2\pi)^D} \frac{1}{(q^2+m^2)((q-p)^2+m^2)} = \lambda^2 \int \frac{d^D q}{(2\pi)^D} \int_0^1 dx \frac{1}{[\underbrace{(q+xp)}_{k}{}^2 + m^2 + x(1-x)p^2]^2}$$

$$= \lambda^2 \int_0^1 dx \int \frac{d^D k}{(2\pi)^D} \frac{1}{[k^2+m^2+x(1-x)p^2]^2}$$

Use this: $\displaystyle\int_0^1 dx \frac{1}{[xA+(1-x)B]^2} = \int_0^1 dx \frac{1}{[x(A-B)+B]^2} = \frac{1}{AB}.$

Here $A = (p-p')^2 + m^2$. $B = p'^2 + m^2$.

$$\Pi_\phi(p) = \lambda^2 \int_0^1 dx \int \frac{d^D k}{(2\pi)^D} \frac{1}{(k^2+\Delta)^2} = \frac{\lambda^2}{(4\pi)^{D/2}} \int_0^1 dx \frac{\Gamma(2-D/2)}{\Gamma(2)} \frac{1}{[m^2 + x(1-x)p^2]^{2-D/2}}.$$

(iv) Let us take $D = 2$ to make the integrals simpler and focus on the physics. The radial integral should be straightforward to compute, and you should be left with

$$\Pi_\phi(p^2) = \frac{\lambda^2}{8\pi m^2} \int_0^1 \frac{dx}{1 + \frac{p^2}{m^2}x(1-x) - i\epsilon}. \tag{4}$$

Show that the integrand has a singularity (for $x \in [0,1]$) iff $p^2 = -p_0^2 + p_1^2 < -4m^2$. In this regime, one has to be careful with the $i\epsilon$ prescription which resolves the singularities. This is a physical effect: we are injecting enough energy p_0 in the ϕ field to create two χ excitations ($p_0 > 2m$). We will precisely be interested in this regime, where Π_ϕ acquires an imaginary part. Show that

$$\mathrm{Im}\,\Pi_\phi(p^2) = \frac{\lambda^2}{4(-p^2)} \frac{\Theta(-p^2 - 4m^2)}{\sqrt{1 + \frac{4m^2}{p^2}}}, \tag{5}$$

where Θ is the Heaviside step function: $\Theta(a) = 1$ if $a > 0$ and 0 otherwise. [**Hint:** you can use $\mathrm{Im}\,\frac{1}{y - i\epsilon} = \pi\delta(y)$].

focus on $1 + \frac{p^2}{m^2}x(1-x) - i\epsilon = 0$.

$\Rightarrow \frac{m^2}{p^2} = x(x-1) \in [-\frac{1}{4}, 0]$.

We have $p^2 < -4m^2$ to have singularity.

For $p_0 > 2m$. $\Pi_\phi(p) = \frac{\lambda^2}{8\pi m^2}\int_0^1 \frac{dx}{1+\frac{p^2}{m^2}x(1-x) - i\epsilon}$

on shell $p^2 = m^2$, $(p-p')^2 = m^2$.

$\mathrm{Im}\,\frac{1}{y - i\epsilon} = \pi\,\delta(y)$

$\mathrm{Im}\,\Pi_\phi(p) = -\frac{1}{4m^2}\int \frac{d^D k}{(2\pi)^D}\,\delta(p^2 - m^2)\,\delta((p-p')^2 - m^2)$.

$\frac{1}{x - i\epsilon} = P\frac{1}{x} + i\pi\delta(x)$.

$\dots = \frac{m\lambda^2}{8\pi}\int \frac{d\Omega}{(2\pi)^3}\frac{\sqrt{1 - \frac{4m^2}{p^2}}}{p^2}$

I know this is not right...

but can not get it right

$\dots = \frac{\lambda^2}{4(-p^2)}\frac{\Theta(-p^2 - 4m^2)}{\sqrt{1 + \frac{4m^2}{p^2}}}$

(v) This imaginary part of the self-energy is particularly interesting, because it cannot have the interpretation of a renormalization of the ϕ mass M^2, which is real. Let us study the ϕ two-point function close to its pole $p^2 \approx -M^2$. Near this pole we can approximate the 1-loop corrected two-point function as

$$G_\phi(p) \simeq \frac{-i}{p^2 + M^2 - \Pi_\phi(-M^2)} \tag{6}$$

Ignoring the real part of $\Pi_\phi(-M^2)$,[1] show that the poles are no longer located at $p_0 = \pm\left(\sqrt{M^2+p_i^2} - i\epsilon\right)$ but rather at

$$p_0 = \pm\left(\sqrt{M^2+p_i^2} - i\Gamma\right), \tag{7}$$

to leading order in λ, with $\Gamma = \frac{\lambda^2}{8M^2\sqrt{M^2+p_i^2}}\frac{1}{\sqrt{1-\frac{4m^2}{M^2}}}$. The $i\epsilon$ has come to life!

(vi) Γ is called the decay rate (or inverse lifetime) of the unstable heavy particle ϕ. Fourier transforming time only $\int \frac{dp_0}{2\pi} e^{ip_0 t}$, compute $G_\phi(t, p_i = 0)$ and argue why that is a reasonable name.

(We will learn to compute particle decay rates after studying scattering. You can consult Srednicki Sec. 25 for some inspiration on this problem).

$$p^2 + m^2 - \Pi_\phi(-M^2) = 0.$$

$$\Pi_\phi(p) = \frac{i}{2}\lambda^2 \int \frac{d^D p'}{(2\pi)^D} \frac{-i}{p'^2 - m^2 - i\varepsilon} \frac{-i}{(p-p')^2 - m^2 - i\varepsilon}$$

$$\Pi_\phi(-M^2) = \frac{i}{2}\lambda^2 \int \frac{d^D p'}{(2\pi)^D} \underbrace{\frac{-i}{p'^2 - m^2 - i\varepsilon}}_{A} \underbrace{\frac{i}{(-M^2 - 2p\cdot p' + p'^2) - m^2 + i\varepsilon}}_{B}$$

$$\int_0^1 dx \frac{1}{x(p'^2 - m^2) + (1-x)[(p-p')^2 - m^2]}. \qquad q \to p - (1-x)p'.$$

$$\Pi_\phi(-M^2) = \frac{i}{2}\lambda^2 \int_0^1 dx \int \frac{d^D q}{(2\pi)^D} \frac{1}{[q^2 + \underbrace{x(1-x)(-M^2) - m^2 + i\varepsilon}_{\Delta}]^2} = \frac{1}{(4\pi)^{\frac{D}{2}}} \frac{\Gamma(2-\frac{D}{2})}{\Gamma(2)} \frac{1}{\Delta^{2-\frac{D}{2}}}. \qquad \text{For } D=2.$$

$$= \frac{i}{2}\lambda^2 \int_0^1 dx \frac{1}{4\pi} \frac{1}{x(1-x)(-M^2) - m^2} \longrightarrow \text{roots}: x = \frac{1 \pm \sqrt{1-\frac{4m^2}{M^2}}}{2}$$

$$\Rightarrow \quad \mathrm{Im}\, \Pi_\phi(-M^2) = \frac{\lambda^2}{8\pi M^2}\sqrt{1-\frac{4m^2}{M^2}}\,.$$

Then $\quad p^2 = -M^2 + \dfrac{\lambda^2}{8\pi M^2}\sqrt{1-\dfrac{4m^2}{M^2}}\,.$

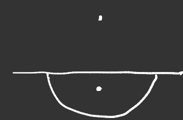
$p_i^2 - p_0^2$

$$\Rightarrow \quad p_0 = \pm\left(\sqrt{M^2+p_i^2} - i\Gamma\right). \qquad \Gamma = \frac{\lambda^2}{8M^2\sqrt{M^2+p_i^2}}\,\frac{1}{\sqrt{1-\frac{4m^2}{M^2}}}\,.$$

(vi) $\quad G_\phi(t, p_i=0) = \displaystyle\int \frac{dp^0}{2\pi}\, e^{-ip^0 t}\, G_\phi(p^0, \vec{p}=0)$

$$= \int \frac{dp^0}{2\pi}\,\frac{-i\,e^{-ip^0 t}}{p^{0\,2}+M^2-\Pi_\phi(p^0)} = \int \frac{dp^0}{2\pi}\,\frac{-i\,e^{-ip^0 t}}{p^{0\,2}+M_R^2-iM_R\Gamma}\,.$$

roots: $\sqrt{M_R^2-\frac{\Gamma^2}{4}} \pm \frac{i}{2}\Gamma\,.$



$$\Rightarrow \quad \mathrm{Res}\left(\sqrt{M_R^2-\frac{\Gamma^2}{4}} - \frac{i}{2}\Gamma\right)$$

$$= e^{-\frac{\Gamma}{2}t}\,\frac{e^{-i\sqrt{M_R^2-\frac{\Gamma^2}{4}}\,t}}{2\sqrt{M_R^2-\frac{\Gamma^2}{4}}}\,.$$

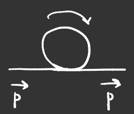
decay rate.

2. ϕ^4 **theory:** compute the leading in λ correction to the connected two-point function of φ in the theory

$$S = -\int d^D x \, \frac{1}{2}(\partial\varphi)^2 + \frac{1}{2}m^2\varphi^2 + \frac{\lambda}{4!}\varphi^4 . \tag{8}$$

In particular, show that in this theory, the leading correction is $O(\lambda)$. Show that at this order in this theory the two-point function does *not* see the multiparticle continuum, i.e. its only non-analyticity is a simple pole in p^2 (unlike what we had found for the ϕ^3 theory). Show however that the mass still gets renormalized, and give the expression for the counterterm δm^2 in terms of your sharp momentum cutoff Λ. Can you guess at what order in λ one will get qualitatively different non-analyticities in p^2, and what they will look like? Draw the diagram, and guess the value of p^2 where you expect a branch point.

Two point function:



$$\langle 0 | T\phi(x)\phi(y) | 0 \rangle = \int \frac{d^D p}{(2\pi)^D} e^{-ip(x-y)} \frac{1}{p^2 + m^2 - \Pi(p^2)}$$

Correction term:

$$\Pi(p^2) = \frac{\lambda}{2} \int \frac{d^D q}{(2\pi)^D} \frac{1}{(q^2+m^2)((q-p)^2+m^2)}$$

One vertex in diagram $\longrightarrow$ $O(\lambda)$.

For multi-particle continuum, we should have on-shell condition:

$$q^2 = -m^2, \quad (q-p)^2 = -m^2 .$$

They cannot be fulfilled simultaneously. Focus on analyticity of $\Pi(p^2)$:

Only non-analyticity $p^2 + m^2 - \Pi(p^2)$ $\Rightarrow$ $m_R^2 = m^2 + \Pi(-m^2)$. Renormalized mass

$\delta m^2 = \Pi(-m^2) = \cdots$ calculate at $D = 4$.

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1. **Renormalization of ϕ^3 in $D = 6$:** We showed that coefficient of the ϕ^3 interaction has dimension $[\lambda_{\phi^3}] = \frac{6-D}{2}$; this interaction is therefore marginal in $D = 6$. Following steps similar to what we did in class for the ϕ^4 interaction [see Peskin Sec. 12.1], show that the ϕ^3 *grows* upon coarse-graining. The crucial diagram will look like Fig. 16.1 in Srednicki, but with double lines (representing the high energy mode $\hat{\phi}$) running in the loop. You can assume that the loop momentum $\sim \Lambda$ is much larger than the external momenta k_1, k_2 and the mass m, as we did for ϕ^4.

   [**Note:** to fully conclude whether λ_{ϕ^3} is marginally relevant or irrelevant, one would have to also find the wavefunction renormalization δZ. This was not necessary for ϕ^4, because there $\delta Z = 0$ at leading order. However, for ϕ^3 theory you showed in PS6, Problem 4, that there *is* a finite δZ at leading order in coupling.]

For ϕ^4 theory $\quad S = -\int d^Dx \left[\frac{1}{2}(\partial_\mu\phi)^2 + \frac{1}{2}m^2\phi^2 + \frac{1}{4!}\lambda\phi^4\right]$.

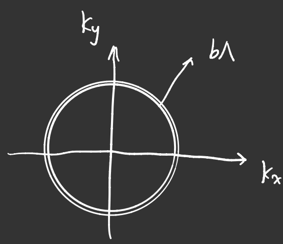
$[\phi] = \frac{D-2}{2}$. $\quad [\lambda] = 2 - 2[\phi] = 4 - D$. $\qquad [\lambda] \overset{?}{\sim} 0$

Wick rotate $\quad t \to -it_E$. $\quad e^{iS} \to e^{-S_E}$.

$$\Rightarrow \quad S_E = \int d^Dx \left[\frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2 + \frac{1}{4!}\lambda\phi^4\right].$$

$$Z = \int D\phi\, e^{-S} = \int \prod_x d\phi(x)\, e^{-S} = \int \prod_k d\phi_k\, e^{-S}.$$ ($|k| > \Lambda$ modes vanish)

$$\Rightarrow \quad Z_\Lambda = \int \prod_{|k|<\Lambda} d\phi_k\, e^{-S}.$$



Coarse graining means further

integrating out $b\Lambda < |k| < \Lambda$. $\quad (1-b \ll 1)$.

$$S[\phi+\tilde{\phi}] = \int d^Dx \left[\frac{1}{2}(\nabla(\tilde{\phi}+\phi))^2 + \frac{1}{2}m^2(\phi+\tilde{\phi})^2 \frac{\lambda}{4!}(\phi+\tilde{\phi})^4\right]$$

$$\phi_{tot}(x) = \int_{k<\Lambda} e^{ikx}\phi_{k,tot} = \int_{k<b\Lambda} e^{ikx}\phi_k + \int_{b\Lambda<k<\Lambda} e^{ikx}\phi_k = \phi_0(x) + \tilde{\phi}(x).$$

$$S[\phi_{tot}] = S[\phi+\tilde{\phi}] = \int d^Dx \left[\frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2 + \frac{1}{2}(\nabla\tilde{\phi})^2 + \frac{1}{2}m^2\tilde{\phi}^2 + \frac{1}{4!}(\phi+\tilde{\phi})^4\right]$$

$\phi, \tilde{\phi}$ decoupled, $\phi_k \tilde{\phi}_{-k}$ vanish

$$Z = \int D\phi\, e^{-S(\phi)} \int D\tilde{\phi}\, \exp\left\{-\int_x \left[\frac{1}{2}(\nabla\tilde{\phi})^2 + \frac{1}{2}m^2\tilde{\phi}^2 + \frac{\lambda}{4!}\left(\tilde{\phi}^4 + 4\tilde{\phi}^3\phi + 6\tilde{\phi}^2\phi^2 + 4\tilde{\phi}\phi^3\right)\right]\right\}$$

$$= \int D\phi\, e^{-S(\phi)} \int D\tilde{\phi} \left[1 - \int_x \frac{\lambda}{4!}\left(4\phi\tilde{\phi}^3 + 6\phi^2\tilde{\phi}^2 + 4\phi^3\tilde{\phi}\right) e^{-S(\tilde{\phi})} + O(\lambda^2)\right]$$

$$\approx \int D\phi\, e^{-S(\phi)} \left[1 - \frac{\lambda}{4!}\int_x \left(4\phi\underbrace{\langle\tilde{\phi}^3\rangle} + 6\phi^2\langle\tilde{\phi}^2\rangle + 4\phi^3\underline{\langle\tilde{\phi}\rangle}\right) + O(\lambda^2)\right]$$

vanish

$$\langle\tilde{\phi}^2\rangle = \langle\tilde{\phi}(x)\tilde{\phi}(x)\rangle = \int_{b\Lambda<k<\Lambda} \frac{1}{k^2+m^2}$$

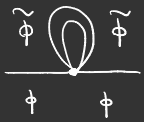
$$= \frac{2\pi^2}{(2\pi)^D}\int_{b\Lambda}^{\Lambda} dk\, \frac{k^{D-1}}{k^2+m^2} \approx \frac{1}{2^{D-1}\pi^{D-2}}\int_{b\Lambda}^{\Lambda} dk\, k \quad (\Lambda \gg m)$$

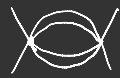
$$= \frac{1}{2^D\pi^{D-2}}\left[\Lambda^2 - (b\Lambda)^2\right] = \frac{\Lambda^2}{2^D\pi^{D-2}}(1-b^2)$$

$$\Rightarrow \quad Z \approx \int D\phi\, e^{-S[\phi]}\left[1 - \int_x \frac{\lambda\Lambda^2}{2^{D+2}\pi^{D-2}}(1-b^2)\,\phi(x)^2 + O(\lambda^2)\right]$$

$$= \int D\phi\, e^{-S[\phi]}\left[e^{-\int_x \frac{1}{2}\Delta m^2\phi^2} + O(\lambda^2)\right].$$

with $\Delta m^2 = \dfrac{\lambda\Lambda^2}{2^{D+1}\pi^{D-2}}(1-b^2)$

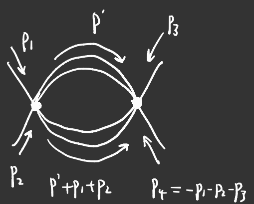




interaction renormalize mass term ... renormalize interactions

$$Z = \int D\phi \, e^{-S[\phi]} \int D\tilde{\phi} \left(1 - S_{int} + \frac{1}{2} S_{int}^2 + \cdots\right) e^{-S(\tilde{\phi})}$$

$$S_{int}^2 \sim \frac{\lambda^2}{4^2} \int_{x,x'} \phi_x^2 \phi_{x'}^2 \langle \tilde{\phi}_x^2 \tilde{\phi}_{x'}^2 \rangle \approx \frac{\lambda^2}{4} \int_{p_1,p_2,p_3} \phi_{p_1} \phi_{p_2} \phi_{p_3} \phi_{-p_1-p_2-p_3} \underbrace{\int_{p'} G(p') G(-p')}$$



$$\approx \int_{b\Lambda < p' < \Lambda} \frac{d^D p'}{(2\pi)^D} \frac{1}{(p')^2} = \frac{S_{D-1}}{(2\pi)^D} \log \frac{1}{b} .$$

We see ϕ^4 adds $\frac{1}{4!} \Delta\lambda \, \phi^4$ to action, with

$$\Delta\lambda = -4! \frac{S_{D-1} \lambda^2}{4^2 (2\pi)^D} \log \frac{1}{b} .$$

$\Delta\lambda < 0$ at $D = 4$. meaning marginally irrelevant.

Even above derivation I cannot quite follow ... so not sure how to apply it to ϕ^3 theory.

Let me still try to put something here with help from online references..

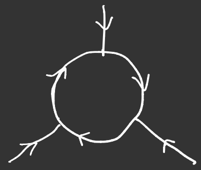


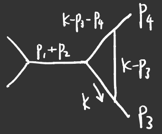
Fig 16.1

For ϕ^3

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - m^2\phi^2 - \frac{\lambda}{3!}\phi^3 .$$

$$\lambda_R = \frac{\lambda_0}{Z_1} . \qquad Z_1 = 1 - \lambda^2 \Delta F(m^2) = 1 - \lambda^2 \Delta F(\mu^2)$$

$$s = \mu^2$$

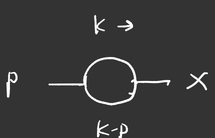


vertex graph coupling

contribution $\quad -i \dfrac{\lambda^4}{s-m^2}\Delta F(s)$

add to $\quad -i\dfrac{\lambda_R^2}{s-m^2}\left(1+\lambda_R^2\left(\Delta F(s) - \Delta F(m^2)\right)\right) . \longrightarrow -i\dfrac{\lambda_R^2(\mu^2)}{s-m^2}\left(1+\lambda_R^2\left(\Delta F(s) - \Delta F(\mu^2)\right)\right) .$

Self-energy correction



$$\frac{i}{p^2-m^2-\Sigma(p^2)} = \frac{iZ}{p^2-m_R^2-\Sigma_R(p^2)} .$$

$$\lambda_R \Rightarrow \frac{Z^{\frac{3}{2}}}{Z_1}\lambda_0 .$$

$$\Sigma(p^2) = i\frac{1}{2}\lambda_R \int \frac{d^4k}{(2\pi)^4}\frac{1}{(k^2-m^2)((k-p)^2-m^2)}$$

$$= i\frac{1}{2}\lambda_R^2 \int \frac{d^4k}{(2\pi)^4}\frac{1}{(k^2-m^2)(k^2-2p\cdot k+p^2-m^2)}$$

$$= -\frac{\lambda_R^2}{32\pi^2}\int_0^1 d\alpha \ln\left(\frac{\Lambda^2}{m^2-p^2\alpha(1-\alpha)}\right)$$

$$Z-1 = \frac{\partial}{\partial p^2}\Sigma(p^2)\Big|_{p^2=m^2} = -\frac{\lambda_R^2}{32\pi^2}\int_0^1 d\alpha \frac{\alpha(1-\alpha)}{m^2(1-\alpha+\alpha^2)}$$

2. **Wilson-Fisher fixed point in $D = 4 - \varepsilon$:** Let us return to ϕ^4 theory. In $D = 4$, integrating out a shell of modes with momenta $b\Lambda < p < \Lambda$ leads to a change in the coupling $\Delta\lambda = -\frac{3\lambda^2}{16\pi^2} \log \frac{1}{b}$. How the coupling changes with scale is called the *beta function* for the coupling

$$\beta_\lambda \equiv \frac{d\lambda}{d \log b} . \tag{1}$$

Now when $D \neq 4$, there is an additional change in the coupling due to its non-zero dimension (see Peskin, Eq. (12.26)). Combining these two effects, one finds that near $D = 4$ the coupling changes as

$$\lambda' = \lambda b^{D-4} - \frac{3\lambda^2}{16\pi^2} \log \frac{1}{b} . \tag{2}$$

Defining $\epsilon = 4 - D > 0$ and assuming[1] $\epsilon \ll 1$, show that there is a zero of the beta function at a particulara value of the coupling that you must find. Argue that your result is trustworthy even though you obtained it from perturbation theory (i.e. assuming λ small). At this fixed point, the physics does not change as one zooms out – it is scale invariant! Explain Fig. 12.2 of Peskin.

If you have time, you can read more about the Wilson-Fisher fixed point in Sec. 12.5 of Peskin. This fixed point, whose existence you've proved for $\epsilon \ll 1$, is believed to become the 3D Ising fixed point (CFT) in $D = 3$ when $\epsilon \to 1$.

$$\lambda' = \lambda b^{-\varepsilon} - \frac{3\lambda^2}{16\pi^2} \log \frac{1}{b} .$$

$$\beta_\lambda = \frac{d\lambda'}{d\log b} = \frac{d\lambda'}{db} \left(\frac{d\log b}{db}\right)^{-1} = \left(-\varepsilon\lambda b^{-\varepsilon-1} + \frac{3\lambda^2}{16\pi^2 \log 10} \frac{1}{b}\right) \cdot b \log 10$$

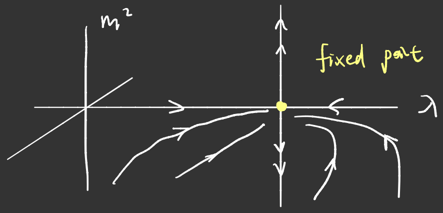
$$= -\varepsilon\lambda b^{-\varepsilon} \log 10 + \frac{3\lambda^2}{16\pi^2} .$$

$\beta_\lambda = 0$ with $\lambda = \frac{16\pi^2}{3} \log 10 \cdot \varepsilon b^{-\varepsilon}$.

at this coupling number, coupling does not change with scale.

This does not depend on small ε.

Fig 12.2 of Peskin :



For case $D<4$. λ is relevant.

ϕ^4 interaction becomes increasingly important at large distances.

We have recursion formula

$$\lambda' = \left[\lambda - \frac{3\lambda^2}{(4\pi)^{\frac{d}{2}}\Gamma(\frac{d}{2})}\frac{b^{d-4}-1}{4-d}\Lambda^{d-4}\right]b^{d-4}.$$

increase due to scaling compensated by nonlinear effect

at fixed point λ being a second fixed point of RG flow.

$d\to 4$, Wilson-Fisher fixed point merges with free field fixed point.

$d\sim 4$, mass parameter m^2 increased by iteration.

m^2 be a relevant parameter near fixed point.

as shown in figure, different flow directions represent different phases showing the effect of m^2 parameter.

3. **'In' and 'out' states:** Read the argument at the end of Sec. 5 in Srednicki, showing that $a_k^\dagger(-\infty)$ does not create multiparticle states.

I will exactly follow the argument in Srednicki.

$a_1^\dagger(\pm\infty)$ not create combined state on $|0\rangle \iff \langle 0|\varphi(0)|0\rangle = 0$.

If nonzero, set $\varphi(x) \to \varphi(x) + v$. with $\langle 0|\varphi(x)|0\rangle = 0$ after shift.

Consider $\langle p|\varphi(x)|0\rangle = \langle p|e^{-iPx}\varphi(0)e^{+iPx}|0\rangle = e^{-ipx}\underbrace{\langle p|\varphi(0)|0\rangle}_{=1\ ?}$.

In interacting theory, to ensure $a_1^\dagger(\pm\infty)$ create normalized one-particle state.

we should have $\langle p|\varphi(0)|0\rangle = 1$. If not, we should rescale.

Consider $\langle p,n|\varphi(0)|0\rangle = \langle p,n|e^{-iPx}\varphi(0)e^{+iPx}|0\rangle$

$$= e^{-ipx}\underbrace{\langle p,n|\varphi(0)|0\rangle}_{=0\ ?} = e^{-ipx}A_n(\vec{p}).$$

Let's write $|\psi\rangle = \sum_n \int d^3p\, \psi_n(\vec{p})\,|p,n\rangle$.

examine $\langle\psi|a_1^\dagger(t)|0\rangle = -i\sum_n\int d^3p\,\psi_n^*(\vec{p})\int d^3k\, f_1(\vec{k})\int d^3x\, e^{ikx}\overleftrightarrow{\partial_0}\underbrace{\langle p,n|\varphi(x)|0\rangle}_{e^{-ipx}A_n(\vec{p})}$

$$= \sum_n\int d^3p\,\psi^*(\vec{p})\int d^3k\, f_1(\vec{k})\int d^3x\,(p^0+k^0)e^{i(k-p)x}A_n(\vec{p}) \qquad (2\pi)^3\delta(\vec{k}-\vec{p})$$

$$= \sum_n\int d^3p\,(2\pi)^3(p^0+k^0)\,\psi_n^*(\vec{p})\,f_1(\vec{p})\,A_n(\vec{p})\,e^{i(p^0-k^0)t}.$$

Riemann – Lebesgue Lemma :

$$\int_{-\infty}^{+\infty} f(x)\, e^{-ikx}\, dx . \qquad f(x) \in L^1(\mathbb{R}) .$$

$|k| \to \infty$. integral $\to 0$.

In above integral we have

$p^0 = \sqrt{\vec{p}^{\,2} + M^2}$, $k^0 = \sqrt{\vec{p}^{\,2} + m^2}$. $\quad p^0 > k^0$ as $M \geq 2m > m$.

As $t \to \pm\infty$, $|p^0 - k^0| \to \infty$.

Thus the integral vanishes and $\langle \varphi | a_i^\dagger(t) | 0 \rangle = 0$.

This means overlap between multiparticle & single wavepacket

goes to zero as $t \to \infty$. the contribution of

multiparticle state could be made as small as we want.

$\Rightarrow$ $a_k^\dagger(-\infty)$ does not create multiparticle states.

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1. **Discovering the 'Higgs boson' :**

Consider a simplistic model of our universe, where we only know the existence of one light scalar particle χ with mass $m = 1$ GeV. However, we have come to suspect that our universe may also contain a much heavier scalar particle, an elusive particle called φ. Thanks to novel high-energy χ colliders reaching near TeV scale, we hope to carefully study $\chi\chi \to \chi\chi$ collisions to find evidence for the existence of φ.

(i) To make sure we understand the 'background' signal, let us first study the χ-only theory. Assume we know from every day life that it has $\chi \to -\chi$ symmetry; show that the most general Lorentz-invariant Lagrangian for χ involving only relevant or marginal terms, in $D = 4$ spacetime dimensions, is

$$\mathcal{L}_\chi = -\frac{1}{2}(\partial\chi)^2 - \frac{1}{2}m^2\chi^2 - \frac{1}{4!}\lambda\chi^4 . \tag{1}$$

[**Note:** we will keep the χ^4 term even though we have shown in class that it is marginally irrelevant.]

(ii) Next we turn to $\chi\chi \to \chi\chi$ scattering in the χ-only theory (1). We are interested in the matrix element $\mathcal{M}$ appearing in the non-trivial part of the S-matrix:

$$\langle f|i\rangle = \delta_{if} + (2\pi)^4\delta^4(k_1 + k_2 - k_1' - k_2')i\mathcal{M}(k_1, k_2 \to k_1', k_2') . \tag{2}$$

We found in class that it can be obtained from the time-ordered connected 4-point function using LSZ:

$$\begin{aligned} i\mathcal{M}(k_1, k_2 \to k_1', k_2') = {} & (k_1^2 + m^2)(k_2^2 + m^2)(k_1'^2 + m^2)(k_2'^2 + m^2) \\ & \times \langle 0|\mathcal{T}\phi(k_1)\phi(k_2)\phi(-k_1')\phi(x=0)|0\rangle_c , \end{aligned} \tag{3}$$

(i) $\chi \to -\chi$ symmetry. $\Rightarrow$ even orders of χ. Z_2 invariance.

$\mathcal{L}_0 = \frac{1}{2}(\partial_\mu\chi)(\partial^\mu\chi) - \frac{1}{2}m^2\chi^2$. $\quad \mathcal{L}_{int} = -\frac{\lambda}{4!}\chi^4 - \frac{\lambda'}{6!}\chi^6 \dots$

For $D=4$. $\mathcal{L}_\chi = -\frac{1}{2}(\partial\chi)^2 - \frac{1}{2}m^2\chi^2 - \frac{\lambda}{4!}\chi^4$.

4 4 4 marginal

(ii) Let's revisit the process.

$$\langle f|i\rangle = i^4 \int_{x_1 x_2 x_1' x_2'} e^{ik_1x_1} e^{ik_2x_2} e^{-ik_1'x_1'} e^{-ik_2'x_2'} \cdot (m^2-\partial_{1'}^2)(m^2-\partial_{2'}^2) \cdot$$

$$\langle 0|T\{\phi(x_1)\phi(x_2)\phi(x_1')\phi(x_2')\}|0\rangle$$

$$= (m^2+k_1^2)(m^2+k_2^2)(m^2+k_1'^2)(m^2+k_2'^2) \int_{x_1x_2x_1'x_2'} e^{i(k_1x_1+k_2x_2-k_1'x_1'-k_2'x_2')} \underline{\langle\phi(x_1)\phi(x_2)\phi(x_1')\phi(x_2')\rangle}$$

$$\underline{\langle\phi_1\phi_2\phi_{1'}\phi_{2'}\rangle_c} + \langle\phi_1\phi_{1'}\phi_2\phi_{2'}\rangle + \langle\phi_1\phi_{2'}\phi_2\phi_{1'}\rangle + \langle\phi_1\phi_2\rangle\langle\phi_{1'}\phi_{2'}\rangle$$

$G(k_1)G(k_2)\delta(k_1-k_1')\delta(k_2-k_2')$ $G(k_1)G(k_2)\delta(k_1-k_2')\delta(k_2-k_1')$ $\propto \delta(k_1+k_2)$

$$\Rightarrow \langle f|i\rangle = \delta_{if} + iT_{if}.$$

$$= \delta_{if} + (m^2+k_1^2)(m^2+k_2^2)(m^2+k_1'^2)(m^2+k_2'^2)\int_{x_1x_2x_1'x_2'} e^{i(k_1x_1+k_2x_2-k_1'x_1'-k_2'x_2')} \langle\phi(x_1)\phi(x_2)\phi(x_1')\phi(x_2')\rangle_c$$

$$\equiv (2\pi)^4\delta^{(4)}(k_1+k_2-k_1'-k_2')\, i\underset{\sim}{M}.$$ compute for χ theory

For one-loop contribution,

$$iM(k_1,k_2\to k_1',k_2') = (k_1^2+m^2)(k_2^2+m^2)(k_1'^2+m^2)(k_2'^2+m^2)\times \underline{\langle 0|T\phi(k_1)\phi(k_2)\phi(-k_1')\phi(x=0)|0\rangle_c}.$$

We should evaluate

$$-i\frac{\lambda}{4!}\int d^4x \langle 0|T\phi(k_1)\phi(k_2)\phi(-k_1')\phi(x=0)|0\rangle$$

① $x_1,x_2\to x_3,x_4$. (s)

② $x_1,x_3\to x_2,x_4$. (t)

③ $x_1,x_4\to x_2,x_3$. (u)

$\langle\phi^4_{\text{external}}\ \frac{1}{2}\left(\frac{i\lambda}{4!}\right)^2\phi^4(x)\phi^4(y)\rangle$,

not just $\langle\phi^4\rangle$.

$$\Rightarrow iM(k_1,k_2\to k_1',k_2') = -i\lambda$$

$$+(-i\lambda)^2\left[iV(s)+iV(t)+iV(u)\right] - i\delta_\lambda + O(\lambda^3)$$

with the tree-level result

$$\mathcal{M}(k_1, k_2 \to k_1', k_2') = -\lambda + O(\lambda^2)\,. \tag{4}$$

Following similar steps as those used in class, obtain the $O(\lambda^2)$ correction. Your result should be Eq. (10.21) in Peskin, namely:

$$i\mathcal{M}(k_1, k_2 \to k_1', k_2') = -i\lambda + (-i\lambda)^2\,[iV(s) + iV(t) + iV(u)] - i\delta_\lambda + O(\lambda^3)\,, \tag{5}$$

where s, t, u are the Mandelstam variables, and with

$$V(-k^2) \equiv -\frac{i}{2}\int \frac{d^4p}{(2\pi)^4} G(p)G(k+p)\,, \qquad G(p) \equiv \frac{-i}{p^2 + m^2}\,. \tag{6}$$

What is δ_λ and why is it necessary? Next let us evaluate $V(-k^2)$. Peskin does this in Eq. (10.23) using dimensional regularization: do it instead using a sharp cutoff Λ on Euclidean momentum as we have done in the past. As a renormalization prescription, we will choose δ_λ to simply cancel the $\log \Lambda$ divergence. Evaluate all integrals except for the one over the Feynman parameter $\int_0^1 dx$. Your answer will be slightly different from Peskin's Eq. (10.25) because of the different renormalization scheme.

(iii) We will be interested in high energy scattering, $s,\ t,\ u \gg m^2$. Show that in this limit,

$$i\mathcal{M}(k_1, k_2 \to k_1', k_2') \simeq -i\lambda - i\frac{\lambda^2}{32\pi^2}\log(stu)\,. \tag{7}$$

(We are only keeping the λ^2 terms that are enhanced by logs of momenta). In this limit, the differential cross-section is approximately angle independent. Show that the total cross-section is given by

$$\sigma(s) \simeq \frac{\lambda^2}{16\pi^2 s}\left(1 + \frac{3\lambda}{16\pi^2}\log s\right)\,. \tag{8}$$

Plot it for the following parameters: $\lambda = 0.2$, and $\sqrt{s} \in [20, 200]$ GeV. Confirm that your one-loop correction is small in this regime.

(iv) Next, we study the 'new physics' signatures that would come from the new particle φ. As a real scalar, the free part of its Lagrangian is

$$\mathcal{L}_\varphi = -\frac{1}{2}(\partial\varphi)^2 - \frac{1}{2}M^2\varphi^2 \tag{9}$$

Now the interesting terms in the action are those that will involve both φ and χ – these will allow the new particle φ to influence the physics of the known particle χ.

δ_λ ensures renormalization condition $iM = -i\lambda$ at $s = 4m^2$, $t = u = 0$.

$$V(-k^2) = -\frac{i}{2}\int \frac{d^4p}{(2\pi)^4} G(p)G(k+p)$$

$$= \frac{1}{2}\int \frac{d^4p}{(2\pi)^4} \frac{1}{p^2+m^2} \frac{1}{\underbrace{(k+p)^2}_{\sim k^2+p^2-m^2}+m^2}. \qquad d^4p = 2\pi^2\int_0^\Lambda p^3 dp$$

$$= \frac{1}{2}\frac{1}{(2\pi)^4}\int_0^\Lambda p^3 dp \frac{1}{k^2}\left(\frac{1}{p^2+m^2} - \frac{1}{p^2+k^2+m^2}\right)$$

Evaluate $\displaystyle\int_0^\Lambda dp \frac{p^3}{\underbrace{p^2+\Delta^2}_{u}} \longrightarrow \int_{\Delta^2}^{\Lambda^2+\Delta^2} du\left(1-\frac{\Delta^2}{u}\right)$

$$= \frac{1}{2}\left[\Lambda^2 - \Delta^2 \ln\left(\frac{\Lambda^2+\Delta^2}{\Delta^2}\right)\right]$$

$$\Rightarrow V(-k^2) = \frac{1}{(2\pi)^4}\frac{1}{2k^2}\left[\frac{1}{2}\Lambda^2 - \frac{1}{2}m^2\ln\left(\frac{\Lambda^2+m^2}{m^2}\right) - \frac{1}{2}\Lambda^2 + \frac{1}{2}(k^2+m^2)\ln\left(\frac{\Lambda^2+k^2+m^2}{k^2+m^2}\right)\right]$$

$$= \frac{1}{32\pi^4 k^2}\left[\frac{1}{2}k^2\ln\left(1+\frac{\Lambda^2}{k^2+m^2}\right) + \frac{1}{2}m^2\ln\left(\frac{\Lambda^2+k^2+m^2}{k^2+m^2}\cdot\frac{m^2}{\Lambda^2+m^2}\right)\right]$$

Compare to Peskin $\displaystyle\frac{1}{32\pi^2}\int_0^1 dx \log\left(\frac{m^2-x(1-x)r}{m^2-x(1-x)4m^2}\right)$. $\quad r = s, t, u$.

(iii) For $r \gg m^2$. $\Rightarrow \displaystyle\frac{1}{32\pi^2}\int_0^1 dx \log\left(1+\frac{x(1-x)(r-4m^2)}{\underline{x(1-x)4m^2-m^2}}\right) \approx \frac{r'-4}{4-\frac{1}{x(1-x)}}$

$x \to y+\frac{1}{2}$ $\Rightarrow \displaystyle\frac{1}{32\pi^2}\int_{-\frac{1}{2}}^{\frac{1}{2}} dy \log\left(\frac{r'+(y^2-\frac{1}{4})^{-1}}{4+(y^2-\frac{1}{4})^{-1}}\right) \approx \frac{1}{32\pi^2}\log(r')$.

$r' = s, t, u$.

$$\Rightarrow \; i\mathcal{M}(k_1,k_2\to k_1',k_2') \simeq -i\lambda - i\frac{\lambda^2}{32\pi^2}\log(stu).$$

$$|\mathcal{M}|^2 = \lambda^2 + \lambda\frac{\lambda^2}{16\pi^2}\log(stu) + \left(\frac{\lambda^2}{32\pi^2}\right)^2\log^2(stu).$$

$$\sigma = \frac{1}{\underline{4E_1E_2|\vec{v}_1-\vec{v}_2|}}\int |\mathcal{M}|^2 \, \underline{d\Phi} \qquad \frac{1}{4(2\pi)^2}\frac{|\vec{k}_1'|}{\sqrt{s}}d\Omega = \frac{1}{16\pi^2}\frac{\sqrt{s-4m^2}}{2\sqrt{s}}d\Omega \approx \frac{d\Omega}{16\pi^2}$$

$$\approx 4E_1E_2\cdot\frac{\sqrt{(k_1\cdot k_2)^2-m_1^2m_2^2}}{E_1E_2} \approx 4s$$

$$\approx 3\log s$$

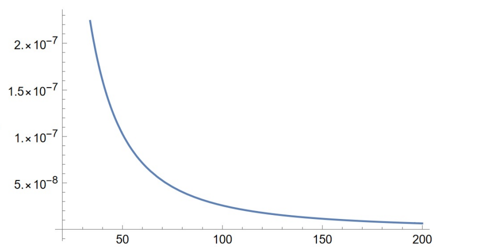
$$\Rightarrow \; \sigma \simeq \frac{1}{4s}\cdot\frac{1}{16\pi^2}\int|\mathcal{M}|^2 d\Omega = \frac{1}{16\pi^2}\cdot\frac{1}{4s}\cdot\left(\lambda^2 + \frac{\lambda^3}{16\pi^2}\underline{\log(stu)}\right)\cdot 4\pi$$

$$= \frac{\lambda^2}{16\pi^2 s}\left(1+\frac{3\lambda}{16\pi^2}\log s\right).$$

```
In[11]:= ClearAll["Global`*"]
         λ = 0.2;
         σ[sr_] = λ^2/(16 π^2 sr^2) (1 + 3 λ/(16 π^2) Log10[sr^2]);

In[10]:= Plot[0.000253303 (1 + 0.00165012 Log[sr^2])/sr^2, {sr, 20, 200}]
```

Out[10]=



1-loop correction is small

in the sense ...

(iv) $\mathcal{L}_\varphi = -\frac{1}{2}(\partial\varphi)^2 - \frac{1}{2}M^2\varphi^2.$

φ does not have $\varphi\to-\varphi$ symmetry. then

$\mathcal{L}_{int} = \sum g\,\chi^n\varphi^m.$ n even. m odd.

Assuming that this new particle does not have a $\varphi \to -\varphi$ symmetry, show that the most relevant term involving both particles is

$$\mathcal{L}_{\text{int}} = -\frac{1}{2} g \varphi \chi^2 . \tag{10}$$

What is the dimension of g?

(v) This interaction will give a correction to the $\chi\chi \to \chi\chi$ matrix element at $O(g^2)$. Find this matrix element and draw the associated Feynman diagrams (use different symbols for lines denoting different particles). Your answer should diverge when s, t, or $u \to M^2$. You run to your experimentalist friend to tell them about this spectacular effect.

(vi) On your way home after your experimentalist friend laughed at your divergent answer, you remember Problem Set 7, where you had found that heavy particles should be unstable, leading to a finite 'width' Γ to their propagators. Assuming for simplicity $M^2 \gg s, t, u \gg m^2$, show that you can take this into account by replacing the heavy particle propagators in your answer by:

$$\frac{-i}{p^2 + M^2} \to \frac{-i}{p^2 + M^2 - i\Gamma M} . \tag{11}$$

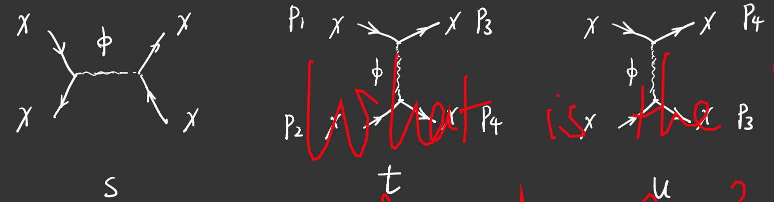
Sketch the diagrams that are now being taken into account. In that problem set we were working in $D = 2$, and had found, in this limit, $\Gamma = \alpha \frac{g^2}{M^3}$, with dimensionless number $\alpha = \frac{1}{8}$. Using dimensional analysis in $D = 4$, guess the expression for Γ up to the dimensionless number α. [We could obtain this number α by repeating Problem Set 7 in $D = 4$, but we will not do this]. Compute the $O(\lambda g^2)$ contribution to the differential cross-section $d\sigma/d\Omega_{\text{CM}}$, and integrate over angles to obtain the total cross section [do not expand out Γ, i.e. keep it in the form (11)]. Finally, plot your final cross-section, keeping the $O(\lambda)$, $O(\lambda^2)$ and $O(\lambda g^2)$ terms, using the values: $\alpha = 5$, $g = 0.8$, $\lambda = 0.2$, $M = 125$ GeV, zooming in on the region $\sqrt{s} \in [100, 160]$ GeV.

Compare your plot to https://home.cern/science/physics/higgs-boson/how. (The energy scale on their y axis is slightly different than ours: their $\sqrt{s}$ is fixed, and instead $\sqrt{t}$ or Ω_{CM} is varied.)

For $D = 4$. $\mathcal{L}_{int} = -\frac{1}{2} g \varphi \chi^2$. $[g] = 1$.

$1 \quad 1 \quad 2$

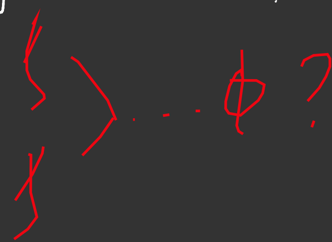
(v)



What is the Feynman rule of $\frac{g}{2}\chi^2\phi$ for the vertex

$$\mathcal{M}' = -\frac{1}{2} g^2 \left(\frac{1}{s-M^2} + \frac{1}{t-M^2} + \frac{1}{u-M^2} \right)$$

diverge with $s, t, u \sim M^2$.



(vi)