Schrödinger 方程包含粒子数守恒

$$H\psi = i\hbar\frac{\partial\psi}{\partial t} \longrightarrow \int d^3x\left(\psi^\dagger H\psi\right) = i\hbar\int d^3x\left(\psi^\dagger\frac{\partial\psi}{\partial t}\right)$$

$$\psi^\dagger H = -i\hbar\frac{\partial\psi^\dagger}{\partial t} \longrightarrow \int d^3x\left(\psi^\dagger H\psi\right) = -i\hbar\int d^3x\left(\frac{\partial\psi^\dagger}{\partial t}\psi\right).$$

$$\Rightarrow \frac{d}{dt}\int d^3x\,\psi^\dagger\psi = 0.$$

而 $[x,p]=0$. $\Delta x\Delta p \ge \frac{\hbar}{2}$. $p^2c^2+m^2c^4=E^2$.

$$\Delta E = \frac{p\Delta p}{E}c^2 \ge \frac{p\hbar c^2}{E\Delta x} \Rightarrow \Delta x \ge \frac{pc}{E}\left(\frac{\hbar c}{\Delta E}\right).$$

无新粒子 $\Delta E \lesssim mc^2$

$$\Rightarrow \Delta x \ge \frac{v}{c}\frac{\hbar}{mc}.$$

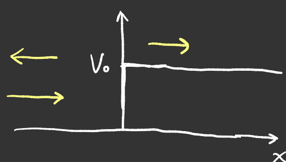
(a) 非相对论粒子. $\frac{v}{c}\ll 1$.

Δx 无太大限制. 粒子可局限在任意小范围内.

(b) 相对论粒子. $\frac{v}{c}\approx 1$. $\Delta x \ge \frac{\hbar}{mc}$.

比 Compton 波长小的空间尺度内将产生新粒子.

Klein 佯谬



K-G eqn $\left(\frac{\partial^2}{\partial t^2}-\nabla^2+m^2\right)\psi(x,t)=0$.

$\psi_L(x,t) = e^{-iEt-ip_1x}+Re^{-iEt+ip_1x}$ $\quad p_1=\sqrt{E^2-m^2}$

$\psi_R(x,t) = Te^{-iEt-ip_2x}$ $\quad p_2=\sqrt{(E-V_0)^2-m^2}$

边条 $\Rightarrow$ $1+R=T$. $(1-R)p_1 = Tp_2$.

$T = \dfrac{2p_1}{p_1+p_2}$. $R = \dfrac{p_1-p_2}{p_1+p_2}$.

① $\frac{v}{c} \ll 1$. $E > V_0+m$. 透射+反射

$E < V_0+m$. 仅反射.

② $\frac{v}{c} \approx 1$. $V_0 > 2m$. $m < V_0 - E$. 范围内 $V_0 \uparrow$. 不应透射.

部分透射部分反射.

自然单位制 $\hbar = 1$. $c=1$.

质量. $[\text{长度}]^{-1}$. $m_e = \dfrac{1}{\frac{\hbar}{m_e c}}$.

$[\text{时间}]^{-1}$. $m_e = \dfrac{1}{\frac{\hbar}{m_e c^2}}$.

能量. $m_e = m_e c^2$.

动量. $m_e = m_e c$.

场论 $\qquad q_i(t) \longrightarrow \phi(\vec{x},t)$

$$S = \int \mathcal{L}\left(\phi(\vec{x},t),\ \partial_\mu\phi\right) d^3x\, dt\,.$$

$$\delta S = \int \left[\frac{\partial \mathcal{L}}{\partial \phi}\delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi)}\delta(\partial_\mu\phi)\right] dx^4 = \int \underbrace{\left[\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi)}\right]}_{=0} \delta\phi\, dx^4\,.$$

E-L eqn

$$\pi(\vec{x},t) = \frac{\partial \mathcal{L}}{\partial(\partial_0\phi)}\,. \qquad \mathcal{H} = \pi\dot{\phi} - \mathcal{L}\,.$$

$$\phi(\vec{x},t) \longrightarrow \phi_i(\vec{x},t)\,. \quad (\text{多个场}) \qquad i=1,2,\cdots n\,.$$

$$\frac{\partial \mathcal{L}}{\partial \phi_i} = \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi_i)}\right). \qquad \pi_i(\vec{x},t) = \frac{\partial \mathcal{L}}{\partial(\partial_0\phi_i)}\,.$$

$$\mathcal{H} = \pi_i\dot{\phi}_i - \mathcal{L}\,.$$

Noether 定理

$$t \longrightarrow t+a\,. \qquad m\frac{d^2\vec{x}}{dt^2} = -\nabla V(\vec{x},t)\,.$$

设 $V(\vec{x},t) = V(\vec{x})$.

$$m\frac{d\vec{x}}{dt}\cdot\left(\frac{d^2\vec{x}}{dt^2}\right) = -\frac{d\vec{x}}{dt}\cdot\nabla V = -\frac{d}{dt}\left[V(\vec{x})\right].$$

$$\Rightarrow \quad \frac{d}{dt}\left[\frac{1}{2}m\left(\frac{d\vec{x}}{dt}\right)^2 + V(\vec{x})\right] = 0\,.$$

E 守恒

经典力学. $\qquad S = \int L(q_i, \dot{q}_i)\, dt\,.$

$$q_i \longrightarrow q_i' = f_i(q_j)\,. \qquad q_i \longrightarrow q_i' \simeq q_i + \delta q_i\,.$$

$$\delta S = \int \delta L\, dt = \int \left[\frac{\partial L}{\partial q_i}\delta q_i + \frac{\partial L}{\partial \dot{q}_i}\delta \dot{q}_i\right] dt\,. \qquad \delta\dot{q}_i = \frac{d}{dt}(\delta q_i)\,.$$

$$\frac{\partial L}{\partial q_i} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) \longrightarrow \quad \delta S = \int \left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)\delta q_i + \frac{\partial L}{\partial \dot{q}_i}\frac{d}{dt}(\delta q_i)\right] dt = \int \frac{d}{dt}\left(\underbrace{\frac{\partial L}{\partial \dot{q}_i}\delta q_i}_{\text{守恒量}}\right) dt$$

若 $\delta L \neq 0$. $\delta L = \frac{d}{dt}K$. $\Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\delta q_i - K\right) = 0$.

场论. $$S = \int \mathcal{L}(\phi, \partial_\mu \phi)\, d^4x.$$

$$\phi(x) \longrightarrow \phi'(x') \qquad x^\mu \longrightarrow x'^\mu$$

$$\delta\phi = \phi'(x') - \phi(x) \qquad \delta x^\mu = x'^\mu - x^\mu$$

$$d^4x' = J\, d^4x \qquad J = \left|\frac{\partial(x_0', x_1', x_2', x_3')}{\partial(x_0, x_1, x_2, x_3)}\right| \quad \text{体积元变化}$$

$$J = \left|\frac{\partial x'^\mu}{\partial x^\nu}\right| \approx \left|g^\mu_\nu + \frac{\partial(\delta x^\mu)}{\partial x^\nu}\right| \approx 1 + \partial_\mu(\delta x^\mu) \quad \text{无穷小变换}$$

$$\det(1+\varepsilon) \approx 1 + \mathrm{Tr}(\varepsilon), \quad |\varepsilon| \ll 1$$

$$d^4x' = d^4x\left(1 + \partial_\mu(\delta x^\mu)\right)$$

$$\delta S = \int \left[\frac{\partial \mathcal{L}}{\partial \phi}\delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\delta(\partial_\mu \phi) + \mathcal{L}\,\partial_\mu(\delta x^\mu)\right] dx^4$$

固定时变分 $$\bar{\delta}\phi(x) = \phi'(x) - \phi(x) = \phi'(x) - \phi'(x') + \phi'(x') - \phi(x)$$

$$= \underline{\phi'(x) - \phi'(x')} - (\partial_\mu \phi)\,\delta x^\mu + \delta\phi$$

$$\delta\phi = \bar{\delta}\phi + (\partial_\mu \phi)\,\delta x^\mu$$

$$\delta(\partial_\mu \phi) = \bar{\delta}(\partial_\mu \phi) + \partial_\nu(\partial_\mu \phi)\,\delta x^\nu$$

$$\Rightarrow \delta S = \int \left\{\frac{\partial \mathcal{L}}{\partial \phi}\left(\bar{\delta}\phi + (\partial_\mu \phi)\delta x^\mu\right) + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\left[\bar{\delta}(\partial_\mu \phi) + \partial_\nu(\partial_\mu \phi)\delta x^\nu\right] + \mathcal{L}\,\partial_\mu(\delta x^\mu)\right\} dx^4$$

$$\xrightarrow[\partial_\mu(\bar{\delta}\phi) = \bar{\delta}(\partial_\mu \phi)]{\frac{\partial \mathcal{L}}{\partial \phi} = \partial^\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\right)} \quad \frac{\partial \mathcal{L}}{\partial \phi}\bar{\delta}\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\bar{\delta}(\partial_\mu \phi) = \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\bar{\delta}\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\partial_\mu(\bar{\delta}\phi) = \partial_\mu\left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\bar{\delta}\phi\right].$$

$$\Rightarrow \left[\frac{\partial \mathcal{L}}{\partial \phi}(\partial_\nu \phi) + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\partial_\nu(\partial_\mu \phi)\right]\delta x^\nu + \mathcal{L}\,\partial_\nu(\delta x^\nu) = (\partial_\nu \mathcal{L})\delta x^\nu + \mathcal{L}\,\partial_\nu(\delta x^\nu) = \partial_\nu(\mathcal{L}\,\delta x^\nu).$$

$$\Rightarrow dS = \int dx^4\, \partial_\mu \underline{\left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\bar{\delta}\phi + \mathcal{L}\,\delta x^\mu\right]}.$$

流守恒

相对论波动方程

K-G 方程 $E = \frac{\vec{p}^2}{2m} + V(\vec{r})$.

$E \to i\frac{\partial}{\partial t}$. $\vec{p} \to -i\vec{\nabla}$.

$\Rightarrow i\hbar\frac{\partial\psi}{\partial t} = \left[-\frac{1}{2m}\nabla^2 + V(\vec{r})\right]\psi$. x与t不平权

Lorentz 协变的 E

$E^2 = \vec{p}^2 + m^2$. $\Rightarrow \boxed{-\partial_0^2\phi = (-\nabla^2+m^2)\phi}$.

或 $(\Box^2 + m^2)\psi = 0$. $\Box = \partial_0^2 - \nabla^2 = \partial^\mu\partial_\mu = \partial^2$.

无概率诠释

$(\partial_0^2 - \nabla^2 + m^2)\phi = 0$.

$(\partial_0^2 - \nabla^2 + m^2)\phi^* = 0$.

$\Rightarrow \frac{\partial\rho}{\partial t} + \nabla\cdot\vec{j} = 0$

$\rho = i(\phi\partial_0\phi^* - \phi\partial_0\phi^*)$ $\vec{j} = i(\phi\nabla\phi^* - \phi^*\nabla\phi)$

$$\frac{dP}{dt} = \int_V \frac{\partial\rho}{\partial t} d^3x = -\int_V \nabla\cdot\vec{j}\, d^3x = -\oint_S \vec{j}\cdot d\vec{s} = 0.$$

$P = \int \rho\, d^3x$ 守恒. 而 P 不正定.

K-G 方程的解

$$(\Box + m^2)\phi(x) = (\partial_0^2 - \nabla^2 + m^2)\phi(x) = 0$$

正能解 $p_0 = \omega_p = \sqrt{\vec{p}^2 + m^2}$

$$\phi_p^{(+)}(x) = \exp(-i\omega_p t + i\vec{p}\cdot\vec{x})$$

负能解 $p_0 = -\omega_p = -\sqrt{\vec{p}^2 + m^2}$

$$\phi_p^{(-)}(x) = \exp(i\omega_p t - i\vec{p}\cdot\vec{x})$$

通解 $$\phi(x) = \int \frac{d^3k}{\sqrt{(2\pi)^3 2\omega_k}}\left[a(k)e^{i\vec{k}\cdot\vec{x} - i\omega_k t} + b(k)e^{-i\vec{k}\cdot\vec{x} + i\omega_k t}\right]$$

内积 $(\partial_0^2-\nabla^2+m^2)\phi_1=0$ 左· ϕ_2^*

$(\partial_0^2-\nabla^2+m^2)\phi_2^*=0$ · ϕ_1

$$\Rightarrow \int d^3x\left\{\left[\phi_2^*\partial_0^2\phi_1-\phi_1\partial_0^2\phi_2^*\right]-\left[\phi_2^*\nabla^2\phi_1-\phi_1\nabla^2\phi_2^*\right]\right\}=0.$$

$$\int d^3x\left\{\partial_0\left[\phi_2^*\partial_0\phi_1-\phi_1\partial_0\phi_2^*\right]-\underbrace{\nabla\cdot\left[\phi_2^*\nabla\phi_1-\phi_1\nabla\phi_2^*\right]}_{\infty\text{处 }0}\right\}=0.$$

$$\Rightarrow \frac{d}{dt}\underbrace{\int d^3x\left[\phi_2^*\partial_0\phi_1-\phi_1\partial_0\phi_2^*\right]}_{\text{内积}}=0.$$

Dirac 方程

假设 $E=\alpha_1p_1+\alpha_2p_2+\alpha_3p_3+\beta m=\vec{\alpha}\cdot\vec{p}+\beta m.$

$$E^2=\underbrace{\frac{1}{2}(\alpha_i\alpha_j+\alpha_j\alpha_i)}_{\delta_{ij}}p_ip_j+\underbrace{(\alpha_i\beta+\beta\alpha_i)}_{0}mp_i+\underbrace{\beta^2}_{1}m^2.$$

$\Downarrow$

$\alpha_i^2=1.$ $\alpha_1\alpha_2=-\alpha_2\alpha_1.$ $\alpha_2=-\alpha_1\alpha_2\alpha_1.$

$\mathrm{Tr}(\alpha_2)=-\mathrm{Tr}(\alpha_1\alpha_2\alpha_1)=-\mathrm{Tr}(\alpha_2\alpha_1^2)=-\mathrm{Tr}(\alpha_2).$ $=0$

均为超越 Pauli 矩阵.

$$\alpha_i=\begin{pmatrix} & \sigma_i\\ \sigma_i & \end{pmatrix}.\qquad \beta=\begin{pmatrix}1 & \\ & -1\end{pmatrix}.$$

$$E=\vec{\alpha}\cdot\vec{p}+\beta m.\quad\Rightarrow\quad i\frac{\partial\psi}{\partial t}=(-i\vec{\alpha}\cdot\nabla+\beta m)\psi.$$

或 $(-i\beta\vec{\alpha}\cdot\nabla+i\beta\partial_t+m)\psi=0.$

定义 $\gamma^0=\begin{pmatrix}1 & \\ & -1\end{pmatrix}.$ $\gamma^i=\begin{pmatrix} & \sigma_i\\ -\sigma_i & \end{pmatrix}.$

$$(-i\gamma^i \partial_i - i\gamma^0 \partial_0 + m)\psi = 0. \qquad (-i\gamma^\mu \partial_\mu + m)\psi = 0.$$

概率诠释 $$-i\frac{\partial \psi^\dagger}{\partial t} = (-i\vec{\alpha}\cdot\nabla + \beta m)\psi^\dagger = \psi^\dagger(i\vec{\alpha}\cdot\nabla + \beta m)$$

$$i\left(\frac{\partial \psi^\dagger}{\partial t}\psi + \psi^\dagger\frac{\partial \psi}{\partial t}\right) = \psi^\dagger(-i\vec{\alpha}\cdot\nabla + \beta m)\psi - \psi^\dagger(i\vec{\alpha}\cdot\nabla + \beta m)\psi.$$

$$i\frac{d}{dt}\int d^3x(\psi^\dagger\psi) = \int\left\{-i\psi^\dagger(\alpha\cdot\nabla)\psi - i\left((\vec{\alpha}\cdot\nabla)\psi^\dagger\right)\psi\right\}d^3x = -i\int \underline{\vec{\nabla}\cdot(\psi^\dagger\vec{\alpha}\psi)}\, d^3x = 0$$

相对论波动方程

$$E = \frac{\vec{p}^2}{2m} + V(\vec{r}) \qquad E \to i\frac{\partial}{\partial t}, \quad \vec{p} \to -i\vec{\nabla}.$$

K-G方程

$$\Rightarrow \quad i\hbar \frac{\partial \psi}{\partial t} = \left[-\frac{1}{2m}\nabla^2 + V(\vec{r}) \right] \psi. \qquad x,t \text{ 不平权}$$

Lorentz 协变的 E.

$$E^2 = \vec{p}^2 + m^2. \quad \Rightarrow \quad \boxed{-\partial_0^2 \phi = (-\nabla^2 + m^2)\phi}$$

或 $(\Box^2 + m^2)\psi = 0.$ $\qquad \Box = \partial_0^2 - \nabla^2 = \partial^\mu \partial_\mu = \partial^2.$

无概率诠释

$$(\partial_0^2 - \nabla^2 + m^2)\phi = 0$$

$$(\partial_0^2 - \nabla^2 + m^2)\phi^* = 0$$

$$\Rightarrow \quad \frac{\partial \rho}{\partial t} + \nabla\cdot\vec{j} = 0$$

$$\rho = i(\phi\, \partial_0 \phi^* - \phi^* \partial_0 \phi)$$

$$\frac{dP}{dt} = \int_V \frac{\partial \rho}{\partial t} d^3x = -\int_V \nabla\cdot\vec{j}\, d^3x = -\oint_S \vec{j}\cdot d\vec{s} = 0.$$

$P = \int \rho\, d^3x$ 守恒 $\qquad$ 而 ρ 不正定.

K-G方程的解

$$(\Box^2 + m^2)\phi(x) = (\partial_0^2 - \nabla^2 + m^2)\phi(x) = 0$$

正能解 $\quad P_0 = \omega_p = \sqrt{\vec{p}^2 + m^2}$

$$\phi_p^{(+)}(x) = \exp(-i\omega_p t + i\vec{p}\cdot\vec{x}).$$

负能解 $\quad P_0 = -\omega_p = -\sqrt{\vec{p}^2 + m^2}$

$$\phi_p^{(-)}(x) = \exp(i\omega_p t - i\vec{p}\cdot\vec{x}).$$

通解 $\quad \phi(x) = \int \frac{d^3k}{\sqrt{(2\pi)^3 2\omega_k}} \left[a(k)\, e^{i\vec{k}\cdot\vec{x} - i\omega_k t} + b(k)\, e^{-i\vec{k}\cdot\vec{x} + i\omega_k t} \right]$

内积 $(\partial_0^2-\nabla^2+m^2)\phi_1=0$ 左 $\cdot\phi_2^*$

$(\partial_0^2-\nabla^2+m^2)\phi_2=0$ $\cdot\phi_1$

$$\Rightarrow \int d^3x\left\{\left[\phi_2^*\partial_0^2\phi_1-\phi_1\partial_0^2\phi_2^*\right]-\left[\phi_2^*\nabla^2\phi_1-\phi_1\nabla^2\phi_2^*\right]\right\}=0$$

$$\Rightarrow \int d^3x\left\{\partial_0\left[\phi_2^*\partial_0\phi_1-\phi_1\partial_0\phi_2^*\right]-\underbrace{\nabla\cdot\left[\phi_2^*\nabla\phi_1-\phi_1\nabla\phi_2^*\right]}_{\infty\text{处}=0}\right\}=0$$

$$\Rightarrow \frac{d}{dt}\underbrace{\int d^3x\left[\phi_2^*\partial_0\phi_1-\phi_1\partial_0\phi_2^*\right]}_{\text{内积}}=0$$

Dirac 方程

假设 $E=\alpha_1p_1+\alpha_2p_2+\alpha_3p_3+\beta m=\vec{\alpha}\cdot\vec{p}+\beta m$.

$$E^2=\underbrace{\tfrac{1}{2}(\alpha_i\alpha_j+\alpha_j\alpha_i)}_{\delta_{ij}}p_ip_j+\underbrace{(\alpha_i\beta+\beta\alpha_i)}_{0}mp_i+\underbrace{\beta^2}_{1}m^2.$$

$\Downarrow$

$\alpha_i^2=1.$ $\quad\alpha_1\alpha_2=-\alpha_2\alpha_1.$ $\quad\alpha_2=-\alpha_1\alpha_2\alpha_1.$

$$Tr(\alpha_2)=-Tr(\alpha_1\alpha_2\alpha_1)=-Tr(\alpha_2\alpha_1^2)=-\underline{Tr(\alpha_2)}.\quad=0$$

均为超越 Pauli 矩阵

$$\alpha_i=\begin{pmatrix} & \sigma_i\\ \sigma_i & \end{pmatrix}\qquad \beta=\begin{pmatrix}1 & \\ & -1\end{pmatrix}$$

$$E=\vec{\alpha}\cdot\vec{p}+\beta m.\ \Rightarrow\ i\frac{\partial\psi}{\partial t}=(-i\vec{\alpha}\cdot\nabla+\beta m)\psi.$$

或 $(-i\beta\vec{\alpha}\cdot\nabla+i\beta\partial_t+m)\psi=0.$

定义 $\gamma^0=\begin{pmatrix}1 & \\ & -1\end{pmatrix}.\qquad \gamma^i=\begin{pmatrix} & \sigma_i\\ -\sigma_i & \end{pmatrix}.$

$$\left(-i\gamma^i\partial_i - i\gamma^0\partial_0 + m\right)\psi = 0. \qquad \left(-i\gamma^\mu\partial_\mu + m\right)\psi = 0$$

概率诠释 $\qquad -i\frac{\partial\psi^\dagger}{\partial t} = \left(-i\vec{\alpha}\cdot\nabla + \beta m\right)\psi^\dagger = \psi^\dagger\left(i\vec{\alpha}\cdot\nabla + \beta m\right)$

$$i\left(\frac{\partial\psi^\dagger}{\partial t}\psi + \psi^\dagger\frac{\partial\psi}{\partial t}\right) = \psi^\dagger\left(-i\vec{\alpha}\cdot\nabla + \beta m\right)\psi - \psi^\dagger\left(i\vec{\alpha}\cdot\nabla + \beta m\right)\psi.$$

$$i\frac{d}{dt}\int d^3x\left(\psi^\dagger\psi\right) = \int\left\{-i\psi^\dagger(\vec{\alpha}\cdot\nabla)\psi - i\left((\vec{\alpha}\cdot\nabla)\psi^\dagger\right)\psi\right\}d^3x = -i\int \underline{\nabla\cdot\left(\psi^\dagger\vec{\alpha}\psi\right)d^3x} = 0.$$

Dirac 方程的解

$$\left(i\gamma^\mu\partial_\mu - m\right)\psi = 0 \qquad \text{设} \quad \psi(x) = e^{-ip\cdot x}\,w(p) \quad \text{平面波解}$$

$$\Rightarrow \quad \left(\underline{\gamma^0 p_0 - \vec{\gamma}\cdot\vec{p}} - m\right)w(p) = 0.$$

$\not{p} = \gamma^\mu p_\mu$

$$\left(p_0 - \vec{\alpha}\cdot\vec{p} - \beta m\right)w(p) = 0. \qquad \vec{\alpha} = \gamma_0\vec{\gamma}, \quad \beta = \gamma_0.$$

即 $\quad \underset{\sim}{H}\,w(p) = p_0\,w(p). \qquad H = \vec{\alpha}\cdot\vec{p} + \beta m.$

$$H = \begin{pmatrix} m & \vec{\sigma}\cdot\vec{p} \\ \vec{\sigma}\cdot\vec{p} & -m \end{pmatrix}. \qquad w(p) = \begin{pmatrix} u \\ l \end{pmatrix}. \qquad \Rightarrow \begin{cases} (p_0 - m)u - (\vec{\sigma}\cdot\vec{p})l = 0 \\ -(\vec{\sigma}\cdot\vec{p})u + (p_0 + m)l = 0. \end{cases}$$

$$\Rightarrow \quad p_0^2 = \vec{p}^2 + m^2. \qquad p_0 = \pm\sqrt{\vec{p}^2 + m^2}.$$

正能解 $\qquad p_0 = E = \sqrt{\vec{p}^2 + m^2}. \qquad l = \frac{\vec{\sigma}\cdot\vec{p}}{E+m}u.$

$$w^{(s)}(p) = N\begin{pmatrix} 1 \\ \frac{\vec{\sigma}\cdot\vec{p}}{E+m} \end{pmatrix}\chi_s.$$

$$\psi = e^{-ipx}w^{(s)}(p) = e^{-iEt}e^{i\vec{p}\cdot\vec{x}}N\begin{pmatrix} 1 \\ \frac{\vec{\sigma}\cdot\vec{p}}{E+m} \end{pmatrix}\chi_s.$$

负能解 $p_0 = -E = -\sqrt{\vec{p}^2 + m^2}$. $\quad u = \frac{-(\vec{\sigma}\cdot\vec{p})}{E+m} \ell$.

$$\omega(p) = N \begin{pmatrix} \frac{-\vec{\sigma}\cdot\vec{p}}{E+m} \\ 1 \end{pmatrix} \chi_s .$$

$$\psi = e^{iEt} e^{i\vec{p}\cdot\vec{x}} N \begin{pmatrix} \frac{-\vec{\sigma}\cdot\vec{p}}{E+m} \\ 1 \end{pmatrix} \chi_s .$$

旋量 spinor

$$u(p,s) = N \begin{pmatrix} 1 \\ \frac{\vec{\sigma}\cdot\vec{p}}{E+m} \end{pmatrix} \chi_s . \qquad v(p,s) = N \begin{pmatrix} \frac{\vec{\sigma}\cdot\vec{p}}{E+m} \\ 1 \end{pmatrix} \chi_s . \qquad N = \sqrt{E+m} .$$

$$u^\dagger(p,s)\, v(-p,s) = 0 .$$

Dirac 共轭

动量空间中 $(\not{p} - m)\psi(p) = 0$. $\quad \not{p} = p^\mu \gamma_\mu$. 非厄米

$$\psi^\dagger(p)\,(\not{p}^\dagger - m) = 0 . \qquad \gamma_0^\dagger = \gamma_0 . \quad \gamma_i^\dagger = -\gamma_i . \quad \gamma_\mu^\dagger = \gamma_0 \gamma_\mu \gamma_0 .$$

$$\psi^\dagger(p)\,(\gamma_0 \gamma_\mu \gamma_0 p^\mu - m) = 0 \quad \text{或} \quad \psi^\dagger(p)\gamma_0(\gamma_\mu p^\mu - m) = 0 .$$

即 $\bar{\psi}(\not{p} - m) = 0$. $\quad \bar{\psi} = \psi^\dagger \gamma_0$. Dirac 共轭

正则量子化

K-G eqn　spin-0

Dirac eqn　spin-$\frac{1}{2}$

Maxwell eqn　spin-1

场量子化　$q_i(t) \longrightarrow \phi(\vec{x},t)$

$$\phi_i(t) = \frac{1}{\Delta V_i}\int_{\Delta V_i} d^3x\, \phi(\vec{x},t).$$

$$L = \int d^3x\, \mathcal{L} = \sum_i \Delta V_i\, \mathcal{L}_i(\phi_i, \partial_0\phi_i).$$

$$p_i(t) = \frac{\partial L}{\partial(\partial_0\phi_i(t))} = \Delta V_i \frac{\partial \mathcal{L}_i}{\partial(\partial_0\phi_i(t))} = \Delta V_i\, \pi_i(t).$$

$$H = \sum_i p_i(t)\,\partial_0\phi_i(t) - L = \sum_i \Delta V_i \left(\pi_i \partial_0\phi_i(t) - \mathcal{L}_i\right) \longrightarrow \int d^3x\, \mathcal{H}.$$

$$\mathcal{H} = \pi_i \partial_0\phi_i(t) - \mathcal{L}.$$

量子化　$[\phi_i(t), p_j(t)] = i\delta_{ij}$.　$[\phi_i(t), \phi_j(t)] = 0$.　$[p_i(t), p_j(t)] = 0$.

$$[\phi_i(t), \pi_j(t)] = i\frac{\delta_{ij}}{\Delta V_i}.$$

$\Delta V_i \to 0$

连续情形　$[\phi(\vec{x},t), \pi(\vec{x}',t)] = i\delta^3(\vec{x}-\vec{x}')$.　$[\phi(\vec{x},t), \phi(\vec{x}',t)] = 0$.　$[\pi(\vec{x},t), \pi(\vec{x}',t)] = 0$.

标量场　ϕ.　$\mathcal{L} = \frac{1}{2}(\partial^\mu\phi)(\partial_\mu\phi) - \frac{\mu^2}{2}\phi^2$.

E-L eqn:　$\partial^\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial^\mu\phi)}\right) - \frac{\partial\mathcal{L}}{\partial\phi} = 0$.　$\Rightarrow$ K-G eqn　$(\partial^\mu\partial_\mu + \mu^2)\phi = 0$.

$$\pi(\vec{x},t) = \frac{\partial\mathcal{L}}{\partial(\partial_0\phi)} = \partial_0\phi.$$

$$\mathcal{H} = \pi\partial_0\phi - \mathcal{L} = \frac{1}{2}\left[(\partial^0\phi)^2 + (\nabla\phi)^2\right] + \frac{1}{2}\mu^2\phi^2.$$

$$[H, \phi(\vec{x},t)] = \int d^3y \, [\mathcal{H}, \phi(\vec{x},t)] = -i\partial_0\phi.$$

模态展开 mode expansion

K-G eqn 通解 $\exp(ik_0 t - \vec{k}\cdot\vec{x})$. $k_0^2 = \vec{k}^2 + \mu^2$.

$$\phi(\vec{x},t) = \int \frac{d^3k}{\sqrt{(2\pi)^3 2\omega_k}} \left[a(\vec{k})\, e^{-ik\cdot x} + a^\dagger(\vec{k})\, e^{ik\cdot x}\right]. \qquad k = \sqrt{\vec{k}^2+\mu^2}.$$

$$\partial_0\phi(\vec{x},t) = \int \frac{d^3k}{\sqrt{(2\pi)^3 2\omega_k}} (-ik_0) \left[a(\vec{k})\, e^{-ik\cdot x} - a^\dagger(\vec{k})\, e^{ik\cdot x}\right]. \qquad k_0 = \sqrt{\vec{k}^2+\mu^2} = \omega_k.$$

$\cdot\, e^{ik'x}$ $\Rightarrow \int e^{ik'x} d^3x \left(\partial_0\phi - ik_0'\phi\right) = \int \frac{d^3k}{\sqrt{(2\pi)^3 2\omega_k}} (-2ik_0)(2\pi)^3 \delta^3(k-k')\, a(k)$

$$\Rightarrow a(k) = i\int d^3x \frac{1}{\sqrt{(2\pi)^3 2\omega_k}} \Big[\underbrace{e^{ik\cdot x}\partial_0\phi - (\partial_0 e^{ik\cdot x})}_{e^{ik\cdot x}\overleftrightarrow{\partial_0}\phi(x)}\Big].$$

$$a^\dagger(k) = -i\int d^3x \frac{e^{-ik\cdot x}}{\sqrt{(2\pi)^3 2\omega_k}} \overleftrightarrow{\partial_0}\phi(x).$$

$$[a(\vec{k}), a^\dagger(\vec{k}')] = \int \frac{d^3x\, d^3x'\, e^{ikx} e^{-ik'x'}}{\sqrt{(2\pi)^3 2\omega_k (2\pi)^3 2\omega_{k'}}} \left[\partial_0\phi(x) - ik_0\phi(x),\ \partial_0\phi(x') - ik_0'\phi(x')\right]$$

$$= \int \frac{d^3x\, d^3x'\, e^{ikx} e^{-ik'x'}}{\sqrt{(2\pi)^3 2\omega_k (2\pi)^3 2\omega_{k'}}} \left(ik_0'(-i) - ik_0 i\right) \delta^3(x-x')$$

$$= \delta^3(\vec{k}-\vec{k}').$$

$$H = \int d^3x\, \mathcal{H} = \frac{1}{2}\int d^3x \left[\dot{\phi}^2 + |\nabla\phi|^2 + \mu^2\phi^2\right]$$

$$= \frac{1}{2}\int d^3k\, \omega_k \left[a^\dagger(\vec{k})\, a(\vec{k}) + a(\vec{k})\, a^\dagger(\vec{k})\right] = \int d^3k\, \underset{\smile}{\mathcal{H}_k}.$$

$$\frac{\omega_k}{2}\left[a^\dagger(k)\, a(k) + a(k)\, a^\dagger(k)\right]$$

$$P_i = \int d^3x\, T_{0i} = \int d^3x \frac{\partial \mathcal{L}}{\partial(\partial_0\phi)} = \int d^3x\, \pi \partial_i \phi.$$

$$[P_i, \phi(\vec{x},t)] = \int d^3y \left[\pi(\vec{y},t)\partial_i\phi(\vec{y},t),\ \phi(\vec{x},t)\right]$$

$$= \int d^3y\ \partial_i\phi(\vec{y},t)(-i)\delta^3(\vec{x}-\vec{y}) = -i\partial_i\phi(\vec{x},t).$$

$$\vec{P} = \frac{1}{2}\int d^3k\, \vec{k}\left[a^\dagger(k)a(k) + a(k)a^\dagger(k)\right] = \int d^3k\, \underset{\sim}{\vec{p}_k}.$$

$$\frac{\vec{k}}{2}\left[a^\dagger(k)a(k) + a(k)a^\dagger(k)\right]$$

关于 $\delta^3(0)$. $\quad a(k)a^\dagger(k) = a^\dagger(k)a(k) + \delta^3(0).$

$$\delta^3(\vec{k}) = \int \frac{d^3x}{(2\pi)^3} e^{i\vec{k}\cdot\vec{x}}. \qquad k\to 0.\quad \delta^3(0) = \frac{1}{(2\pi)^3}\int d^3x = \frac{V}{(2\pi)^3}.$$

$$H = \int d^3k\, \omega_k\left[a^\dagger(k)a(k) + \frac{V}{2\cdot(2\pi)^3}\right] = \int d^3k\ \omega_k\, a^\dagger(k)a(k) + V\int\frac{d^3k}{(2\pi)^3}\cdot\frac{\omega_k}{2}.$$

引入正规乘积 (normal ordering)

$$\langle 0| :f(a,a^\dagger): |0\rangle = 0.$$

$$H = \frac{1}{2}\int d^3k\, \omega_k : \left[a^\dagger(k)a(k) + a(k)a^\dagger(k)\right]: = \int d^3k\, \omega_k\, a^\dagger(k)a(k).$$

$$\vec{P} = \cdots = \int d^3k\, \vec{p}_k\, a^\dagger(k)a(k).$$

在壳 on shell

$$|\vec{k}\rangle = \sqrt{(2\pi)^3 2\omega_k}\, a^\dagger(k)|0\rangle. \qquad H|\vec{k}\rangle = \omega_k|\vec{k}\rangle.\quad \vec{P}|\vec{k}\rangle = \vec{k}|\vec{k}\rangle.\quad \omega_k = \sqrt{\vec{k}^2 + \mu^2}.$$

$$|\vec{k}_1,\vec{k}_2\rangle = \sqrt{(2\pi)^3 2\omega_{k_1}}\sqrt{(2\pi)^3 2\omega_{k_2}}\, a^\dagger(\vec{k}_1)a^\dagger(\vec{k}_2)|0\rangle. \quad \text{双粒子态}$$

$$|\vec{k}_1,\vec{k}_2,\cdots\vec{k}_n\rangle = \sqrt{(2\pi)^3 2\omega_{k_1}}\cdots\sqrt{(2\pi)^3 2\omega_{k_n}}\, a^\dagger(\vec{k}_1)\cdots a^\dagger(k_n)|0\rangle. \quad \text{多}$$

$$\phi(x)|0\rangle = \int\frac{d^3k}{\sqrt{(2\pi)^3 2\omega_k}} a^\dagger(\vec{k})e^{ik\cdot x}|0\rangle = \int\frac{d^3k}{(2\pi)^3 2\omega_k} e^{ikx}|k\rangle.$$

Bose 统计

$$|\Phi\rangle = \left[C_0|0\rangle + \sum_{n=1}^{\infty} \int d^3k_1 \cdots d^3k_n \, \underbrace{C_n(k_1, k_2 \cdots k_n)}_{\text{动量空间波函数}} a^\dagger(\vec{k}_1) \cdots a^\dagger(k_n) |0\rangle \right]$$

$$[a^\dagger(k_i), a^\dagger(k_j)] = 0. \qquad C_n(k_1, \cdots k_i, \cdots k_j \cdots, k_n) = C_n(k_1, \cdots, k_j, \cdots, k_i, \cdots k_n).$$

概率幅 $y \to x$

$$\langle 0| \phi(x)\phi(y) |0\rangle = \int \frac{d^3k \, d^3k'}{(2\pi)^3 \sqrt{2\omega_k 2\omega_{k'}}} \langle 0| a(k) e^{-ikx} \, a^\dagger(k') e^{ik'y} |0\rangle$$

$$= \int \frac{d^3k \, d^3k'}{(2\pi)^3 \, 2\omega_k} \delta^3(k-k') \, e^{-ikx+ik'y} = \int \frac{d^3k}{(2\pi)^3 \, 2\omega_k} e^{-ik(x-y)}$$

$$i\Delta(x-y) = [\phi(x), \phi(y)] = \int \frac{d^3k}{(2\pi)^3 2\omega_k} \left[e^{-ik\cdot(x-y)} - e^{ik(x+y)} \right].$$

类空间隔 $(x-y)^2 < 0$. 存在参考系 $x-y = (0, \vec{x}-\vec{y})$.

$$\langle 0| [\phi(x), \phi(y)] |0\rangle = \int \frac{d^3k}{(2\pi)^3 2\omega_k} \left[e^{i\vec{k}\cdot(\vec{x}-\vec{y})} - e^{-i\vec{k}\cdot(\vec{x}-\vec{y})} \right] = 0.$$

设有两个标量场

$$\mathcal{L} = \frac{1}{2}(\partial^\mu \phi_1)(\partial_\mu \phi_1) - \frac{\mu_1^2}{2}\phi_1^2 + \frac{1}{2}(\partial^\mu \phi_2)(\partial_\mu \phi_2) - \frac{\mu_2^2}{2}\phi_2^2.$$

$$\partial^\mu \frac{\partial \mathcal{L}}{\partial(\partial^\mu \phi_i)} - \frac{\partial \mathcal{L}}{\partial \phi_i} = 0.$$

K-G eqn

$$\partial^\mu \partial_\mu \phi_1 + \mu_1^2 \phi_1 = 0$$

$$\partial^\mu \partial_\mu \phi_2 + \mu_2^2 \phi_2 = 0$$

$\mu_1^2 = \mu_2^2$ 时.

$$\mathcal{L} = \frac{1}{2}\left[(\partial^\mu\phi_1)(\partial_\mu\phi_1) + (\partial^\mu\phi_2)(\partial_\mu\phi_2)\right] - \frac{\mu^2}{2}(\phi_1^2+\phi_2^2)$$

在 ϕ_1. ϕ_2 空间转动下不变

$$\phi_1 \to \phi_1' = \cos\theta\,\phi_1 + \sin\theta\,\phi_2.$$

$$\phi_2 \to \phi_2' = -\sin\theta\,\phi_1 + \cos\theta\,\phi_2.$$

O(2) 对称性

考虑无穷小转动

$$\delta\phi_1 = \phi_1' - \phi_1 = \theta\phi_2. \qquad \delta\phi_2 = \phi_2' - \phi_2 = -\theta\phi_1.$$

守恒流 $$j_\mu = \frac{\partial\mathcal{L}}{\partial(\partial^\mu\phi_i)}\delta\phi_i = \left[(\partial_\mu\phi_1)\phi_2 - (\partial_\mu\phi_2)\phi_1\right]\theta.$$

引入复标量场 $$\phi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2).$$

U(1) 对称性

$$\Rightarrow \quad \mathcal{L} = (\partial^\mu\phi^\dagger)(\partial_\mu\phi) - \mu^2\phi^\dagger\phi. \qquad \phi \to \phi' = e^{-i\theta}\phi.$$

$$j_\mu = \frac{\partial\mathcal{L}}{\partial(\partial^\mu\phi)}\delta\phi + \frac{\partial\mathcal{L}}{\partial(\partial^\mu\phi^\dagger)}\delta\phi^\dagger = i\left[(\partial^\mu\phi^\dagger)\phi - (\partial^\mu\phi)\phi^\dagger\right]\theta.$$

模态展开

$$\phi(\vec{x},t) = \int\frac{d^3k}{\sqrt{(2\pi)^3 2\omega_k}}\left[a(\vec{k})e^{-ik\cdot x} + b^\dagger(\vec{k})e^{ik\cdot x}\right]. \qquad k_0 = \sqrt{\vec{k}^2+\mu^2}.$$

$$a(\vec{k}) = \frac{1}{\sqrt{2}}\left[a_1(\vec{k}) + i a_2(\vec{k})\right]. \qquad b(\vec{k}) = \frac{1}{\sqrt{2}}\left[a_1(\vec{k}) - i a_2(\vec{k})\right]$$

$$\left[a(\vec{k}), a^\dagger(\vec{k}')\right] = \delta^3(\vec{k}-\vec{k}'). \qquad \left[b(\vec{k}), b^\dagger(\vec{k}')\right] = \delta^3(\vec{k}-\vec{k}'). \qquad \left[a(\vec{k}), b(\vec{k}')\right] = 0.$$

$$H = \int d^3k\left[a^\dagger(k)a(k) + b^\dagger(k)b(k)\right].$$

$$\vec{P}_k = \int d^3k \; \vec{p}_k \left[a^\dagger(k)\, a(k) + b^\dagger(k)\, b(k)\right].$$

$$Q = \int d^3x \; j_0 = i\int d^3x \left[(\partial_0 \phi^\dagger)\phi - (\partial_0\phi)\phi^\dagger\right] = \int d^3k \left[a^\dagger(k)\, a(k) - b^\dagger(k)\, b(k)\right].$$

Dirac 方程

$$(i\gamma^\mu \partial_\mu - m)\psi = 0.$$

$$\psi(x) = e^{-ip\cdot x} w(p).$$

$$\Rightarrow (\not{p} - m) w(p) = 0. \quad \not{p} = \gamma^\mu p_\mu = \gamma^0 p_0 - \vec{\gamma}\cdot\vec{p}.$$

$$(p_0 - \vec{\alpha}\cdot\vec{p} - \beta m) w(p) = 0. \quad \vec{\alpha} = \gamma_0\vec{\gamma}. \quad \beta = \gamma_0.$$

$$\Rightarrow H w(p) = p_0 w(p). \quad H = \vec{\alpha}\cdot\vec{p} + \beta m.$$

旋量 spinor

$$u(p,s) = N\begin{pmatrix} 1 \\ \frac{\vec{\sigma}\cdot\vec{p}}{E+m}\end{pmatrix}\chi_s. \quad v(p,s) = N\begin{pmatrix} \frac{\vec{\sigma}\cdot\vec{p}}{E+m} \\ 1\end{pmatrix}\chi_s.$$

$$\psi = e^{-iEt} e^{i\vec{p}\cdot\vec{x}} N\begin{pmatrix} 1 \\ \frac{\vec{\sigma}\cdot\vec{p}}{E+m}\end{pmatrix}\chi_s. \quad \psi = e^{iEt} e^{-i\vec{p}\cdot\vec{x}} N\begin{pmatrix} \frac{\vec{\sigma}\cdot\vec{p}}{E+m} \\ 1\end{pmatrix}\chi_s.$$

$$u^\dagger(p,s')\, v(-p,s) = 0.$$

$$\psi(\vec{x},t) = \sum_s \int \frac{d^3p}{\sqrt{(2\pi)^3 2E_p}}\left[b(p,s)\, u(p,s)\, e^{-ip\cdot x} + d^\dagger(p,s)\, v(p,s)\, e^{ip\cdot x}\right].$$

$$\xrightarrow{\cdot u^\dagger(p',s')\, e^{-p'x}}$$

$$\int u^\dagger(p',s)\, e^{-p'x}\psi(\vec{x},t)\, d^3x = \sum_s \int \frac{d^3p}{\sqrt{(2\pi)^3 2E_p}}\left[b(p,s)\, u^\dagger(p',s')\, u(p,s)\, \delta^3(p-p') + d^\dagger(p,s)\, \underbrace{u^\dagger(p',s')\, v(p,s)}_{=0}\, \delta^3(p+p')\right]$$

$$\Rightarrow b(p,s) = \int \frac{d^3x\, e^{ip\cdot x}}{\sqrt{(2\pi)^3 2E_p}}\, u^\dagger(p,s)\, \psi(\vec{x},t).$$

$$d^\dagger(p,s) = \int \frac{d^3x\, e^{ip\cdot x}}{\sqrt{(2\pi)^3 2E_p}}\, v(p,s)\, \psi(\vec{x},t).$$

$$H = \vec{\alpha}\cdot\vec{p} + \beta m, \quad \vec{L} = \vec{r}\times\vec{p} \text{ 不守恒}.$$

$$\vec{S} = \frac{i}{4}\vec{\alpha}\times\vec{\alpha}. \quad [\vec{L}+\vec{S}, H] = 0.$$

螺旋度 Helicity $\lambda = \vec{J}\cdot\hat{p} = \vec{S}\cdot\hat{p}$. $\hat{p} = \frac{\vec{p}}{|\vec{p}|}$.

$$u(p,\pm) = \sqrt{2m}\begin{pmatrix} 1 \\ \frac{\vec{\sigma}\cdot\vec{p}}{E+m}\end{pmatrix}\chi_\pm . \qquad (\vec{\sigma}\cdot\hat{p})\chi_\pm = \pm\chi_\pm .$$

$$\lambda\, u(p,\pm) = (\vec{S}\cdot\hat{p})\, u(p,\pm) = \begin{pmatrix} \frac{\vec{\sigma}\cdot\hat{p}}{2} & \\ & \frac{\vec{\sigma}\cdot\hat{p}}{2}\end{pmatrix} u(p,\pm) = \pm\frac{1}{2}u(p,\pm).$$

手征性 chirality

$$\psi_R \equiv \frac{1}{2}(1+\gamma_5)\psi . \qquad \psi_L \equiv \frac{1}{2}(1-\gamma_5)\psi .$$

$$\gamma_5\psi_R = \psi_R . \qquad \gamma_5\psi_L = -\psi_L . \qquad \gamma_5 = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}.$$

费米场

$$(i\gamma^\mu\partial_\mu - m)\psi = 0 . \qquad \bar{\psi}(-i\gamma^\mu\overleftarrow{\partial_\mu} - m) = 0 .$$

$$\mathcal{L} = \bar{\psi}_\alpha (i\gamma^\mu\partial_\mu - m)_{\alpha\beta}\psi_\beta .$$

$$\frac{\partial\mathcal{L}}{\partial\psi^\dagger_\gamma} = (\gamma^0)_{\gamma\alpha}(i\gamma^\mu\partial_\mu - m)_{\alpha\beta}\psi_\beta . \qquad \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi^\dagger_\gamma)} = 0 .$$

$$\pi_\alpha = \frac{\partial\mathcal{L}}{\partial(\partial_0\psi_\alpha)} = i\psi^\dagger_\alpha .$$

反对易关系

$$\{\pi_\alpha(\vec{x},t),\ \psi_\beta(\vec{y},t)\} = i\delta^3(\vec{x}-\vec{y})\,\delta_{\alpha\beta} .$$

$$\{\psi_\alpha(\vec{x},t),\ \psi_\beta(\vec{y},t)\} = 0 . \qquad \{\pi_\alpha(\vec{x},t),\ \pi_\beta(\vec{y},t)\} = 0 .$$

$$\mathcal{H} = \sum_\alpha \pi_\alpha\partial_0\psi_\alpha - \mathcal{L} = i\psi^\dagger\partial_0\psi - \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi = \bar{\psi}(i\vec{\gamma}\cdot\vec{\nabla} + m)\psi .$$

模态展开

$$\psi(\vec{x},t) = \sum_s \int \frac{d^3p}{\sqrt{(2\pi)^3 2E_p}}\left[b(p,s)\,u(p,s)\,e^{-ip\cdot x} + d^\dagger(p,s)\,v(p,s)\,e^{ip\cdot x}\right] .$$

$$b(p,s) = \int \frac{d^3x\, e^{ip\cdot x}}{\sqrt{(2\pi)^3 2E_p}}\, u^\dagger(p,s)\,\psi(\vec{x},t) . \qquad b^\dagger(p,s) = \int \frac{d^3x\, e^{-ip\cdot x}}{\sqrt{(2\pi)^3 2E_p}}\, \psi^\dagger(\vec{x},t)\,u(p,s) .$$

$$d^{\dagger}(p,s) = \int \frac{d^3x\, e^{-ip\cdot x}}{\sqrt{(2\pi)^3 2E_p}} v^{\dagger}(p,s)\, \psi(\vec{x},t) \qquad d(p,s) = \int \frac{d^3x\, e^{ip\cdot x}}{\sqrt{(2\pi)^3 2E_p}} \psi^{\dagger}(\vec{x},t)\, v(p,s).$$

$$\{b(p,s),\, b^{\dagger}(p',s')\} = \int \frac{d^3x\, e^{ip\cdot x}}{\sqrt{(2\pi)^3 2E_p}} \int \frac{d^3x'\, e^{-ip'\cdot x'}}{\sqrt{(2\pi)^3 2E_{p'}}} \{u^{\dagger}(p,s)\psi(\vec{x},t),\, \psi^{\dagger}(\vec{x}',t)u(p',s')\}$$

$$= \int \frac{d^3x\, e^{ip\cdot x}}{\sqrt{(2\pi)^3 2E_p}} \int \frac{d^3x\, e^{-ip'\cdot x'}}{\sqrt{(2\pi)^3 2E_{p'}}} (2\pi)^3\, \delta^3(x-x')\, u^{\dagger}(p,s)\, u(p',s')$$

$$= \delta_{ss'}\, \delta^3(\vec{p}-\vec{p}').$$

$$\{d(p,s),\, d^{\dagger}(p',s')\} = \delta_{ss'}\, \delta^3(\vec{p}-\vec{p}').$$

$$H = \int d^3x\, \mathcal{H} = \int \bar{\psi}(i\vec{\gamma}\cdot\nabla + m)\psi\, d^3x = i\int \psi^{\dagger}\partial_0\psi\, d^3x$$

$$= \sum_s \int d^3p\, E_p\left[b^{\dagger}(p,s)\, b(p,s) - d(p,s)\, d^{\dagger}(p,s)\right] = \sum_s \int d^3p\, \mathcal{H}_{ps}.$$

$$\vec{P} = \sum_s \int d^3p\, \vec{P}_{ps}. \qquad \vec{P}_{ps} = \vec{p}\left[b^{\dagger}(p,s)\, b(p,s) - d(p,s)\, d^{\dagger}(p,s)\right].$$

$$[H,\, b^{\dagger}(p,s)] = \sum_{s'} \int d^3p' \left[b^{\dagger}(p',s')\, b(p',s'),\, b^{\dagger}(p,s)\right] E_{p'}$$

$$= \int d^3p'\, E_{p'} \sum_{s'} \left[b^{\dagger}(p',s')\{b(p',s'),\, b^{\dagger}(p,s)\} - \{b^{\dagger}(p',s'),\, b^{\dagger}(p,s)\}\, b(p',s')\right]$$

$$= b^{\dagger}(p,s)\, E_p. \quad \text{产生算符}$$

$$[\vec{P},\, b^{\dagger}(p,s)] = \vec{p}\, b^{\dagger}(p,s).$$

对称性

$$\mathcal{L} = \bar{\psi}(i\gamma^{\mu}\partial_{\mu} - m)\psi.$$

$$\psi(x) \to e^{i\alpha}\psi(x). \qquad \psi^{\dagger}(x) \to \psi^{\dagger}(x)\, e^{-i\alpha} \quad \text{下不变}.$$

$$\delta\psi = i\alpha\psi. \qquad \delta\psi^{\dagger} = -i\alpha\psi^{\dagger}.$$

$$j_\mu = \frac{\partial \mathcal{L}}{\partial(\partial^\mu \psi_\alpha)} \delta\psi_\alpha = -\alpha \bar{\psi} \gamma_\mu \psi .$$

守恒荷 $Q = \int j_0(x)\, d^3x = \int d^3x \, :\psi^\dagger(\vec{x},t)\psi(\vec{x},t):$

$$= \int d^3x \sum_{ss'} \int \frac{d^3p}{\sqrt{(2\pi)^3 2E_p}} :\left[b^\dagger(p',s') u^\dagger(p',s') e^{ip'\cdot x} + d(p',s') v^\dagger(p',s') e^{-ip'\cdot x} \right]$$

$$\times \int \frac{d^3p}{\sqrt{(2\pi)^3 2E_p}} \left[b(p,s) u(p,s) e^{-ip\cdot x} + d^\dagger(p,s) v^\dagger(p',s') e^{-ip'\cdot x} \right]:$$

$$= \sum_s \int d^3p \, :\left[b^\dagger(p,s) b(p,s) + d(p,s) d^\dagger(p,s) \right]:$$

$$= \sum_s \int d^3p \, :\left[b^\dagger(p,s) b(p,s) - d^\dagger(p,s) d(p,s) \right]: .$$ 带相反的号

电磁场

$$\nabla\cdot\vec{E}=\frac{\rho}{\varepsilon_0}.\qquad \nabla\cdot\vec{B}=0$$

$$\nabla\times\vec{E}+\frac{\partial\vec{B}}{\partial t}=0.\qquad \frac{1}{\mu_0}\nabla\times\vec{B}=\varepsilon_0\frac{\partial\vec{E}}{\partial t}+\vec{J}.$$

$$\Rightarrow\ \nabla\cdot\frac{\partial\vec{E}}{\partial t}=\frac{1}{\varepsilon_0}\frac{\partial\rho}{\partial t}.\qquad \frac{1}{\mu_0}\nabla\cdot\nabla\times\vec{B}=\varepsilon_0\frac{\partial\nabla\cdot\vec{E}}{\partial t}+\nabla\cdot\vec{J}.\qquad \frac{\partial\rho}{\partial t}+\nabla\cdot\vec{J}=0.$$

写为闵氏时空中的张量

$$J^\mu=(\rho,\vec{J}).\qquad\Rightarrow\qquad \frac{\partial\rho}{\partial t}+\nabla\cdot\vec{J}=\partial_\mu J^\mu=0.$$

用 $\vec{A}$. ϕ 表示 Maxwell 方程组

$$\vec{B}=\nabla\times\vec{A}.\qquad \vec{E}=-\nabla\phi-\frac{\partial\vec{A}}{\partial t}.$$

$$\Rightarrow\ -\nabla^2\phi-\frac{\partial}{\partial t}\nabla\cdot\vec{A}=\frac{\rho}{\varepsilon_0}.\qquad \frac{1}{\mu_0}\left[\nabla(\nabla\cdot\vec{A})-\nabla^2\vec{A}\right]=-\varepsilon_0\frac{\partial}{\partial t}\left(\nabla\phi-\frac{\partial\vec{A}}{\partial t}\right)+\vec{J}.$$

如选取规范

$$\nabla\cdot\vec{A}+\frac{\partial\phi}{\partial t}=0.$$

$$\Rightarrow\ \left(\frac{\partial^2}{\partial t^2}-\nabla^2\right)\phi=\frac{\rho}{\varepsilon_0}.\qquad \left(\frac{\partial^2}{\partial t^2}-\nabla^2\right)\vec{A}=\mu_0\vec{J}.$$

有四矢量 $A^\mu=(\phi,\vec{A})$. $\qquad\left(\frac{\partial^2}{\partial t^2}-\nabla^2\right)A^\mu=J^\mu.$

$$\vec{E}=-\nabla A^0-\partial_0\vec{A}.\qquad E^i=\partial^iA^0-\partial^0A^i.$$

构造 $\quad F^{\mu\nu}=\partial^\mu A^\nu-\partial^\nu A^\mu.$

$$E^i=-F^{0i}.$$

$$F^{ij}=\partial^iA^j-\partial^jA^i=\varepsilon^{ijk}B^k.$$

Maxwell 方程组写为

$$\partial_\mu F^{\mu\nu}=J^\nu.\qquad \partial_\mu\tilde{F}^{\mu\nu}=0.\qquad \tilde{F}^{\mu\nu}=\varepsilon^{\mu\nu\alpha\beta}F_{\alpha\beta}.$$

Lorentz 变换

$$F^{\mu\nu}\longrightarrow F'^{\mu\nu}=\Lambda^\mu_\alpha\Lambda^\nu_\beta F^{\alpha\beta}.$$

$F^{\mu\nu}$ 在 $A^\mu \to A^\mu + \partial^\mu \alpha$. $\alpha=\alpha(x)$ 下不变.

取 $\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} = \frac{1}{2}(\vec{E}^2-\vec{B}^2)$.

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} = -(\partial^\mu A^\nu - \partial^\nu A^\mu), \quad \frac{\partial \mathcal{L}}{\partial A_\nu} = 0.$$

E-L eqn $\Rightarrow$ Maxwell eqn

$$\partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)}\right) = \frac{\partial \mathcal{L}}{\partial A_\nu} \Rightarrow \partial_\mu(\partial^\mu A^\nu - \partial^\nu A^\mu) = \partial_\mu F^{\mu\nu} = 0.$$

$$\pi_0 = \frac{\partial L}{\partial(\partial_0 A_0)} = 0. \quad \pi^i(x) = \frac{\partial L}{\partial(\partial_0 A_i)} = -F^{0i} = E^i.$$

A_0 并无共轭动量. 不是真正自由度.

$$\mathcal{H} = \pi^k \dot{A}_k - \mathcal{L} = (\partial^k A^0 - \partial^0 A^k)\partial_0 A_k + \frac{1}{2}\partial_\mu A_\nu(\partial^\mu A^\nu - \partial^\nu A^\mu) = \frac{1}{2}(\vec{E}^2+\vec{B}^2) + (\vec{E}\cdot\vec{\nabla})A_0.$$

因 $\vec{\nabla}\cdot\vec{E}=0$. 有

$$H = \int d^3x\, \mathcal{H} = \frac{1}{2}\int d^3x(\vec{E}^2+\vec{B}^2).$$

量子化 $\quad [\pi^i(\vec{x},t), A^j(\vec{y},t)] = -i\delta_{ij}\delta^3(\vec{x}-\vec{y})$. $\times$

与 $\nabla\cdot\vec{E}=0$ 矛盾. $[\nabla\cdot\vec{E}(x,t), A_j(x,t)] = -i\partial_j\delta^3(x-y) \neq 0$.

$$\partial_j \delta^3(\vec{x}-\vec{y}) = i\int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot(\vec{x}-\vec{y})} k_j.$$

$$\delta_{ij}^{tr}\delta^3(\vec{x}-\vec{y}) = \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot(\vec{x}-\vec{y})}\left(\delta_{ij} - \frac{k_i k_j}{k^2}\right).$$

则 $\partial_i \delta_{ij}^{tr}\delta^3(\vec{x}-\vec{y}) = i\int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot(\vec{x}-\vec{y})} k_i\left(\delta_{ij} - \frac{k_i k_j}{k^2}\right) = 0.$

对易子 $\quad [E^i(x,t), A_j(y,t)] = -i\delta_{ij}^{tr}(\vec{x}-\vec{y})$

且 $\quad [E^i(x,t), \nabla\cdot\vec{A}(y,t)] = 0.$

选取辐射规范 radiation gauge

$$A_0 = 0. \quad \nabla\cdot\vec{A} = 0.$$

$$\pi^i = \partial^i A^0 - \partial^0 A^i = -\partial^0 A^i.$$

$$\left[\partial_0 A^i(\vec{x},t),\ A^j(\vec{y},t)\right] = i\,\delta^{tr}_{ij}(\vec{x}-\vec{y}).$$

$$\nabla\times\vec{B} = 0. \qquad \nabla\times\vec{E} + \frac{\partial\vec{B}}{\partial t} = 0$$

$$\nabla\cdot\vec{E} = 0. \qquad \frac{1}{\mu_0}\nabla\times\vec{B} - \varepsilon_0\frac{\partial\vec{E}}{\partial t} = 0.$$

$$\vec{B} = \nabla\times\vec{A}. \qquad \vec{E} = -\nabla\phi - \frac{\partial\vec{A}}{\partial t}.$$

$$\Rightarrow\ F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu. \qquad F^{0i} = \partial^0 A^i - \partial^i A^0 = -E^i.$$

$$\partial_\nu F^{\mu\nu} = 0. \qquad F^{ij} = \partial^i A^j - \partial^j A^i = -\varepsilon_{ijk}B_k.$$

模态展开

$$\partial_\nu(\partial^\nu A^\mu - \partial^\mu A^\nu) = \Box A^\mu - \partial^\mu(\partial_\nu A^\nu) = 0.$$

辐射规范 $\partial_\nu A^\nu = 0.$ $\quad \Box\vec{A} = 0.$ 零质量 K-G eqn

$$A_i(\vec{x},t) = \int\frac{d^3k}{\sqrt{(2\pi)^3 2\omega}}\sum_\lambda \varepsilon_i(k,\lambda)\left[a(k,\lambda)e^{-ikx} + a^\dagger(k,\lambda)e^{ikx}\right]. \qquad \omega = |k_0| = |\vec{k}|.$$

$\nabla\cdot\vec{A} = 0$ 且 $\vec{k}\cdot\vec{\varepsilon}(k,\lambda) = 0.$ $\quad \lambda = 1,2.$

令 $\vec{\varepsilon}(k,\lambda)\cdot\vec{\varepsilon}(k,\lambda') = \delta_{\lambda\lambda'}.$ $\quad \vec{\varepsilon}(-k,1) = -\vec{\varepsilon}(k,1).$ $\quad \vec{\varepsilon}(-k,2) =$

$$a(k,\lambda) = i\int\frac{d^3k}{\sqrt{(2\pi)^3 2\omega}}\left[e^{ik\cdot x}\overleftrightarrow{\partial_0}\,\vec{\varepsilon}(k,\lambda)\cdot\vec{A}(x)\right].$$

$$a^\dagger(k,\lambda) = -i\int\frac{d^3k}{\sqrt{(2\pi)^3 2\omega}}\left[e^{-ik\cdot x}\overleftrightarrow{\partial_0}\,\vec{\varepsilon}(k,\lambda)\cdot\vec{A}(x)\right].$$

$$\left[a(k,\lambda),\ a^\dagger(k',\lambda')\right] = \delta_{\lambda\lambda'}\delta^3(\vec{k}-\vec{k}'). \qquad \left[a(k,\lambda),\ a(k',\lambda')\right] = 0. \qquad \left[a^\dagger(k,\lambda),\ a^\dagger(k',\lambda')\right] = 0.$$

$$H = \frac{1}{2}\int d^3x\, :(E^2+B^2): = \int d^3k\,\omega\sum_\lambda a^\dagger(k,\lambda)a(k,\lambda)$$

$$\vec{P} = \int d^3x\, :E\times B: = \int d^3k\,\vec{k}\sum_\lambda a^\dagger(k,\lambda)a(k,\lambda)$$

U(1) 局域对称性

$$m\frac{d^2\vec{x}}{dt^2} = e(\vec{E}+\vec{v}\times\vec{B}).$$

$$L = \frac{1}{2}m\vec{v}^2 + e\vec{A}\cdot\vec{v} - eA_0.$$

$$\frac{\partial L}{\partial v_i} = mv_i + eA_i. \qquad \frac{\partial L}{\partial x_i} = e\frac{\partial A_j}{\partial x_i}v_j - e\frac{\partial A_0}{\partial x_i}.$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial v_i}\right) = m\frac{dv_i}{dt} + e\frac{\partial A_i}{\partial x_j}\frac{dx_j}{dt} + e\frac{\partial A_i}{\partial t}.$$

E-L eqn 给出

$$m\frac{dv_i}{dt} + e\frac{\partial A_i}{\partial x_j}\frac{dx_j}{dt} + e\frac{\partial A_i}{\partial t} = e\frac{\partial A_j}{\partial x_i}v_j - e\frac{\partial A_0}{\partial x_i}.$$

$$(\vec{v}\times\vec{B})_i = \varepsilon_{ijk}v_jB_k = \varepsilon_{ijk}\varepsilon_{klm}\partial_l A_m$$

$$= v_j(\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})\partial_l A_m = v_j(\partial_i A_j - \partial_j A_i).$$

$$\Rightarrow m\frac{dv_i}{dt} = -e\frac{\partial A_i}{\partial x_j}v_j - e\frac{\partial A_i}{\partial t} + e\frac{\partial A_j}{\partial x_j}v_j - e\frac{\partial A_0}{\partial x_i}.$$

即 $m\frac{dv_i}{dt} = e(\partial_i A_j - \partial_j A_i)v_j + e(-\partial_i A_0 - \partial_0 A_i) = e(\vec{E}+\vec{v}\times\vec{B}).$

$$p_i = \frac{\partial L}{\partial v_i} = mv_i + eA_i. \quad\Rightarrow\quad v_i = \frac{1}{m}(p_i - eA_i).$$

$$H = p_iv_i - L = p_iv_i - \frac{1}{2}m\vec{v}^2 - e\vec{A}\cdot\vec{v} + eA_0$$

$$= \frac{1}{2m}(\vec{p}-e\vec{A})^2 + eA_0.$$

$$\vec{p}\to\vec{p}-e\vec{A}. \qquad H\to H-eA_0.$$

$p^\mu \longrightarrow p^\mu - eA^\mu.$ 最小替换原则

Schrödinger 方程

$$\left[-\frac{1}{2m}(\nabla - ie\vec{A})^2 + eA_0\right]\psi = i\frac{\partial\psi}{\partial t}.$$

对 $A^\mu \to A^\mu + \partial^\mu \alpha$ 或 $\vec{A} \to \vec{A} - \nabla\alpha$. $A_0 \to A_0 + \partial_0 \alpha$ 并不是不变.

同时作相位变换 $\psi \to \psi' = e^{-ie\alpha}\psi$.

定义协变微分 covariant derivative

$$\vec{D}\psi = (\vec{\partial} - ie\vec{A})\psi.$$

$$\vec{D}\psi' = (\vec{\partial} - ie\vec{A}')\psi' = e^{-ie\alpha}\left[\vec{\partial} - ie\vec{\nabla}\alpha - ie(\vec{A} - \vec{\nabla}\alpha)\right]\psi = e^{-ie\alpha}(\vec{D}\psi).$$

$$\vec{D}^2\psi' = e^{-ie\alpha}(\vec{D}^2\psi).$$

$$D_0\psi = (\partial_0 + ieA_0)\psi.$$

$$D_0\psi' = e^{-ie\alpha}(\partial_0 + ie\partial_0\alpha - ieA_0 - ie\partial_0\alpha)\psi = e^{-ie\alpha}D_0\psi.$$

Schrödinger 方程

$$\left[-\frac{1}{2m}(\vec{\nabla} - ie\vec{A}')^2 + eA_0'\right]\psi' = i\frac{\partial\psi'}{\partial t}.$$

$$\Rightarrow \quad e^{-ie\alpha}\left[-\frac{1}{2m}(\vec{\nabla} - ie\vec{A})^2 + eA_0\right]\psi = e^{-ie\alpha}\, i\frac{\partial\psi}{\partial t}.$$

$\alpha = \alpha(\vec{x}, t)$. $U(1)$ 变换.