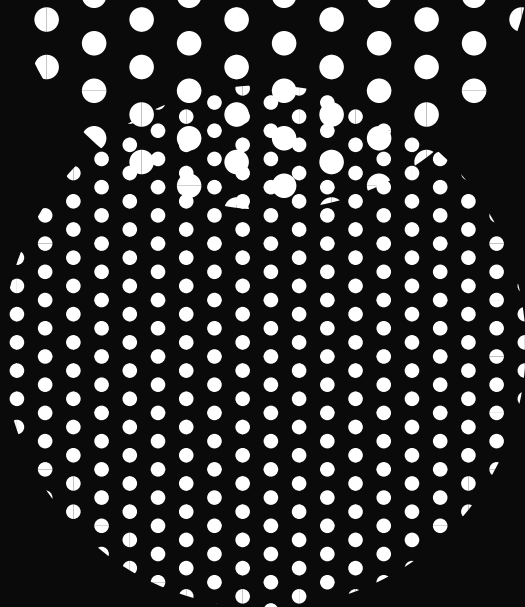




量子光学基础巩固

QM 回顾. $i\hbar \frac{d|\psi\rangle}{dt} = \hat{H}|\psi\rangle.$

H 不含时. $|\psi(t)\rangle = U(t)|\psi(0)\rangle = e^{-iHt/\hbar} |\psi(0)\rangle.$

H 含时. $U(t) = \hat{T} \exp \left[\frac{1}{i\hbar} \int_0^t d\tau \hat{H}(\tau) \right] \quad \Delta$

$$= 1 + \frac{1}{i\hbar} \int_0^t d\tau \hat{H}(\tau) + \left(\frac{1}{i\hbar} \right)^2 \int_0^t d\tau_1 \int_0^{\tau_1} d\tau_2 \hat{H}(\tau_1) \hat{H}(\tau_2) \dots$$

$$t \geq \tau_1 \geq \tau_2 \dots$$

当不同时刻 H 对易时,

$$\int_0^t d\tau_1 \int_0^{\tau_1} d\tau_2 \dots \int_0^{\tau_{n-1}} d\tau_n \hat{H}(\tau_1) \hat{H}(\tau_2) \dots \hat{H}(\tau_n) = \frac{1}{n!} \left[\int_0^t d\tau \hat{H}(\tau) \right]^n.$$

$$U(t) = \hat{T} \exp \left[\frac{1}{i\hbar} \int_0^t d\tau H(\tau) \right] = \exp \left[\frac{1}{i\hbar} \int_0^t d\tau \hat{H}(\tau) \right].$$

不确定关系

任意可观测测量 M

$$(\Delta M)^2 = \langle \Delta \hat{M}^2 \rangle = \langle \psi | [\hat{M} - \langle \hat{M} \rangle]^2 | \psi \rangle = \langle \psi | \hat{M}^2 | \psi \rangle - \langle \psi | \hat{M} | \psi \rangle^2$$

引入 $|\psi_1\rangle = [\hat{A} - \langle \hat{A} \rangle] |\psi\rangle.$

$$|\psi_2\rangle = [\hat{B} - \langle \hat{B} \rangle] |\psi\rangle.$$

利用 $\langle \psi_1 | \psi_1 \rangle \langle \psi_2 | \psi_2 \rangle \geq \langle \psi_1 | \psi_2 \rangle \langle \psi_2 | \psi_1 \rangle$. 得

$$(\Delta A)^2 (\Delta B)^2 \geq \frac{1}{4} [\langle C \rangle^2 + \langle F \rangle^2].$$

$$\hat{C} = \frac{1}{i} [\hat{A}, \hat{B}] \quad \hat{F} = \hat{A} \hat{B} + \hat{B} \hat{A} - 2\langle \hat{A} \rangle \langle \hat{B} \rangle.$$

密度矩阵

$$\langle \hat{A} \rangle = \sum_n p_n \langle \psi_n | \hat{A} | \psi_n \rangle = \sum_n \sum_a p_n \langle \psi_n | \hat{A} | a \rangle \langle a | \psi_n \rangle$$

$$= \sum_a \langle a | \left(\sum_n p_n |\psi_n\rangle \langle \psi_n| \right) \hat{A} | a \rangle = \text{Tr}[\rho \hat{A}]$$

$$\langle \psi | \rho | \psi \rangle = \sum_n p_n |\langle \psi | \psi_n \rangle|^2 \geq 0$$

归一化的系统 $\text{Tr}[\rho] = \sum_n p_n \text{Tr}[|\psi_n\rangle \langle \psi_n|] = \sum_n p_n = 1$

混乱度参数 $P = 1 - \text{Tr}[\rho^2]$ 纯度 $0 < P < 1$

纯态 $P = 0$ 混态 $\text{Tr}[\rho^2] = \sum_n p_n^2 \text{Tr}[|\psi_n\rangle \langle \psi_n|] \leq \sum_n p_n = 1$

Von Neumann 熵 $S = -\text{Tr}[\rho \ln \rho]$ $P \leq S$

关于绘景的补充

$$|\psi\rangle \quad \bar{A} = \langle \psi | A | \psi \rangle$$

$$\bar{A}' = \langle \psi | U^\dagger U A U^\dagger U | \psi \rangle = \bar{A}$$

$$|\psi\rangle \longrightarrow |\tilde{\psi}\rangle = U |\psi\rangle$$

不变

$$A \longrightarrow \tilde{A} = U A U^\dagger$$

Heisenberg Picture

$$A_H(t) = U^\dagger(t) A_S U(t) \quad |\psi\rangle_H = U^\dagger(t) \psi_S(t) = \psi_S(0)$$

$$\frac{dA_H}{dt} = \frac{dU^\dagger}{dt} A_S U + U^\dagger \frac{dA_S}{dt} U + U^\dagger A_S \frac{dU}{dt}$$

$$= iH U^\dagger A_S U + U^\dagger \frac{\partial A_S}{\partial t} U + U^\dagger A_S \cdot (-iHU)$$

$$= i[H, A_H] + \left(\frac{\partial A}{\partial t}\right)_H.$$

H 不显含时间时, $U(t) = e^{-iHt}$ 且

$$A_H(t) = e^{iHt} A_S e^{-iHt}.$$

Interaction Picture

$$H = H_0 + V.$$

$$A_I(t) = U_0^\dagger(t) A_S U_0(t).$$

$$|\psi\rangle_I = U_0^\dagger |\psi_S(t)\rangle = U_0^\dagger(t) U(t) |\psi_0\rangle.$$

$$\frac{dA_I}{dt} = \frac{dU_0^\dagger}{dt} A_S U_0 + U_0^\dagger \frac{\partial A_S}{\partial t} U_0 + U_0^\dagger A_S \frac{dU_0}{dt}$$

$$= iH_0 U_0^\dagger A_S U_0 + \frac{\partial A_I}{\partial t} + U_0^\dagger A_S (-iH U_0)$$

$$= i[H_0, A_I] + \frac{\partial A_I}{\partial t}.$$

$$\frac{d|\psi_I\rangle}{dt} = \frac{dU_0^\dagger}{dt} U |\psi_0\rangle + U_0^\dagger \frac{dU}{dt} |\psi_0\rangle$$

$$= iH_0 U_0^\dagger U |\psi_0\rangle + U_0^\dagger \cdot (-iH U) |\psi_0\rangle$$

$$= iH_0 U_0^\dagger U |\psi_0\rangle - iU_0^\dagger H_0 U |\psi_0\rangle - iU_0^\dagger V U |\psi_0\rangle$$

$$= -i \underline{U_0^\dagger V U_0} U_0^\dagger U |\psi_0\rangle = -iV_I |\psi_I\rangle.$$

更一般的情形

$$\text{令 } |\phi(t)\rangle = \tilde{U}(t) |\psi(t)\rangle$$

$$\begin{aligned} i\hbar \frac{d|\phi(t)\rangle}{dt} &= i\hbar \left(\frac{d\tilde{U}(t)}{dt} |\psi(t)\rangle + \tilde{U}(t) \frac{d|\psi(t)\rangle}{dt} \right) \\ &= i\hbar \frac{d\tilde{U}(t)}{dt} \tilde{U}^\dagger |\phi\rangle + \tilde{U}(t) H \tilde{U}^\dagger |\phi\rangle \end{aligned}$$

$$H_{\text{eff}} = \left(\tilde{U} H \tilde{U}^\dagger + i\hbar \frac{d\tilde{U}}{dt} \tilde{U}^\dagger \right) |\phi(t)\rangle.$$

$$\tilde{U}(t) \sim \hat{T} \exp \left[-\frac{i}{\hbar} \int_0^t d\tau \hat{H}(\tau) \right] \quad \text{Heisenberg Picture}$$

$$\sim \exp(iH_0 t / \hbar) \quad \text{Interaction Picture}$$

两体系统

$$|\psi\rangle = \sum_{mn} c_{mn} |a_n\rangle \otimes |b_n\rangle.$$

$$\text{SVD.} \quad C = U \cdot D \cdot V.$$

$$M \times N \quad M \times M \quad M \times N \quad N \times N$$

$$c_{mn} = \sum_k U_{mk} D_{kk} V_{kn}.$$

$$|\psi\rangle = \sum_k D_{kk} \left(\sum_n U_{nk} |a_n\rangle \right) \otimes \left(\sum_n V_{kn} |b_n\rangle \right).$$

$|k\rangle_A$
 $|k\rangle_B$

Schmidt decomposition

谐振子

$$i\hbar \frac{d}{dt} |\psi(x,t)\rangle = \hat{H} |\psi(x,t)\rangle$$

$$H = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2 \hat{x}^2$$

$$|\psi(x,t)\rangle = |\phi(x)\rangle e^{-iE_n t/\hbar} \quad E_n \text{ 的本征态}$$

$$\left(-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + \frac{1}{2}m\omega^2 x^2\right) |\phi(x)\rangle = E_n |\phi(x)\rangle$$

$$E_n = (n + \frac{1}{2})\hbar\omega$$

$$\lambda \quad l_T = \sqrt{\frac{\hbar}{m\omega}}$$

$$\hat{a} = \frac{1}{\sqrt{2\hbar m\omega}} (m\omega \hat{x} + i\hat{p}) \quad [\hat{a}, \hat{a}^\dagger] = 1$$

$$\hat{a}^\dagger = \frac{1}{\sqrt{2\hbar m\omega}} (m\omega \hat{x} - i\hat{p}) \quad H = \frac{1}{2}\hbar\omega (\hat{a}^\dagger \hat{a} + \hat{a} \hat{a}^\dagger) = \hbar\omega (\hat{a} \hat{a}^\dagger + \frac{1}{2})$$

$$\hat{a}|0\rangle = 0 \quad \hat{a}^\dagger \hat{a}|n\rangle = n|n\rangle$$

$$\hat{a}|n\rangle = \sqrt{n}|n-1\rangle \quad \hat{a}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$$

$$a_{nn} = \langle n|\hat{a}|n\rangle = \sqrt{n} \delta_{n,n-1}$$

$$a_{nn}^\dagger = \sqrt{n+1} \delta_{n,n+1}$$

电磁场量子化

$$\text{自由空间} \quad \vec{E}(\vec{r}, t) = i \sum_{\vec{k}, \sigma} \hat{e}_\sigma \sqrt{\frac{\hbar\omega_k}{2\epsilon_0 V}} \left(\alpha_k e^{-i\omega_k t + i\vec{k}\cdot\vec{r}} - \alpha_k^* e^{i\omega_k t - i\vec{k}\cdot\vec{r}} \right)$$

$$\vec{H}(\vec{r}, t) = i \sum_{\vec{k}, \sigma} \frac{c \vec{k} \times \vec{e}_\sigma}{\omega_k} \sqrt{\frac{\hbar\omega_k}{2\mu_0 V}} \left(\alpha_k e^{-i\omega_k t + i\vec{k}\cdot\vec{r}} - \alpha_k^* e^{i\omega_k t - i\vec{k}\cdot\vec{r}} \right)$$

量子化 $\vec{A}(\vec{r}, t) = \sum_{\vec{k}, \sigma} \hat{e}_{\sigma} \sqrt{\frac{\hbar}{2\epsilon_0 \omega_k V}} \left(\hat{a}_{\vec{k}, \sigma} e^{-i\omega_k t + i\vec{k} \cdot \vec{r}} + \hat{a}_{\vec{k}, \sigma}^{\dagger} e^{i\omega_k t - i\vec{k} \cdot \vec{r}} \right)$

$$\vec{E}(\vec{r}, t) = i \sum_{\vec{k}, \sigma} \hat{e}_{\sigma} \sqrt{\frac{\hbar \omega_k}{2\epsilon_0 V}} \left(\hat{a}_{\vec{k}, \sigma} e^{-i\omega_k t + i\vec{k} \cdot \vec{r}} - \hat{a}_{\vec{k}, \sigma}^{\dagger} e^{i\omega_k t - i\vec{k} \cdot \vec{r}} \right)$$

$$\vec{H}(\vec{r}, t) = i \sum_{\vec{k}, \sigma} \frac{c \vec{k} \times \hat{e}_{\sigma}}{\omega_k} \sqrt{\frac{\hbar \omega_k}{2\mu_0 V}} \left(\hat{a}_{\vec{k}, \sigma} e^{-i\omega_k t + i\vec{k} \cdot \vec{r}} - \hat{a}_{\vec{k}, \sigma}^{\dagger} e^{i\omega_k t - i\vec{k} \cdot \vec{r}} \right)$$

正负频形式 $\vec{E}(\vec{r}, t) = \vec{E}^{(+)}(\vec{r}, t) + \vec{E}^{(-)}(\vec{r}, t)$

$$\vec{E}^{(+)}(\vec{r}, t) = i \sum_{\vec{k}, \sigma} \hat{e}_{\sigma} \sqrt{\frac{\hbar \omega_k}{2\epsilon_0 V}} \left(\hat{a}_{\vec{k}, \sigma} e^{-i\omega_k t + i\vec{k} \cdot \vec{r}} \right)$$

$$\vec{E}^{(-)}(\vec{r}, t) = [\vec{E}^{(+)}(\vec{r}, t)]^{\dagger}$$

场动量算符 $\vec{P} = \frac{1}{c} \int_V \vec{E} \times \vec{H} d\tau = \frac{1}{2} \sum_{\vec{k}, \sigma} \hbar \vec{k} \left(\hat{a}_{\vec{k}, \sigma} \hat{a}_{\vec{k}, \sigma}^{\dagger} + \hat{a}_{\vec{k}, \sigma}^{\dagger} \hat{a}_{\vec{k}, \sigma} \right)$

热平衡光场

下 正则系综

第 n 能级占据概率 $P_n = \frac{\exp(-E_n/k_B T)}{\sum_n \exp(-E_n/k_B T)}$

对单模光场 $\hat{H} = \hbar \omega (n + \frac{1}{2})$

$$\hat{\rho}_T = \frac{\exp(-\hat{H}/k_B T)}{\text{Tr}[\exp(-\hat{H}/k_B T)]} = \frac{\exp(-\hat{H}/k_B T)}{\mathcal{Z}} = \sum_{n=0}^{\infty} P_n |n\rangle \langle n|$$

$$\mathcal{Z} = \text{Tr}[\exp(-\hat{H}/k_B T)]$$

$$\begin{aligned}
&= \sum_{n=0}^{\infty} \langle n | \exp(-\hat{H}/k_B T) | n \rangle = \sum_n \exp(-\underline{E_n}/k_B T) \quad (n+\frac{1}{2}) \hbar \omega \\
&= \sum_n \exp\left(-n \hbar \omega / k_B T\right) \cdot \exp\left(-\frac{\hbar \omega}{2 k_B T}\right) \\
&= \frac{\exp(-\hbar \omega / 2 k_B T)}{1 - \exp(-\hbar \omega / k_B T)}.
\end{aligned}$$

平均光子数 $\bar{n} = \text{Tr}[\hat{n} \hat{\rho}_T] = \sum_{n=0}^{\infty} \langle n | \hat{n} \hat{\rho}_T | n \rangle$

$$\begin{aligned}
&= \exp(-\hbar \omega / k_B T) \sum_n n \exp(-n \hbar \omega / k_B T) / Z \\
&= \frac{\exp(-\hbar \omega / k_B T)}{1 - \exp(-\hbar \omega / k_B T)} = \frac{1}{\exp(\hbar \omega / k_B T) - 1}.
\end{aligned}$$

ρ_T 的平均光子数表示.

$$\hat{\rho}_T = \frac{1}{1 + \bar{n}} \sum_{n=0}^{\infty} \left(\frac{\bar{n}}{1 + \bar{n}} \right)^n |n\rangle \langle n|, \quad P_n = \frac{\bar{n}^n}{(1 + \bar{n})^{n+1}}.$$

热光场的涨落

$$\langle \hat{n}^2 \rangle = \text{Tr}[\hat{n}^2 \rho_T] = 2\bar{n}^2 + \bar{n}.$$

$$(\Delta n)^2 = \langle \Delta \hat{n}^2 \rangle = \langle \hat{n}^2 \rangle - \langle \hat{n} \rangle^2 = \bar{n}^2 + \bar{n}.$$

$$\bar{n} \gg 1 \text{ 时, } \Delta n \approx \bar{n} + \frac{1}{2}.$$

$$\bar{n} \ll 1 \text{ 时, } \Delta n \approx \sqrt{\bar{n}}.$$

能量密度

$$\begin{aligned} U(\omega) d\omega &= \rho(\omega) \bar{n} \hbar \omega d\omega = \frac{\hbar \omega^3}{\pi^2 c^3} \frac{1}{\exp(\hbar \omega / kT) - 1} d\omega \\ &= \frac{8\pi \hbar \nu^3}{c^3} \frac{1}{\exp(\hbar \nu / kT) - 1} d\nu. \end{aligned}$$

量子相位算符 (有限维)

定义 $(M+1)$ 维空间中的参考相位态

$$|\theta_0\rangle = \frac{1}{\sqrt{M+1}} \sum_{n=0}^M e^{in\theta_0} |n\rangle.$$

$$\text{正交基 } |\theta_k\rangle = \frac{1}{\sqrt{M+1}} \sum_{n=0}^M e^{in\theta_k} |n\rangle, \quad \theta_k = \theta_0 + \frac{2k\pi}{M+1}.$$

$$\Rightarrow |n\rangle = \frac{1}{\sqrt{M+1}} \sum_{k=0}^M e^{-in\theta_k} |\theta_k\rangle. \quad (\text{相位涨落最大})$$

$$e^{i\hat{\phi}_0} = \sum_k e^{i\theta_k} |\theta_k\rangle \langle \theta_k|.$$

$$\text{有 } e^{i\hat{\phi}_0} |\theta_k\rangle = e^{i\theta_k} |\theta_k\rangle, \quad \hat{\phi}_0 = \sum_k \theta_k |\theta_k\rangle \langle \theta_k|.$$

态 $|\gamma\rangle = \sum_{n=0}^M c_n |n\rangle$ 的相位分布函数为

$$\begin{aligned} W(\theta_k) &= |\langle \theta_k | \gamma \rangle|^2 = \frac{1}{M+1} \left| \sum_{n=0}^M c_n e^{-in\theta_k} \right|^2 = \frac{1}{M+1} \left| \sum_{n=0}^M c_n e^{-i \frac{2\pi n k}{M+1}} \right|^2 \\ &= \frac{1}{2\pi} \left| \sum_{n=0}^M c_n e^{-i \frac{2\pi n k}{M+1}} \right|^2 \frac{2\pi}{M+1}. \end{aligned}$$

$M \rightarrow \infty$ 积分形式

$$W(\theta) d\theta = \frac{1}{2\pi} \left| \sum_{n=0}^M c_n e^{-in\theta} \right|^2 d\theta \quad \text{利用 } W(\theta) \text{ 求平均}$$

$$\langle \psi | f(\hat{\phi}_0) | \psi \rangle = \sum_k f(\theta_k) W(\theta_k) = \int_0^{2\pi} f(\theta) W(\theta) d\theta$$

$$W(\theta) = \frac{1}{2\pi} |\langle \theta | \psi \rangle|^2 = \frac{1}{2\pi}$$

光子波函数

$$\begin{cases} \nabla \times \vec{H} = \frac{\partial \vec{D}}{\partial t} \\ \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \\ \nabla \cdot \vec{B} = 0 \\ \nabla \cdot \vec{D} = 0 \end{cases} \quad \begin{aligned} &\text{引入 } \vec{F}_{\pm} = \frac{1}{\sqrt{2}} (\sqrt{\epsilon_0} \vec{E} \pm i\sqrt{\mu_0} \vec{H}) \\ &\text{Maxwell 方程改写为} \\ &i\hbar \frac{\partial}{\partial t} \vec{F}_{\pm} = \pm c(\hat{s} \cdot \hat{p}) \vec{F}_{\pm} \end{aligned}$$

$$\text{引入 } F = \frac{1}{\sqrt{2}} \begin{bmatrix} \vec{F}_+ \\ \vec{F}_- \end{bmatrix} \quad \text{有 } i\hbar \frac{\partial}{\partial t} F = c \hat{\sigma}_z \otimes (\hat{s} \cdot \hat{p}) F$$

$$\text{光子反粒子为本身} \Rightarrow F^{(-)}(\vec{r}, t) = (\hat{\sigma}_z \otimes I_2) F^{(+)*}(\vec{r}, t)$$

$$\text{光子波函数 } \psi(\vec{r}, t) = F^{(+)}(\vec{r}, t) = \frac{1}{\sqrt{2}} \begin{bmatrix} \vec{F}^{(+)} \\ \vec{F}^{(-)} \end{bmatrix}$$

$$H = c \hat{\sigma}_z \otimes (\hat{s} \cdot \hat{p}) \quad \text{无良好空间位置算符定义}$$

二次量子化的光子波函数

$$\text{坐标表象 } \psi(\vec{r}, t) = \langle \vec{r} | \psi(t) \rangle$$

$$|\vec{r}\rangle = \hat{\psi}^\dagger(\vec{r}) |0\rangle$$

$$\hat{\psi}(\vec{r}) = \sum_m c_m^\dagger e^{i\omega_m t} \phi_m^*(\vec{r})$$

$$\Rightarrow \psi(\vec{r}, t) = \langle 0 | \hat{\psi}(\vec{r}) | \psi(t) \rangle$$

对光电探测. $(\vec{r}, t)$ 处的光强

$$P_{\psi}(\vec{r}, t) \propto \langle \psi | \hat{E}^{(-)}(\vec{r}, t) \left(\sum_{\{n\}} |\{n\}\rangle \langle \{n\}| \right) \hat{E}^{(+)}(\vec{r}, t) | \psi \rangle$$

$$= \langle \psi | \hat{E}^{(-)}(\vec{r}, t) | 0 \rangle \underbrace{\langle 0 | \hat{E}^{(+)}(\vec{r}, t) | \psi \rangle}_{\psi_E(t)}$$

单光子电场分量

$$\psi_E(t) = \langle 0 | \hat{E}^{(+)}(\vec{r}, t) | \psi \rangle = \sqrt{\frac{\hbar \omega_k}{2 \epsilon_0 V}} \langle 0 | i \sum_{\vec{k}, \sigma} \hat{e}_{\sigma} \hat{a}_{\vec{k}, \sigma} e^{-i\omega_k t + i\vec{k} \cdot \vec{r}} | \psi \rangle$$

$$= \sqrt{\frac{\hbar \omega_k}{2 \epsilon_0 V}} \underbrace{\vec{y}_E(\vec{r}, t)}_{\text{光电探测概率幅空间分布}} \langle 0 | i \sum_{\vec{k}, \sigma} \hat{e}_{\sigma} \hat{a}_{\vec{k}, \sigma} \frac{e^{-i\omega_k t + i\vec{k} \cdot \vec{r}}}{\sqrt{V}} | \psi \rangle$$

光电探测概率幅空间分布

磁场分量

$$\psi_H(t) = \sqrt{\frac{\hbar \omega}{2 \mu_0}} \vec{y}_H(\vec{r}, t)$$

$$\vec{y}_H(\vec{r}, t) = \langle 0 | i \sum_{\vec{k}, \sigma} \frac{\vec{k} \times \hat{e}_{\sigma}}{|\vec{k}|} \hat{a}_{\vec{k}, \sigma} \frac{e^{-i\omega_k t + i\vec{k} \cdot \vec{r}}}{\sqrt{V}} | \psi \rangle$$

相干态

$$\hat{a} |\alpha\rangle = \alpha |\alpha\rangle$$

非厄米 复数

$$\langle n | \hat{a} | \alpha \rangle = \sqrt{n+1} \langle n+1 | \alpha \rangle = \alpha \langle n | \alpha \rangle$$

$$\langle n | \alpha \rangle = \langle 0 | \frac{\hat{a}^n}{n!} | \alpha \rangle = \frac{\alpha^n}{n!} \langle 0 | \alpha \rangle$$

$$\Rightarrow \sum_n \langle \alpha | n \rangle \langle n | \alpha \rangle = 1$$

$$1 = |\langle 0|\alpha\rangle|^2 \sum_n \frac{|\alpha|^{2n}}{n!} = |\langle 0|\alpha\rangle|^2 e^{|\alpha|^2}$$

$$\langle 0|\alpha\rangle = e^{-|\alpha|^2/2}$$

$$|\alpha\rangle = \sum_n \frac{\alpha^n}{\sqrt{n!}} \langle 0|\alpha\rangle |n\rangle = \exp(-\frac{1}{2}|\alpha|^2) \sum_n \frac{\alpha^n}{\sqrt{n!}} |n\rangle$$

$$= \exp(-\frac{1}{2}|\alpha|^2) \sum_n \frac{(\alpha \hat{a}^\dagger)^n}{n!} |0\rangle = \exp(-\frac{1}{2}|\alpha|^2) e^{\alpha \hat{a}^\dagger} |0\rangle$$

BCH 定理

$$\text{非对易算符 } \hat{A}, \hat{B} \quad e^{\hat{A}} e^{\hat{B}} = e^{\hat{A}+\hat{B} + \frac{1}{2}[\hat{A}, \hat{B}] + \dots}$$

$$\text{若 } [\hat{A}, [\hat{A}, \hat{B}]] = [\hat{B}, [\hat{A}, \hat{B}]] = 0$$

$$e^{\hat{A}+\hat{B}} = e^{\hat{A}} e^{\hat{B}} e^{-\frac{1}{2}[\hat{A}, \hat{B}]} = e^{\hat{B}} e^{\hat{A}} e^{\frac{1}{2}[\hat{A}, \hat{B}]}$$

平移算符

$$\hat{D}(\alpha) = \exp(\alpha \hat{a}^\dagger - \alpha^* \hat{a})$$

$$= \exp(\alpha \hat{a}^\dagger) \exp(-\alpha^* \hat{a}) \exp(-|\alpha|^2/2)$$

$$= \exp(-\alpha^* \hat{a}) \exp(\alpha \hat{a}^\dagger) \exp(|\alpha|^2/2)$$

$$\hat{D}(\alpha) |0\rangle = \exp(\alpha \hat{a}^\dagger) \exp(-\alpha^* \hat{a}) \exp(-|\alpha|^2/2) |0\rangle$$

$$= \exp(-|\alpha|^2/2) \exp(\alpha \hat{a}^\dagger) |0\rangle$$

$$D^\dagger(\alpha) |\alpha\rangle = \hat{D}(-\alpha) |\alpha\rangle = |0\rangle$$

平均光子数

$$\begin{aligned}
 \langle n \rangle &= \text{Tr}[\rho \hat{a}^\dagger \hat{a}] = \text{Tr}[|\alpha\rangle\langle\alpha| \hat{a}^\dagger \hat{a}] \\
 &= e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{|\alpha|^{2n}}{n!} \underbrace{\text{Tr}[|n\rangle\langle n| \hat{a}^\dagger \hat{a}]}_n \\
 &= e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{|\alpha|^{2n}}{n!} \cdot n = |\alpha|^2
 \end{aligned}$$

光子数态 $|k\rangle$ 占比为 Poisson 分布

$$P_k(\alpha) = e^{-|\alpha|^2} \frac{|\alpha|^{2k}}{k!}$$

不确定关系 $l_T = \sqrt{\frac{\hbar}{m\omega}}$ 对 $|\alpha\rangle$

$$\hat{x} = \frac{1}{\sqrt{2}} l_T (\hat{a} + \hat{a}^\dagger) \quad \hat{p} = \frac{1}{i\sqrt{2}} \frac{\hbar}{l_T} (\hat{a} - \hat{a}^\dagger)$$

$$\langle \hat{x} \rangle = \frac{l_T}{\sqrt{2}} (\alpha + \alpha^*) \quad \langle \hat{p} \rangle = \frac{1}{i\sqrt{2}} \frac{\hbar}{l_T} (\alpha - \alpha^*)$$

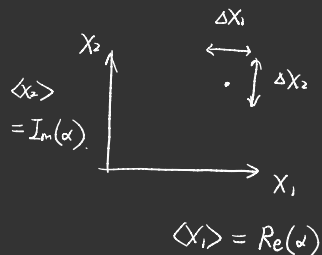
$$\Delta x = \frac{l_T}{\sqrt{2}} \quad \Delta p = \frac{\hbar}{\sqrt{2} l_T}$$

可用 $\hat{X}_1 = \frac{1}{2}(\hat{a} + \hat{a}^\dagger)$ $\hat{X}_2 = \frac{1}{2i}(\hat{a} - \hat{a}^\dagger)$ 刻画相干态起伏

$$[\hat{X}_1, \hat{X}_2] = \frac{i}{2} \quad \Delta X_1 = \Delta X_2 = \frac{1}{2}$$

$$\langle X_1 \rangle = \text{Re}(\alpha)$$

$$\langle X_2 \rangle = \text{Im}(\alpha)$$



相干态波函数

$$\hat{a} = \frac{1}{\sqrt{2}} \left(\frac{\hat{x}}{l_T} + i \frac{l_T}{\hbar} \hat{p} \right) = \frac{1}{\sqrt{2} \hbar m \omega} (m \omega \hat{x} + i \hat{p})$$

$$\hat{a} |\alpha\rangle = \alpha |\alpha\rangle = \frac{1}{\sqrt{2}} \left(\frac{\hat{x}}{l_T} + i \frac{l_T}{\hbar} \hat{p} \right) |\alpha\rangle$$

$$\Rightarrow \sqrt{2} \alpha \langle x' | \alpha \rangle = \langle x' | \left[\frac{\hat{x}}{l_T} + \frac{\partial}{\partial(x/l_T)} \right] |\alpha\rangle$$

$$\Rightarrow \frac{d \langle x' | \alpha \rangle}{\langle x' | \alpha \rangle} = \left(\sqrt{2} \alpha - \frac{x'}{l_T} \right) d \left(\frac{x'}{l_T} \right)$$

$$\langle x' | \alpha \rangle = N \exp \left(-\frac{x'^2}{2l_T^2} + \frac{\sqrt{2} \alpha x'}{l_T} \right)$$

$$\begin{aligned} \psi_\alpha(x) = \langle x' | \alpha \rangle &= \left(\frac{1}{\pi l_T^2} \right)^{\frac{1}{4}} e^{i\gamma} \exp \left[-\frac{x'^2}{l_T^2} + \frac{\sqrt{2} \alpha x'}{l_T} - \frac{1}{4} (\alpha + \alpha^*)^2 \right] \\ &= \left(\frac{1}{\pi l_T^2} \right)^{\frac{1}{4}} e^{i\gamma} \exp \left[-\frac{(x' - \langle x \rangle)^2}{2l_T^2} + i \frac{\langle p \rangle x}{\hbar} \right] \end{aligned}$$

相干态的超完备性

$$\begin{aligned} \langle \alpha | \beta \rangle &= \exp \left(-\frac{1}{2} |\alpha|^2 - \frac{1}{2} |\beta|^2 \right) \sum_{n,m} \frac{\alpha^{*n} \beta^m}{\sqrt{n! m!}} \langle n | m \rangle \\ &= \exp \left(-\frac{1}{2} |\alpha|^2 - \frac{1}{2} |\beta|^2 + \alpha^* \beta \right) \quad \text{非正交} \end{aligned}$$

$$\mathcal{J} = \int d^2 \alpha |\alpha\rangle \langle \alpha| = \sum_{n,n'} \frac{|n\rangle \langle n'|}{\sqrt{n! n!}} \int d^2 \alpha \alpha^{*n} \alpha^{n'} \exp(-|\alpha|^2)$$

$$\text{取 } \alpha = |\alpha| e^{i\phi}, \quad d^2 \alpha = |\alpha| d|\alpha| d\phi.$$

$$\cdots \mathcal{J} = \pi \mathcal{I}.$$

$$I = \frac{1}{\pi} \int d^2\alpha |\alpha\rangle\langle\alpha| \quad \text{超完备}$$

相干态表象展开

$$|\alpha\rangle = \int d^2\beta \frac{\langle\beta|\alpha\rangle}{\pi} |\beta\rangle \quad \text{相干态的相干态展开}$$

$$|f\rangle = \frac{1}{\pi} \int d^2\alpha \langle\alpha|f\rangle |\alpha\rangle = \frac{1}{\pi} \int d^2\alpha |\alpha\rangle f(\alpha^*) \exp\left(-\frac{|\alpha|^2}{2}\right) \quad \text{任意态矢}$$

存在无穷多个等效展开

$$\langle g|f\rangle = \frac{1}{\pi} \int \int d^2\alpha d^2\beta [g(\alpha^*)]^* f(\alpha^*) e^{-|\alpha|^2} \quad \text{标积展开}$$

$$\hat{T} = I \cdot \hat{T} \cdot I = \frac{1}{\pi^2} \int d^2\alpha d^2\beta |\alpha\rangle\langle\alpha| \hat{T} |\beta\rangle\langle\beta|$$

$$= \frac{1}{\pi^2} \int d^2\alpha d^2\beta |\alpha\rangle\langle\beta| \underbrace{T(\alpha^*, \beta)} \exp\left(-\frac{|\alpha|^2 + |\beta|^2}{2}\right)$$

算符展开

$$\hat{a}^\dagger |\alpha\rangle\langle\alpha| = \left(\alpha^* + \frac{\partial}{\partial\alpha}\right) |\alpha\rangle\langle\alpha|$$

$$|\alpha\rangle\langle\alpha| \hat{a} = \left(\alpha + \frac{\partial}{\partial\alpha^*}\right) |\alpha\rangle\langle\alpha|$$

$$\text{Tr}(\hat{O}) = \frac{1}{\pi} \int d^2\alpha \langle\alpha| \hat{O} |\alpha\rangle \quad \text{迹的展开}$$

$$\text{Tr}[D(\lambda)] = \frac{1}{\pi} \int d^2\alpha \langle\alpha| \hat{D}(\lambda) |\alpha\rangle = \frac{1}{\pi} \int d^2\alpha \langle\alpha| \underbrace{D^\dagger(\alpha) \hat{D}(\lambda) D(\alpha)} |\alpha\rangle$$

$$D^\dagger(\alpha) \hat{D}(\lambda) D(\alpha) = \exp[\hat{D}^\dagger(\alpha) (\lambda \hat{a}^\dagger - \lambda^* \hat{a}) \hat{D}(\alpha)] = e^{\lambda\alpha^* - \lambda^*\alpha} \hat{D}(\lambda)$$

$$\text{Tr}[\hat{D}(\lambda)] = \frac{1}{\pi} \langle 0 | \hat{D}(\lambda) | 0 \rangle = \int d^2\alpha e^{\lambda\alpha^* - \lambda^*\alpha} = \pi e^{-|\lambda|^2/2} \delta(\lambda_r) \delta(\lambda_i)$$

$$\delta^{(2)}(\alpha) = \frac{1}{\pi^2} \int d^2\beta \exp(\alpha\beta^* - \alpha^*\beta) \quad \alpha = \alpha_r + i\alpha_i$$

忽略 $e^{-|\lambda|^2/2}$ 则 $\text{Tr}[\hat{D}(\lambda)] = \pi \delta^{(2)}(\lambda)$ $\text{Tr}[\hat{D}(\omega) \hat{D}(\rho)] = \pi \delta^{(2)}(\omega - \rho)$

平移算符可作为一组基矢, 展开算符 $\hat{O}$

$$\hat{O} = \frac{1}{\pi} \int d^2\omega O(\omega) \hat{D}^\dagger(\omega) \quad O(\omega) = \text{Tr}[\hat{O} \hat{D}(\omega)]$$

Husimi-Q 分布函数

$$Q(\alpha, \alpha^*) = \frac{1}{\pi} \langle \alpha | \rho | \alpha \rangle \quad \text{表为 } \hat{a} = (\hat{x} + i\hat{p})/\sqrt{2}$$

$Q(\alpha, \alpha^*)$ 可表示在 $x-p$ 复平面上, 有 $\alpha = (x + ip)/\sqrt{2}$

归一化 $\int d^2\alpha Q(\alpha, \alpha^*) = 1$

计算反正规排列如 $\hat{a}^k (\hat{a}^\dagger)^l$

$$\langle \hat{a}^k (\hat{a}^\dagger)^l \rangle = \text{Tr}[(\hat{a}^\dagger)^l \rho \hat{a}^k] = \frac{1}{\pi} \int d^2\alpha \langle \alpha | (\hat{a}^\dagger)^l \rho \hat{a}^k | \alpha \rangle = \int d^2\alpha \alpha^k (\alpha^*)^l Q(\alpha, \alpha^*)$$

相干态演化

$$\hat{H}_0 = \hbar\omega \hat{a}^\dagger \hat{a} \quad (\text{去除 } \frac{\hbar\omega}{2})$$

$$|\alpha(t)\rangle = \exp\left(\frac{\hat{H}\hat{t}}{i\hbar}\right) |\alpha_0\rangle = \exp(-i\omega \hat{a}^\dagger \hat{a} t) \exp(-|\alpha_0|^2/2) \sum_n \frac{\alpha_0^n}{\sqrt{n!}} |n\rangle$$

$$= e^{-\frac{1}{2}|\alpha_0|^2} \sum_n \frac{(\alpha_0 e^{-i\omega t})^n}{\sqrt{n!}} |n\rangle$$

$$= |\alpha_0 e^{-i\omega t}\rangle$$

驱动谐振子产生相干态, 受迫 Hamiltonian.

$$\hat{H} = \hat{H}_0 + \hat{H}_I = \hbar\omega \hat{a}^\dagger \hat{a} + \hbar \underline{g(t) (\hat{a}^\dagger + \hat{a})} \propto \hat{x} E(t) \text{ 的驱动}$$

$$i\hbar \frac{\partial |\psi(t)\rangle}{\partial t} = \hat{H} |\psi(t)\rangle.$$

$$\text{令 } |\phi(t)\rangle = \exp\left(\frac{i\hat{H}_0 t}{\hbar}\right) |\psi(t)\rangle.$$

$$i\hbar \frac{\partial |\phi(t)\rangle}{\partial t} = \exp\left(\frac{i\hat{H}_0 t}{\hbar}\right) \hat{H}_I(t) \exp\left(\frac{-i\hat{H}_0 t}{\hbar}\right) |\phi(t)\rangle$$

$$\text{又 } \exp\left(\frac{i\hat{H}_0 t}{\hbar}\right) \hat{a} \exp\left(\frac{-i\hat{H}_0 t}{\hbar}\right) = \hat{a} e^{-i\omega t}$$

$$\exp\left(\frac{i\hat{H}_0 t}{\hbar}\right) \hat{a}^\dagger \exp\left(\frac{-i\hat{H}_0 t}{\hbar}\right) = \hat{a}^\dagger e^{-i\omega t}$$

$$\text{得 } \frac{\partial |\phi(t)\rangle}{\partial t} = -ig(t) (\hat{a}^\dagger e^{i\omega t} + \hat{a} e^{-i\omega t}) |\phi(t)\rangle$$

$$|\phi(t)\rangle = \hat{T} \exp\left\{-i \int_0^t g(\tau) (\hat{a}^\dagger e^{i\omega\tau} + \hat{a} e^{-i\omega\tau}) d\tau\right\}$$

$$= \exp\left\{\hat{a}^\dagger e^{-i\omega t} \alpha(t) - \hat{a} e^{i\omega t} \alpha^*(t) + iF(t)\right\} |\phi(t)\rangle$$

$$\alpha(t) = -i \int_0^t d\tau e^{i\omega(t-\tau)} g(\tau)$$

$$F(t) = \frac{i}{2} \int_0^t d\tau \int_0^t d\tau' \text{sgn}(\tau-\tau') g(\tau) g(\tau') e^{i\omega(\tau-\tau')}$$

$$|\psi(t)\rangle = \exp\left[\hat{a}^\dagger \alpha(t) - \hat{a} \alpha^*(t) + iF(t)\right] \exp\left(\frac{-i\hat{H}_0 t}{\hbar}\right) |\psi(t)\rangle$$

$$|\psi(0)\rangle = |0\rangle \text{ 时 } |\psi(t)\rangle = \exp[iF(t)] \underline{|\alpha(t)\rangle}.$$

相干态.

压缩相干态

$$\hat{S}(\xi) = \exp\left(\frac{\xi^*}{2} \hat{a}^2 - \frac{\xi}{2} \hat{a}^{\dagger 2}\right) \quad \xi = r \exp(i\theta)$$

$$\hat{H} = \frac{i\hbar}{2} (g^* \hat{a}^2 - g \hat{a}^{\dagger 2})$$

$$|\psi(t)\rangle = \exp\left(\frac{i\hat{H}t}{\hbar}\right) |\psi\rangle = \exp\left(\frac{\xi^*}{2} \hat{a}^2 - \frac{\xi}{2} \hat{a}^{\dagger 2}\right) |\psi\rangle$$

$$\hat{S}(\xi) = \hat{R}\left(-\frac{\theta}{2}\right) \hat{S}(r) \hat{R}\left(\frac{\theta}{2}\right)$$

$$\hat{S}(r) = \exp\left[\frac{r}{2} (\hat{a}^2 - \hat{a}^{\dagger 2})\right]$$

Bogoliubov 变换

$$|\alpha, r\rangle = \hat{D}(\alpha) \hat{S}(r) |\psi\rangle = \hat{D}(\alpha) |\psi\rangle_r \quad \text{squeezed vacuum}$$

$$\exists \lambda \quad \hat{b} = \hat{S}(r) \hat{a} \hat{S}^\dagger(r) = \mu \hat{a} + \nu \hat{a}^\dagger$$

$$\mu = \cosh r, \quad \nu = \sinh r, \quad |\mu|^2 - |\nu|^2 = 1$$

$$[\hat{b}, \hat{b}^\dagger] = 1 \quad \text{“准光子”算符}$$

$$\text{粒子数 } N_g = \hat{b}^\dagger \hat{b} = \hat{S}(r) \hat{a}^\dagger \hat{a} \hat{S}(r)$$

$$\text{又 } \hat{a} = \mu^* \hat{b} - \nu \hat{b}^\dagger, \quad \hat{a}^\dagger = \mu \hat{b}^\dagger - \nu^* \hat{b}$$

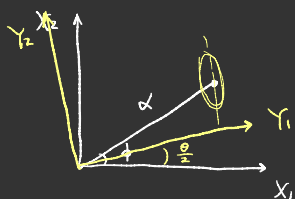
$$|\alpha, r\rangle = \hat{D}(\alpha) |\psi\rangle_r = \exp(\beta \hat{b}^\dagger - \beta^* \hat{b}) |\psi\rangle_r \quad \text{压缩态为准光子空间的相干态}$$

压缩相干态的涨落

定义正交相位振幅算符

$$\hat{X}_\phi = \frac{1}{2} (\hat{a} e^{-i\phi} + \hat{a}^\dagger e^{i\phi}) = \hat{X}_1 \cos\phi + \hat{X}_2 \sin\phi$$

另一主轴 $\hat{Y}_1 + i\hat{Y}_2 = (\hat{X}_1 + i\hat{X}_2) e^{-i\theta/2} = \hat{a} e^{-i\theta/2}$



$$\sqrt{(\Delta Y_1)^2} = \frac{1}{2} e^{-r}$$

$$\sqrt{(\Delta Y_2)^2} = \frac{1}{2} e^r$$

对单模电场

$$E = i \sqrt{\frac{\hbar\omega}{2\epsilon_0 V}} \left(\hat{a} e^{-i\omega t + i\vec{k}\cdot\vec{r}} - \hat{a}^\dagger e^{i\omega t - i\vec{k}\cdot\vec{r}} \right)$$

$$= 2 \sqrt{\frac{\hbar\omega}{2\epsilon_0 V}} \left[\hat{X}_1 \sinh(\omega t - \vec{k}\cdot\vec{r}) - \hat{X}_2 \cosh(\omega t - \vec{k}\cdot\vec{r}) \right]$$

$$(\Delta E)^2 = \frac{2\hbar\omega}{\epsilon_0 V} \left[(\Delta X_1)^2 \sinh^2(\omega t - \vec{k}\cdot\vec{r}) + (\Delta X_2)^2 \cosh^2(\omega t - \vec{k}\cdot\vec{r}) \right]$$

压缩相干态的粒子数分布

利用 $\hat{S}(\xi) = e^{\frac{1}{2}\theta \hat{a}^\dagger \hat{a}} e^{\frac{r}{2}(\hat{a}^2 - \hat{a}^{\dagger 2})} e^{-\frac{1}{2}\theta \hat{a}^\dagger \hat{a}}$

$$= e^{-\frac{1}{2}\hat{a}^{\dagger 2}} e^{i\theta \tanh r} e^{-\frac{1}{2}(\hat{a}^\dagger \hat{a} + \hat{a} \hat{a}^\dagger)} |n(\cosh r)\rangle e^{\frac{1}{2}\hat{a}^2} e^{-i\theta \tanh r}$$

$$\langle \alpha | \hat{S}(\xi) | 0 \rangle = \langle \alpha | \frac{1}{\sqrt{\cosh r}} \exp\left(-\frac{1}{2}\hat{a}^{\dagger 2} e^{i\theta \tanh r}\right) | 0 \rangle$$

$$= \frac{1}{\sqrt{\cosh r}} \exp\left(-\frac{1}{2}\alpha^{*2} e^{i\theta \tanh r}\right) \langle \alpha | 0 \rangle$$

$$= \frac{1}{\sqrt{\cosh r}} \exp\left(-\frac{1}{2}\alpha^{*2} e^{i\theta \tanh r} - \frac{1}{2}|\alpha|^2\right)$$

$$\begin{aligned}\langle \alpha | \beta, \xi \rangle &= \langle \alpha | \hat{D}(\beta) \hat{S}(\xi) | 0 \rangle = \langle \alpha - \beta | \exp \left[-\frac{1}{2} (\alpha \beta^* - \alpha^* \beta) \right] \hat{S}(\xi) | 0 \rangle \\ &= \frac{1}{\sqrt{\cosh r}} \exp \left\{ -\frac{1}{2} [(\alpha^* - \beta^*)^2 e^{i\theta} \tanh r + |\alpha - \beta|^2 + (\alpha \beta^* - \alpha^* \beta)] \right\}\end{aligned}$$

可求出 $\langle n | \beta, \xi \rangle$.

$$\text{又 } \hat{S}^\dagger(\xi) \hat{a} \hat{S}(\xi) = \hat{a} \cosh r - \hat{a}^\dagger e^{i\theta} \sinh r.$$

$$\hat{S}^\dagger(\xi) \hat{D}^\dagger(\beta) \hat{a} \hat{D}(\beta) \hat{S}(\xi) = \hat{a} \cosh r - \hat{a}^\dagger e^{i\theta} \sinh r + \beta.$$

$$\langle \hat{N} \rangle = \langle \beta, \xi | \hat{a}^\dagger \hat{a} | \beta, \xi \rangle = |\beta|^2 + \sinh^2 r.$$

$$\langle \Delta \hat{N}^2 \rangle = |\beta|^2 \left[e^{-2r} \cos^2 \left(\frac{\theta}{2} - \phi \right) + e^{2r} \sin^2 \left(\frac{\theta}{2} - \phi \right) \right] + 2 \sinh^2 r \cos^2 hr.$$

$$\text{对真空态 } \beta = 0, \quad \langle \Delta \hat{N}^2 \rangle = 2 \sinh^2 hr \cos^2 hr \geq \langle \hat{N} \rangle = \sinh^2 hr.$$

$$\text{对大的 } r, \quad \theta = \phi/2 \text{ 时} \quad \langle \Delta \hat{N}^2 \rangle - \langle \hat{N} \rangle^2 = |\beta|^2 (e^{-2r} - 1) + \sinh^2 hr (2 \cos^2 hr - 1)$$

双模压缩态

两个独立模式场算符 $\hat{a}_1, \hat{a}_2$

$$\hat{S}_{12}(\xi) = \exp(\xi^* \hat{a}_1 \hat{a}_2 - \xi \hat{a}_1^\dagger \hat{a}_2^\dagger)$$

一般压缩真空态展开

$$|0, \xi\rangle = \exp\left(\frac{1}{2}\theta \hat{a}^\dagger \hat{a}\right) \exp\left[\frac{r}{2}(\hat{a}^* - \hat{a}^{\dagger 2})\right] |0\rangle$$

$$= \sqrt{\text{sech } r} \sum_{n=0}^{\infty} \frac{\sqrt{(2n)!}}{n! 2^n} \left[-\exp(i\theta) \tanh r \right]^n |2n\rangle$$

双模压缩真空态展开

$$|\xi\rangle = \frac{1}{\cosh r} \sum_{n=0}^{\infty} \left(-e^{i\theta} \tanh r \right)^n |n,n\rangle$$

$$\exists \lambda \quad \hat{a}_1 = \frac{1}{\sqrt{2}} (\hat{c} - e^{i\delta} \hat{d})$$

$$\hat{a}_2 = \frac{1}{\sqrt{2}} (\hat{d} + e^{-i\delta} \hat{c})$$

$$\hat{S}_{12}(\xi) = \exp \left[-\frac{1}{2} \xi \left(\hat{c}^{\dagger 2} e^{i\delta} - \hat{d}^2 e^{-i\delta} \right) + \frac{1}{2} \xi^* \left(\hat{c}^{\dagger 2} e^{-i\delta} - \hat{d}^2 e^{i\delta} \right) \right]$$

$$= \hat{S}_c(\xi e^{i\delta}) \hat{S}_d(-\xi e^{-i\delta})$$

$\hat{c}, \hat{d}$ 模式上 squeezed vacuum state 直积

奇偶相干态

$$|\alpha\rangle_o = N_o (|\alpha\rangle - |- \alpha\rangle) = (\sinh |\alpha|^2)^{-\frac{1}{2}} \sum_{n=0}^{\infty} \frac{\alpha^{2n+1}}{\sqrt{(2n+1)!}} |2n+1\rangle$$

$$|\alpha\rangle_e = N_e (|\alpha\rangle + |- \alpha\rangle) = (\cosh |\alpha|^2)^{-\frac{1}{2}} \sum_{n=0}^{\infty} \frac{\alpha^{2n}}{\sqrt{(2n)!}} |2n\rangle$$

$$|\alpha\rangle = e^{-\frac{1}{2}|\alpha|^2} \left[\sqrt{\sinh |\alpha|^2} |\alpha\rangle_o + \sqrt{\cosh |\alpha|^2} |\alpha\rangle_e \right]$$

$$\hat{a}^2 |\alpha\rangle_{o(e)} = \hat{a}^2 N_{o(e)} (|\alpha\rangle \mp |- \alpha\rangle) = \alpha^2 |\alpha\rangle_{o(e)}$$

$$\hat{a} |\alpha\rangle_o = \alpha \coth^{\frac{1}{2}} |\alpha|^2 |\alpha\rangle_e$$

$$\hat{a} |\alpha\rangle_e = \alpha \tanh^{\frac{1}{2}} |\alpha|^2 |\alpha\rangle_o$$

涨落特性

正交相位振幅

$$(\Delta X_q)_o^2 = \frac{1}{4} + \frac{1}{2} |\alpha|^2 \left[\cos(2\phi - 2\theta) + \underbrace{\coth |\alpha|^2}_{\geq 1} \right]$$

$$(\Delta X_q)_e^2 = \frac{1}{4} + \frac{1}{2} |\alpha|^2 \left[\cos(2\phi - 2\theta) + \underbrace{\tanh |\alpha|^2}_{\leq 1} \right]$$

最小不确定态

$$[\hat{A}, \hat{B}] = i\hat{C} = [\Delta\hat{A}, \Delta\hat{B}]$$

$$\langle(\Delta\hat{A})^2\rangle\langle(\Delta\hat{B})^2\rangle \geq |\langle\Delta\hat{A}\Delta\hat{B}\rangle|^2$$

$$\begin{aligned}\langle(\Delta\hat{A})^2\rangle\langle(\Delta\hat{B})^2\rangle &= \underbrace{\left|\frac{1}{2}\langle[\Delta\hat{A}, \Delta\hat{B}]\rangle\right|^2}_{\geq \frac{1}{4}|\langle\hat{C}\rangle|^2} + \left|\frac{1}{2}\langle\{\Delta\hat{A}, \Delta\hat{B}\}\rangle\right|^2 \\ &\geq \frac{1}{4}|\langle\hat{C}\rangle|^2\end{aligned}$$

$$\Delta\hat{A}|\psi\rangle = -i\lambda\Delta\hat{B}|\psi\rangle \text{ 时,}$$

$$\langle\{\Delta\hat{A}, \Delta\hat{B}\}\rangle = i\lambda^*\langle(\Delta\hat{B})^2\rangle + (-i\lambda)\langle(\Delta\hat{B})^2\rangle = i(\lambda^* - \lambda)\langle(\Delta\hat{B})^2\rangle$$

$$\langle[\Delta\hat{A}, \Delta\hat{B}]\rangle = i(\lambda^* + \lambda)\langle(\Delta\hat{B})^2\rangle = i\langle\hat{C}\rangle$$

$$\Rightarrow \langle\{\Delta\hat{A}, \Delta\hat{B}\}\rangle = \frac{\lambda^* - \lambda}{\lambda^* + \lambda}\langle\hat{C}\rangle = \frac{\text{Im}\lambda}{\text{Re}\lambda}\langle\hat{C}\rangle$$

$$\langle(\Delta\hat{A})^2\rangle = |\lambda|^2\langle(\Delta\hat{B})^2\rangle$$

对 $\hat{a}, \hat{a}^\dagger$ 定义

$$\hat{A} = \frac{1}{2}(\hat{a} + \hat{a}^\dagger), \quad \hat{B} = \frac{1}{2i}(\hat{a} - \hat{a}^\dagger)$$

对应最小不确定态

$$(\hat{A} + i\lambda\hat{B})|\psi\rangle = (\langle\hat{A}\rangle + i\lambda\langle\hat{B}\rangle)|\psi\rangle$$

$$(\mu\hat{a} + \nu\hat{a}^\dagger)|\psi\rangle = \alpha|\psi\rangle$$

$$\mu = \frac{1}{2}\left(\frac{1}{\lambda} + \sqrt{\lambda}\right), \quad \nu = \frac{1}{2}\left(\frac{1}{\lambda} - \sqrt{\lambda}\right)$$

$\lambda=1$: 相干态

$\lambda\neq 1$: 压缩态

对奇、偶相干态

$$[\hat{k}_+, \hat{k}_-] = -2\hat{k}_0, \quad [\hat{k}_0, \hat{k}_{\pm}] = \pm \hat{k}_{\pm}$$

$$\text{取 } \hat{k}_+ = \frac{1}{2}(\hat{a}^\dagger)^2, \quad \hat{k}_- = \frac{1}{2}\hat{a}^2, \quad \hat{k}_0 = \hat{a}^\dagger \hat{a} + \frac{1}{2}$$

$$\text{定义 } \hat{A} = \hat{k}_x = \frac{1}{2}(\hat{k}_+ + \hat{k}_-) = \frac{\hat{a}^2 + (\hat{a}^\dagger)^2}{4}$$

$$\hat{B} = \hat{k}_y = \frac{1}{2i}(\hat{k}_+ - \hat{k}_-) = \frac{-\hat{a}^2 + (\hat{a}^\dagger)^2}{4}$$

$$\lambda = -1 \text{ 时, } \hat{A} + i\lambda \hat{B} = \hat{k}_x - i\hat{k}_y = \hat{k}_- = \frac{1}{2}\hat{a}^2, \quad \text{奇偶相干态}$$

$$\text{重新定义 } \hat{k}_+ = \hat{a}^\dagger \hat{b}^\dagger, \quad \hat{k}_- = \hat{a} \hat{b}, \quad \hat{k}_0 = \frac{1}{2}(\hat{a}^\dagger \hat{a} + \hat{b} \hat{b}^\dagger)$$

$$\text{配对称干态 } |\Phi(\xi, \eta)\rangle = N(\xi, \eta) \sum_{n=0}^{\infty} \frac{\xi^n}{\sqrt{n!(n+\eta)!}} |n+\eta, n\rangle$$

$$\text{有 } \hat{a} \hat{b} |\Phi(\xi, \eta)\rangle = \xi |\Phi(\xi, \eta)\rangle$$

$$(\hat{a}^\dagger \hat{a} - \hat{b}^\dagger \hat{b}) |\Phi(\xi, \eta)\rangle = \eta |\Phi(\xi, \eta)\rangle$$

$$\text{对比 } |\alpha, \beta\rangle = e^{-\frac{(|\alpha|^2 + |\beta|^2)}{2}} \sum_{m,n=0}^{\infty} \frac{\alpha^m \beta^n}{\sqrt{m!n!}} |m, n\rangle$$

Wigner 函数

$$W(x, p) = \frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} d\xi \exp\left(\frac{-i}{\hbar} p \xi\right) \langle x + \frac{1}{2}\xi | \rho | x - \frac{1}{2}\xi \rangle$$

$$\begin{aligned} v = -\frac{\xi}{\hbar} &= \frac{1}{2\pi} \int_{-\infty}^{\infty} dv \langle x - \frac{1}{2}\hbar v | \rho | x + \frac{1}{2}\hbar v \rangle e^{ipv} \\ &= \frac{1}{(2\pi)^2} \iint dv dx \langle x | \rho | x + \hbar v \rangle 2\pi \delta(x - x' - \hbar v/2) e^{ipv} \end{aligned}$$

$$= \frac{1}{(2\pi)^2} \iint dv du \underbrace{\text{Tr}[\rho e^{-iu\hat{x} - iv\hat{p}}]}_{\chi(u, v)} e^{i(u\alpha + vp)}$$

$$\chi(u, v)$$

$$\chi(u, v) = \text{Tr}[\rho e^{-iu\hat{x} - iv\hat{p}}] = \text{Tr}[\rho e^{-i(\eta\hat{a}^\dagger + \eta^*\hat{a})}] = \chi(\eta, \eta^*)$$

$$\eta = \frac{1}{\sqrt{2}} \left(\hat{p}_T u + i \frac{\hbar v}{\hat{x}_T} \right)$$

$$\chi(u, v) = \text{Tr}[\rho e^{-iu\hat{x} - iv\hat{p}}] = \text{Tr}[\rho e^{\lambda\hat{a}^\dagger - \lambda^*\hat{a}}] = \text{Tr}[\rho \hat{D}(\lambda)] = \chi(\lambda, \lambda^*)$$

$$\lambda = -i\eta \quad \overline{W}(\alpha, \alpha^*) = \frac{1}{\pi^2} \int d^2\lambda \chi(\lambda, \lambda^*) e^{-\lambda\alpha^* + \lambda^*\alpha}$$

Weyl 变换

Fourier Transform

$$\tilde{W}(x, p) = \int_{-\infty}^{\infty} d\xi \exp\left(\frac{-i}{\hbar} p \xi\right) \langle x + \frac{1}{2}\xi | \hat{A} | x - \frac{1}{2}\xi \rangle$$

$$= \int_{-\infty}^{\infty} du \exp\left(\frac{i}{\hbar} x u\right) \langle p + \frac{1}{2}u | \hat{A} | p - \frac{1}{2}u \rangle$$

$$W(x, p) = \tilde{\rho} / 2\pi\hbar$$

$$P\text{-表示} \quad \chi_p(\lambda) = \text{Tr} [e^{\lambda \hat{a}^\dagger} e^{-\lambda^* \hat{a}}]$$

$$Q\text{-表示} \quad \chi_q(\lambda) = \text{Tr} [e^{\lambda^* \hat{a}} e^{\lambda \hat{a}^\dagger}]$$

$$W\text{-表示} \quad \chi_w(\lambda) = \text{Tr} [e^{\lambda \hat{a}^\dagger - \lambda^* \hat{a}}]$$

$$Q(\alpha) = \frac{1}{\pi} \langle \alpha | \rho | \alpha \rangle = \frac{1}{\pi} \int d^2\beta \, P(\beta) e^{-|\alpha-\beta|^2}$$

$$P(\alpha) = \exp\left(-\frac{\partial^2}{\partial \alpha^* \partial \alpha}\right) Q(\alpha, \alpha^*)$$

$$W(\alpha) = \frac{2}{\pi} \int d^2\beta \, P(\beta) e^{-2|\alpha-\beta|^2}$$

$$P(\alpha) = \exp\left(-\frac{1}{2} \frac{\partial^2}{\partial \alpha^* \partial \alpha}\right) W(\alpha, \alpha^*)$$

自旋相干态

$$\{\hat{J}_x, \hat{J}_z\} \text{ 共同本征态: } |j, m\rangle = \underbrace{\binom{2j}{j+m}}^{-\frac{1}{2}} \frac{(\hat{J}_+)^{j+m}}{(j+m)!} |j, -j\rangle \quad \frac{(2j)!}{(j+m)!(j-m)!}$$

$$\hat{J}_\pm = \hat{J}_x \pm i\hat{J}_y, \quad \hat{J}_\pm |j, m\rangle = \sqrt{j(j+1) - m(m\pm 1)} |j, m\pm 1\rangle$$

自旋相干态:

$$\begin{aligned} |\theta, \phi\rangle &= \bigotimes_{\ell=1}^N \left(\cos\frac{\theta}{2} |0\rangle_\ell + e^{i\phi} \sin\frac{\theta}{2} |1\rangle_\ell \right) \\ &= \sum_{m=-j}^j \binom{2j}{j+m}^{\frac{1}{2}} \left(\cos\frac{\theta}{2}\right)^{j+m} \left(e^{i\phi} \sin\frac{\theta}{2}\right)^{j-m} |j, m\rangle \end{aligned}$$

$$= \left(\frac{1}{1+|n|^2} \right)^j \sum_{m=-j}^j \binom{2j}{j+m}^{\frac{1}{2}} \eta^{j-m} |j, m\rangle. \quad \eta = e^{i\phi} \tan \frac{\theta}{2}$$

$$|\theta=0, \phi\rangle = |0\rangle_1 |0\rangle_2 \dots |0\rangle_N = |j, j\rangle$$

$$|\theta=\pi, \phi\rangle = e^{iN\phi} |1\rangle_1 |1\rangle_2 \dots |1\rangle_N = e^{iN\phi} |j, j\rangle$$

$$\text{且 } \cos \frac{\theta}{2} |0\rangle + e^{i\phi} \sin \frac{\theta}{2} |1\rangle = e^{-i\theta \hat{\sigma}_z \cdot \hat{n}/2} |0\rangle$$

$$\begin{aligned} |\theta, \phi\rangle &= \bigotimes_{\ell=1}^N e^{-i\theta \hat{\sigma}_z^{(\ell)} \cdot \hat{n}/2} |0\rangle_{\ell} = \bigotimes_{\ell=1}^N e^{(\xi \hat{\sigma}_+^{(\ell)} - \xi^* \hat{\sigma}_-^{(\ell)})} |0\rangle_{\ell} \\ &= \underbrace{\exp(\xi \hat{J}_+ - \xi^* \hat{J}_-)}_{\hat{R}(\theta, \phi)} |j, j\rangle. \quad \xi = -\frac{\theta}{2} e^{-i\phi} \end{aligned}$$

自旋态的 Husimi-Q 函数

$$Q(\theta, \phi) = |\langle \theta, \phi | \psi \rangle|^2$$

$$\langle (\hat{S}_x^{(i)})^2 \rangle = \langle (\hat{S}_1^{(i)})^2 \rangle = \frac{N}{4}. \quad \text{自旋相干态涨落 1 方向各向同性}$$

压缩系数 ...

光场与原子相互作用

$$\vec{E}(\vec{r}, t) = -\nabla\phi(\vec{r}, t) - \frac{\partial \vec{A}(\vec{r}, t)}{\partial t}$$

$$\vec{B}(\vec{r}, t) = \nabla \times \vec{A}(\vec{r}, t)$$

$\vec{A}(\vec{r}, t)$, $\phi(\vec{r}, t)$ 具有规范不变性

$$\vec{A}'(\vec{r}, t) = \vec{A}(\vec{r}, t) + \nabla \xi(\vec{r}, t)$$

$$\Phi'(\vec{r}, t) = \Phi(\vec{r}, t) - \frac{\partial \xi(\vec{r}, t)}{\partial t}$$

带电粒子与电磁场耦合

$$\hat{H} = \frac{1}{2m} [\vec{p} - e\vec{A}(\vec{r}, t)]^2 + e\Phi(\vec{r}, t) + V(\vec{r})$$

$$\hat{H}'(\vec{r}, t) = \frac{1}{2m} [\vec{p} - e\vec{A}'(\vec{r}, t)]^2 + e\Phi'(\vec{r}, t) + V(\vec{r})$$

$$\psi'(\vec{r}, t) = R\psi(\vec{r}, t)$$

$$R = \exp[ie\xi(\vec{r}, t)/\hbar]$$

不同相因子

不同规范切换

可观测量应具有规范不变性。

改写 Schrödinger 方程

$$\hat{\pi} = m \frac{\partial \hat{H}}{\partial \vec{p}} = \vec{p} - e\vec{A}(\vec{r}, t)$$

$$\underbrace{\left[i\hbar \frac{\partial}{\partial t} - e\Phi(\vec{r}, t) \right]}_{\hat{E}^g(\vec{r}, t)} \psi^g(\vec{r}, t) = \underbrace{\left[\frac{\hat{\pi}^2}{2m} + V(\vec{r}) \right]}_{\hat{\varepsilon}^g} \psi^g(\vec{r}, t)$$

$\hat{E}^g(\vec{r}, t)$

$\hat{\varepsilon}^g$ 系统能量算符

规范不变 Schrödinger 方程

也可以认为是由相对论协变形式得到

$$P^\mu = (E, \vec{\pi}) \rightarrow i\hbar D^\mu = i\hbar \left(\partial^\mu + i\frac{e}{\hbar} A^\mu \right)$$

库仑规范

$$\Phi^c(\vec{r}, t) = 0, \quad \nabla \cdot \vec{A}^c(\vec{r}, t) = 0.$$

$$\hat{E}^c(\vec{r}, t) = i\hbar \frac{\partial}{\partial t}$$

$$\hat{\mathcal{E}}^c(\vec{r}, t) = \frac{1}{2m} [\vec{p} - e\vec{A}^c(\vec{r}, t)]^2 + V(\vec{r}) \equiv \hat{H}^c$$

$$i\hbar \frac{\partial}{\partial t} \psi^c(\vec{r}, t) = \hat{H}^c \psi^c(\vec{r}, t)$$

若利用自由原子 $\hat{H}_0$ 本征态展开 $\psi^c(\vec{r}, t)$

$$\psi^c(\vec{r}, t) = \sum_i \alpha_i \phi_i(\vec{r}, t)$$

$$i\hbar \frac{d}{dt} \alpha_i = E_i \alpha_i + \sum_j \alpha_j \langle \phi_i(t) | \hat{\mathcal{E}}_{int}^c(\vec{r}, t) | \phi_j(t) \rangle$$

$$\hat{\mathcal{E}}_{int}^c(\vec{r}, t) = -\frac{e}{m} \vec{A}^c(\vec{r}, t) \cdot \vec{p} + \frac{e^2}{2m} [\vec{A}^c(\vec{r}, t)]^2$$

跃迁矩阵元 $T_{ij} = \langle \phi_i(t) | \mathcal{E}_{int}^c(\vec{r}, t) | \phi_j(t) \rangle$

对手面波 $\vec{A}^c(\vec{r}, t) = \vec{A}_0^c e^{i(\vec{k} \cdot \vec{r} - \omega t)} + c.c.$

原子位置 $\vec{r}_0$ 附近 $\vec{A}^c(\vec{r}, t) \simeq \vec{A}^c(\vec{r}_0, t)$ $[\vec{A}^c]^2$ 不引起跃迁

$$T_{ij}^c = \langle \phi_i | -\frac{e}{m} \vec{A}^c(\vec{r}_0, t) \cdot \vec{p} | \phi_j \rangle$$

电场规范

电偶极近似下选择规范变换

$$\chi^{ec}(\vec{r}, t) = -\vec{A}^c(\vec{r}_0, t) \cdot \vec{r}$$

$$\vec{A}^c(\vec{r}, t) = \vec{A}^c(\vec{r}, t) + \nabla \chi^{ec} = 0$$

$$\Phi^c(\vec{r}, t) = -\frac{\partial}{\partial t} \chi^{ec}(\vec{r}, t) = \vec{r} \cdot \frac{\partial}{\partial t} \vec{A}^c(\vec{r}, t) = -\vec{r} \cdot \vec{E}(\vec{r}, t)$$

$$\Rightarrow \vec{E}^e(\vec{r}, t) = i\hbar \frac{\partial}{\partial t} - q\Phi^e(\vec{r}, t) = i\hbar \frac{\partial}{\partial t} + q\vec{r} \cdot \vec{E}(\vec{r}, t)$$

$$\hat{\xi}^e(\vec{r}, t) = \frac{\vec{p}^2}{2m} + V(\vec{r}) = \hat{H}_0$$

电场规范

$$\hat{\xi}^e(\vec{r}) \phi_i(\vec{r}) = E_i \phi_i(\vec{r})$$

$$i\hbar \frac{\partial}{\partial t} \psi^e(\vec{r}, t) = \left[\frac{\vec{p}^2}{2m} + V(\vec{r}) - q\vec{r} \cdot \vec{E}(\vec{r}, t) \right] \psi^e(\vec{r}, t)$$

$$\text{引} \lambda \quad \hat{H}_I = -\vec{d} \cdot \vec{E}(\vec{r}, t) \quad \vec{d} = q\vec{r} = -e\vec{r}$$

$$T_{ij}^e = \langle \phi_i | -q\vec{r} \cdot \vec{E}(\vec{r}, t) | \phi_j \rangle$$

与库仑规范量子态的关系

$$\psi^e(\vec{r}, t) = R^{ec} \psi^c(\vec{r}, t) = \exp[iq\chi^{ec}(\vec{r}, t)/\hbar] \psi^c(\vec{r}, t) = \exp[-iq\vec{r} \cdot \vec{A}^c(\vec{r}, t)/\hbar] \psi^c(\vec{r}, t)$$

矩阵元关系

$$\vec{E}(\vec{r}, t) = \vec{E}_0 \sin \omega t = E(\vec{r}, \omega) e^{-i\omega t} + E^*(\vec{r}, \omega) e^{i\omega t}$$

$$\vec{A}^c(\vec{r}, t) = \frac{\vec{E}_0}{\omega} \cos \omega t = \vec{A}^c(\vec{r}, \omega) e^{-i\omega t} + (\vec{A}^c)^*(\vec{r}, \omega) e^{i\omega t}$$

$$\vec{A}^c(\vec{r}, \omega) = -i\vec{E}(\vec{r}, \omega)/\omega \quad \text{又} \quad \vec{p} = \frac{m}{i\hbar} [\vec{r}, \hat{H}_0]$$

$$\langle \phi_i | -\frac{q}{m} \vec{p} \cdot \vec{A}^c(\vec{r}, \omega) | \phi_j \rangle = \underbrace{\frac{E_i - E_j}{\hbar \omega}} \langle \phi_i | -q\vec{r} \cdot \vec{E}(\vec{r}, \omega) | \phi_j \rangle$$

经典光场与原子相互作用



$$\hat{H}_0 = \hbar\omega_e |e\rangle\langle e| + \hbar\omega_g |g\rangle\langle g|$$

$$|\phi(t)\rangle = \tilde{U}(t) |\psi(t)\rangle$$

$$i\hbar \frac{d|\phi(t)\rangle}{dt} = \underbrace{(\tilde{U} H \tilde{U}^\dagger + i\hbar \frac{d\tilde{U}}{dt} \tilde{U}^\dagger)}_{H_{\text{eff}}} |\phi(t)\rangle$$

$$\textcircled{1} \tilde{U} = e^{iH_0 t/\hbar} \quad H_{\text{eff}} = H_0 + i\hbar \frac{d\tilde{U}}{dt} \tilde{U}^\dagger = 0$$

$$\textcircled{2} \tilde{U} = e^{i\frac{\omega t}{2} \sigma_z} \quad H_{\text{eff}} = H_0 + i\hbar \frac{d\tilde{U}}{dt} \tilde{U}^\dagger = H_0 - \frac{\hbar\omega}{2} \sigma_z$$

引入相互作用 $V = -\frac{\hbar}{2} \left[\Omega_{eg} (e^{i\omega t} + e^{-i\omega t}) |e\rangle\langle g| + \Omega_{ge} (e^{i\omega t} + e^{-i\omega t}) |g\rangle\langle e| \right]$

$$\textcircled{1} \tilde{U} = e^{iH_0 t/\hbar} \quad H_{\text{eff}} = H_0 + e^{iH_0 t/\hbar} V e^{-iH_0 t/\hbar} - H_0 = e^{iH_0 t/\hbar} V e^{-iH_0 t/\hbar}$$

$$H_0 = \hbar \frac{\omega_e + \omega_g}{2} \mathbb{1} + \frac{\hbar}{2} (\Delta + \omega) \sigma_z \quad H_{\text{eff}} = e^{i\frac{t}{2}(\Delta + \omega) \sigma_z} V e^{-i\frac{t}{2}(\Delta + \omega) \sigma_z}$$

$$H_{\text{eff}} = \begin{pmatrix} e^{i\frac{(\Delta + \omega)}{2}t} & 0 \\ 0 & e^{-i\frac{(\Delta + \omega)}{2}t} \end{pmatrix} \cdot \left(-\frac{\hbar}{2}\right) \begin{pmatrix} \Omega_{eg}(e^{i\omega t} + e^{-i\omega t}) & 0 \\ \Omega_{ge}(e^{i\omega t} + e^{-i\omega t}) & 0 \end{pmatrix} \begin{pmatrix} e^{-i\frac{(\Delta + \omega)}{2}t} & 0 \\ 0 & e^{i\frac{(\Delta + \omega)}{2}t} \end{pmatrix}$$

$$= -\frac{\hbar}{2} \begin{pmatrix} \Omega_{eg}(e^{i2\omega t + i\Delta t} + e^{i\Delta t}) & 0 \\ \Omega_{ge}(e^{i\Delta t} + e^{-i2\omega t - \Delta t}) & 0 \end{pmatrix}$$

$$\textcircled{2} U = e^{i\frac{\omega t}{2} \sigma_z} \quad H_{\text{eff}} = e^{i\frac{\omega t}{2} \sigma_z} H e^{-i\frac{\omega t}{2} \sigma_z} - \frac{\hbar\omega}{2} \sigma_z$$

$$H = \hbar \frac{\omega_e + \omega_g}{2} \mathbb{1} + \frac{\hbar}{2} (\Delta + \omega) \sigma_z - \frac{\hbar}{2} \begin{pmatrix} \Omega_{eg}(e^{i\omega t} + e^{-i\omega t}) & 0 \\ \Omega_{ge}(e^{i\omega t} + e^{-i\omega t}) & 0 \end{pmatrix}$$

$$= \frac{\hbar}{2} \frac{\omega_e + \omega_g}{2} \mathbb{1} + \frac{\hbar}{2} \begin{pmatrix} (\Delta + \omega) & -\Omega_{eg}(e^{2i\omega t} + 1) \\ \Omega_{ge}(1 + e^{-2i\omega t}) & -(\Delta + \omega) \end{pmatrix}$$

$$H_{\text{eff}} = \frac{\hbar}{2} \frac{\omega_e + \omega_g - \omega}{2} \mathbb{1} + \frac{\hbar}{2} \begin{pmatrix} (\Delta + \omega) & -\Omega_{eg}(e^{2i\omega t} + 1) \\ \Omega_{ge}(1 + e^{-2i\omega t}) & -(\Delta + \omega) \end{pmatrix}$$

$$= \frac{\hbar}{2} (\omega_e + \omega_g) \mathbb{1} + \frac{\hbar}{2} \begin{pmatrix} \Delta & -\Omega_{eg}(1 + \underbrace{e^{2i\omega t}}) \\ \Omega_{ge}(1 + \underbrace{e^{-2i\omega t}}) & -\Delta \end{pmatrix}$$

$$\approx \frac{\hbar}{2} (\omega_e + \omega_g) \mathbb{1} + \frac{\hbar}{2} \Delta \sigma_z - \frac{\hbar}{2} (\Omega_{eg} |e\rangle\langle g| + \Omega_{ge} |g\rangle\langle e|)$$

$$\text{取 } \tilde{H}_{\text{eff}} = \frac{\hbar}{2} \left[\Delta \sigma_z - \Omega_R (e^{i\phi} \hat{\sigma}_+ + e^{-i\phi} \hat{\sigma}_-) \right] \quad \begin{matrix} \Delta = \omega_{eg} - \omega \\ |\Omega_{eg}| = |\Omega_{ge}| = \Omega_R \end{matrix}$$

$$|\tilde{\psi}\rangle = C_e(t) |e\rangle + C_g(t) |g\rangle$$

$$\begin{bmatrix} \dot{C}_e(t) \\ \dot{C}_g(t) \end{bmatrix} = -\frac{i}{2} \begin{bmatrix} \Delta & -\Omega_R e^{i\phi} \\ -\Omega_R e^{-i\phi} & -\Delta \end{bmatrix} \begin{bmatrix} C_e(t) \\ C_g(t) \end{bmatrix} \quad \Omega = \sqrt{\Omega_R^2 + \Delta^2}$$

$$\Rightarrow \begin{bmatrix} C_e(t) \\ C_g(t) \end{bmatrix} = \begin{bmatrix} \cos\left(\frac{\Omega t}{2}\right) - i \frac{\Delta}{2} \sinh\left(\frac{\Omega t}{2}\right) & i \frac{\Omega_R e^{i\phi}}{\Omega} \sinh\left(\frac{\Omega t}{2}\right) \\ i \frac{\Omega_R e^{-i\phi}}{\Omega} \sinh\left(\frac{\Omega t}{2}\right) & \cos\left(\frac{\Omega t}{2}\right) + i \frac{\Delta}{2} \sinh\left(\frac{\Omega t}{2}\right) \end{bmatrix} \begin{bmatrix} C_e(0) \\ C_g(0) \end{bmatrix}$$

$$\text{取 } C_e(0) = 1$$

$$W(t) = \langle \hat{\sigma}_z \rangle = |C_e(t)|^2 - |C_g(t)|^2$$

$$= \frac{\Delta - \Omega_R^2}{\Omega^2} \sinh^2\left(\frac{\Omega t}{2}\right) + \cos^2\left(\frac{\Omega t}{2}\right)$$

$$\Delta = 0 \quad \Omega = \Omega_R$$

$$W(t) = \cos^2(\Omega_R t)$$

光学 Bloch 方程

$$i\hbar \frac{d}{dt} \hat{\rho}(t) = [\tilde{H}, \hat{\rho}(t)]$$

$$\frac{d}{dt} \rho_{ee}(t) = \frac{1}{i\hbar} (\tilde{H}_{eg} \rho_{ge} - \rho_{eg} \tilde{H}_{ge}) = \frac{i}{2} (\Omega_R e^{i\phi} \rho_{ge} - \Omega_R e^{-i\phi} \rho_{eg})$$

$$\rho_{gg}(t) = 1 - \rho_{ee}(t)$$

$$\frac{d}{dt} \rho_{eg}(t) = \frac{1}{i\hbar} [(\tilde{H}_{ee} - \tilde{H}_{gg}) \rho_{eg} - \tilde{H}_{eg} (\rho_{ee} - \rho_{gg})] = -i\Delta \rho_{eg} - i\frac{\Omega_R e^{i\phi}}{2} (\rho_{ee} - \rho_{gg})$$

$$\rho = \frac{1}{2}I + \rho_{eg} \hat{\sigma}_+ + \rho_{ge} \hat{\sigma}_- + \frac{1}{2}(\rho_{ee} - \rho_{gg}) \hat{\sigma}_z$$

$$= \frac{1}{2} \left(I + \underline{R_1} \hat{\sigma}_x + \underline{R_2} \hat{\sigma}_y + \underline{R_3} \hat{\sigma}_z \right)$$

$$\begin{matrix} \rho_{eg} + \rho_{ge} & \rho_{eg} - \rho_{ge} & \rho_{ee} - \rho_{gg} \end{matrix}$$

$$\Rightarrow \dot{\underline{R}} = \frac{1}{\hbar} \begin{pmatrix} R_e \langle g | \tilde{H} | e \rangle \\ I_m \langle g | \tilde{H} | e \rangle \\ \hbar \Delta / 2 \end{pmatrix} = \begin{pmatrix} -\Omega \cos \phi \\ \Omega \sin \phi \\ \Delta \end{pmatrix}$$

引入耗散 对角元 Γ

$$\text{非对角元 } \gamma_{\pm} = \Gamma/2 + \gamma_c$$

$$\frac{d}{dt} \rho_{ee}(t) = -\Gamma \rho_{ee} + \frac{i}{2} (\Omega_R e^{i\phi} \rho_{ge} - \Omega_R e^{-i\phi} \rho_{eg})$$

$$\frac{d}{dt} \rho_{gg}(t) = \Gamma \rho_{ee} - \frac{i}{2} (\Omega_R e^{i\phi} \rho_{ge} - \Omega_R e^{-i\phi} \rho_{eg})$$

$$\frac{d}{dt} \rho_{eg}(t) = -(\gamma_L + i\Delta) \rho_{eg} - i\frac{\Omega_R e^{i\phi}}{2} (\rho_{ee} - \rho_{gg})$$

$$\vec{R} = - \begin{pmatrix} R_1/T_2 \\ R_2/T_2 \\ (R_3+1)/T_1 \end{pmatrix} + \vec{Q} \cdot \vec{R}$$

$$T = 1/T_1, \quad \gamma_c = 1/T_c$$

$$\gamma_1 = \frac{1}{T_2} = \frac{1}{2T_1} + \frac{1}{T_c}$$

稳态

$$\rho_{ee} = \frac{\gamma_{\perp} |\Omega_R|^2}{2[\Gamma(\gamma_{\perp}^* + \Delta^2) + \gamma_{\perp} |\Omega_R|^2]}$$

$$\rho_{eg} = -i \frac{\Omega_R e^{i\phi} (\gamma_{\perp} - i\Delta)}{2[\Gamma(\gamma_{\perp}^* + \Delta^2) + \gamma_{\perp} |\Omega_R|^2]}$$

$$3) \lambda \quad S = \frac{|\Omega_R|^2 / \Gamma \gamma_{\perp}}{1 + \Delta^2 / \gamma_{\perp}^2} \sim \frac{1}{\Delta^2 + \gamma_{\perp}^2}$$

$$\rho_{ee} = \frac{1}{2} \frac{S}{1+S}, \quad \rho_{g} - \rho_{ee} = \frac{1}{1+S}, \quad |\rho_{eg}|^2 = \frac{\Gamma}{4\gamma_{\perp}} \frac{S}{(1+S)^2}$$

Landau-Zenner 模型

能级间距 $\propto t$

$$\hat{H}_{LZ} = \frac{1}{2} (vt \hat{\sigma}_z + \Omega \hat{\sigma}_x) = \begin{pmatrix} vt/2 & \Omega/2 \\ \Omega/2 & -vt/2 \end{pmatrix}$$

$$\hat{E}_{\pm}(t) = \pm \frac{1}{2} \sqrt{v^2 t^2 + \Omega^2}$$

$$|\psi(t)\rangle = C_1(t) |\uparrow\rangle + C_2(t) |\downarrow\rangle$$

$$\Rightarrow \quad \dot{C}_{\uparrow} = -\frac{i}{2} vt C_1(t) - \frac{i}{2} \Omega C_2(t)$$

$$\dot{C}_{\downarrow} = -\frac{i}{2} \Omega C_1(t) + \frac{i}{2} vt C_2(t)$$

$$\Rightarrow \quad \ddot{C}_{\uparrow} + \left(\frac{\Omega^2 + v^2 t^2}{4} + \frac{i}{2} v \right) C_{\uparrow} = 0$$

$$C_T(t \rightarrow \infty) = e^{-\pi k^2/4\nu}$$

布居与能级变化速度关系

另: 设 $C_T(t) = |C_T(t)| e^{-i\phi(t)}$

$$\left(-\dot{\phi}^2 - i\ddot{\phi} + \frac{1}{2}\nu + \frac{\Omega^2 + \nu^2 t^2}{4} \right) C_T = 0$$

$$\Rightarrow \dot{\phi} = \pm \frac{1}{2} \sqrt{\Omega^2 + \nu^2 t^2}$$

$$\ddot{\phi} = \frac{\nu}{2}$$

$$\left(\frac{\dot{C}_T}{C_T} \right)_{t \rightarrow \pm\infty} = -i\dot{\phi}_{t \rightarrow \pm\infty} \simeq -i \left(\pm \nu t + \frac{1}{4} \frac{\Omega^2}{\nu t} \right)$$

$$\int_{-\infty}^{+\infty} \frac{\dot{C}_T}{C_T} dt = - \int_C \frac{\dot{C}_T}{C_T}(z) dz$$

...

$$- \int_C \frac{\dot{C}_T}{C_T}(z) dz = \mp \frac{\pi \Omega^2}{4\nu}$$

得 $C_T(+\infty) = \underbrace{C_T(-\infty)}_{=1} \exp\left(\mp \frac{\pi \Omega^2}{4\nu}\right)$

$$P_{L2} = |C_T(+\infty)|^2 = \exp(-\pi \Omega^2/2\nu)$$

量子光场与原子相互作用

Heisenberg 表象 $\hat{E}(\vec{r}, t) = i \sum_{\vec{k}, \sigma} \hat{e}_{\sigma} \sqrt{\frac{\hbar \omega_k}{2 \epsilon_0 V}} \left(\hat{a}_{\vec{k}, \sigma} e^{-i \omega_k t} - \hat{a}_{\vec{k}, \sigma}^{\dagger} e^{i \omega_k t} \right)$

Schrödinger 表象 $\hat{E}(\vec{r}, t) = i \sum_{\vec{k}, \sigma} \hat{e}_{\sigma} \sqrt{\frac{\hbar \omega_k}{2 \epsilon_0 V}} \left(\hat{a}_{\vec{k}, \sigma} - \hat{a}_{\vec{k}, \sigma}^{\dagger} \right)$

代入得 $\hat{H} = \sum_i E_i |i\rangle\langle i| + \hbar \sum_{\vec{k}, \sigma} \omega_k \left(\hat{a}_{\vec{k}, \sigma}^{\dagger} \hat{a}_{\vec{k}, \sigma} + \frac{1}{2} \right) + i \hbar \sum_{\vec{k}, \sigma} \sum_{i,j} \underbrace{g_{\vec{k}, ij}}_{\substack{\text{偶极矩} \\ \text{耦合}}} |i\rangle\langle j| \left(\hat{a}_{\vec{k}, \sigma} - \hat{a}_{\vec{k}, \sigma}^{\dagger} \right)$

对 TLS $\hat{H} = \frac{1}{2} \hbar \omega_0 (|e\rangle\langle e| - |g\rangle\langle g|) + \hbar \omega (\hat{a}^{\dagger} \hat{a} + \frac{1}{2}) - \frac{\omega_k}{\sqrt{2 \epsilon_0 \hbar V}} \vec{d}_{ij} \cdot \hat{e}_{\sigma} \hat{a}_{\vec{k}, \sigma}$
 $\vec{d}_{ij} = \langle i | \vec{d} | j \rangle$
 $+ \hbar \left[\lambda |e\rangle\langle g| (\hat{a} - \hat{a}^{\dagger}) + \lambda^* |g\rangle\langle e| (\hat{a} - \hat{a}^{\dagger}) \right]$

$$\lambda = -i \sqrt{\frac{\omega_k}{2 \epsilon_0 \hbar V}} \vec{d}_{eg} \cdot \hat{e}_{\sigma}$$

Jaynes-Cummings 模型 $\omega_0 \sim \omega$

$$\hat{H}_{JC} = \frac{1}{2} \hbar \omega_0 \hat{\sigma}_z + \hbar \omega \hat{a}^{\dagger} \hat{a} + \hbar \lambda (\hat{\sigma}_+ \hat{a} + \hat{\sigma}_- \hat{a}^{\dagger})$$

总激发数 $\hat{N}_e = \hat{a}^{\dagger} \hat{a} + |e\rangle\langle e| = \hat{a}^{\dagger} \hat{a} + I - |g\rangle\langle g|$ $[H, N_e] = 0$

$$\Rightarrow \hat{H} = \underbrace{\hbar \omega N_e - \frac{\hbar \omega}{2} I}_{\hat{H}_I} + \underbrace{\frac{\hbar \Delta}{2} \hat{\sigma}_z + \hbar \lambda (\hat{\sigma}_+ \hat{a} + \hat{\sigma}_- \hat{a}^{\dagger})}_{\hat{H}_{II}^*}$$

$$|\psi(t)\rangle = \left[C_n^e(t) |e, n\rangle + C_{n+1}^g(t) |g, n+1\rangle \right] e^{-i(n+\frac{1}{2})\omega t}$$

代入解得 $\begin{bmatrix} C_n^e(t) \\ C_{n+1}^g(t) \end{bmatrix} = \begin{bmatrix} \cos\left(\frac{\Omega_n t}{2}\right) - i \frac{\Delta}{\Omega_n} \sin\left(\frac{\Omega_n t}{2}\right) & -i \frac{\lambda_n}{\Omega_n} \sinh\left(\frac{\Omega_n t}{2}\right) \\ -i \frac{\lambda_n}{\Omega_n} \sinh\left(\frac{\Omega_n t}{2}\right) & \cos\left(\frac{\Omega_n t}{2}\right) + i \frac{\Delta}{\Omega_n} \sinh\left(\frac{\Omega_n t}{2}\right) \end{bmatrix} \cdot \begin{bmatrix} C_n^e(0) \\ C_{n+1}^g(0) \end{bmatrix}$

若 $C_{e,n}(0) = 1$ $C_{g,n+1}(0) = 0$ $\Delta = 0$

$$|\psi(t)\rangle = \left[\cos\left(\frac{\Omega_n t}{2}\right) |e, n\rangle - i \sinh\left(\frac{\Omega_n t}{2}\right) |g, n+1\rangle \right] e^{-i(n+\frac{1}{2})\omega t}$$

$$W(t) = |C_n^e(t)|^2 - |C_{n+1}^g(t)|^2 = \cos(\Omega_n t)$$

一般地 $|\psi(0)\rangle = \sum_{n=0}^{\infty} C_n |n\rangle$ 光场 相干态 $C_n = e^{-|\alpha|^2/2} \alpha^n / \sqrt{n!}$

$$|\psi(t)\rangle = \sum_{n=0}^{\infty} C_n e^{-i(n+\frac{1}{2})\omega t} \left[\cos\left(\frac{\Omega_n t}{2}\right) |e\rangle \otimes |n\rangle - i \sin\left(\frac{\Omega_n t}{2}\right) |g\rangle \otimes |n+1\rangle \right]$$

$$W(t) = \langle \psi(t) | \hat{\sigma}_z | \psi(t) \rangle = \sum_{n=0}^{\infty} |C_n|^2 \cos(\Omega_n t)$$

$$= e^{-|\alpha|^2} \sum_{n=0}^{\infty} \frac{|\alpha|^{2n}}{n!} \cos(2\lambda\sqrt{n+1} t) \quad \text{相干态}$$

collapse and revival

缀饰态 Jc 模型中 $\hat{N}_e$ 守恒. 不同激发数构成的子空间彼此相互独立.

对于 $N_e = n+1$, $\hat{H}_n$ 由 $\{|e, n\rangle, |g, n+1\rangle\}$ 张成.

$$\hat{H}_n = \begin{bmatrix} (n+\frac{1}{2})\hbar\omega + \frac{1}{2}\hbar\Delta & \hbar\lambda\sqrt{n+1} \\ \hbar\lambda\sqrt{n+1} & (n+\frac{1}{2})\hbar\omega - \frac{1}{2}\hbar\Delta \end{bmatrix}$$

$$E_{n,\pm} = (n+\frac{1}{2})\hbar\omega \pm \frac{1}{2}\hbar \Omega_n(\Delta) \quad \sqrt{\Delta^2 + 4\lambda^2(n+1)} \quad \text{正规模式频率}$$

多光子过程

双光子耦合 Hamiltonian

$$\hat{H} = \hbar\omega \hat{a}^\dagger \hat{a} + \frac{1}{2}\hbar\omega_s \hat{\sigma}_z + \hbar\lambda (\hat{\sigma}_+ \hat{a}^{\dagger 2} + \hat{\sigma}_- \hat{a}^2)$$

$$\hat{H}_I = \hbar\omega \hat{a}^\dagger \hat{a} + \hbar\omega_s \hat{\sigma}_z$$

$$\hat{H}_{II} = \frac{\hbar\Delta^{(s)}}{2} \hat{\sigma}_z + \hbar\lambda (\hat{\sigma}_+ \hat{a}^2 + \hat{\sigma}_- \hat{a}^{\dagger 2}) \quad \Delta^{(s)} = \omega_s - 2\omega$$

$$|n, +\rangle^{(s)} = \cos\theta_n^{(s)} |g, n+2\rangle + \sin\theta_n^{(s)} |e, n\rangle$$

$$|n, -\rangle = -\sin\theta_n^{(s)} |g, n+2\rangle + \cos\theta_n^{(s)} |e, n\rangle$$

$$\hat{H}_{II} |n, \pm\rangle = \pm \frac{1}{2} \Omega_n^{(0)} |n, \pm\rangle$$

$$\Omega_n^{(0)} = \sqrt{(\Delta^{(0)})^2 + 4\lambda^2 (n+1)(n+2)}$$

$$E = \hbar[(n+1)\omega \pm \frac{1}{2}\Omega_n]$$

$k \geq 3$ 多光子耦合过程

$$\hat{H} = \hbar\omega \hat{a}^\dagger \hat{a} + \frac{1}{2}\hbar\omega_s \hat{\sigma}_z + \hbar\lambda (\hat{\sigma}_+ \hat{a}^k + \hat{\sigma}_- \hat{a}^{\dagger k})$$

$$\hat{H}_I = \hbar\omega (\hat{a}^\dagger \hat{a} + \frac{k}{2}\hat{\sigma}_z)$$

$$\hat{H}_{II} = \frac{\hbar\Delta^{(0)}}{2} \hat{\sigma}_z + \hbar\lambda (\hat{\sigma}_+ \hat{a}^k + \hat{\sigma}_- \hat{a}^{\dagger k})$$

$$\text{构造 } |n, +\rangle^{(k)} = \cos\theta_n^{(k)} |g, n+k\rangle + \sin\theta_n^{(k)} |e, n\rangle$$

$$|n, -\rangle^{(k)} = -\sin\theta_n^{(k)} |g, n+k\rangle + \cos\theta_n^{(k)} |e, n\rangle$$

$$\text{有 } \hat{H}_I |n, \pm\rangle^{(k)} = (n + \frac{k}{2}) \hbar\omega |n, \pm\rangle^{(k)}$$

$$\hat{H}_{II} |n, \pm\rangle^{(k)} = \pm \frac{1}{2} \Omega_n^{(k)} |n, \pm\rangle^{(k)}$$

$$\Omega_n^{(k)} = \sqrt{(\Delta^{(k)})^2 + 4\lambda^2 (n+1)(n+2)\dots(n+k)}$$

$$|\psi(t)\rangle = e^{-i\hat{H}_I t/\hbar} e^{-i\hat{H}_{II} t/\hbar} |\psi(0)\rangle \quad \text{展开为微扰级数...}$$

大失谐近似 $\Delta \gg \lambda$

$$\text{正则变换方法 } \hat{H} = \hat{H}_0 + \epsilon \hat{V}$$

$$\text{假设 } \hat{H}_0 |m\rangle = E_m |m\rangle, \quad \langle m | \hat{V} | m \rangle = 0$$

$$\text{若不满足则引入 } \hat{H}' = H_0 + \sum_m \langle m | V | m \rangle$$

$$\hat{V}' = \hat{V} - \sum_m \langle m | V | m \rangle$$

寻找算符 $\hat{S}$.

$$\begin{aligned}\hat{H}_{\text{eff}} &= e^{\varepsilon \hat{S}} \hat{H} e^{-\varepsilon \hat{S}} = \hat{H} + \varepsilon [\hat{S}, \hat{H}] + \frac{1}{2} \varepsilon^2 [\hat{S}, [\hat{S}, \hat{H}]] + o(\varepsilon^3) \\ &= \hat{H}_0 + \varepsilon (\hat{V} + [\hat{S}, \hat{H}_0]) + \frac{1}{2} \varepsilon^2 [\hat{S}, [\hat{S}, \hat{H}_0]] + \varepsilon^2 [\hat{S}, \hat{V}] + o(\varepsilon^3)\end{aligned}$$

若 $[\hat{S}, \hat{H}_0] = -\hat{V}$ 则

$$\hat{H}_{\text{eff}} = \hat{H}_0 + \frac{1}{2} \varepsilon^2 [\hat{S}, \hat{V}] + o(\varepsilon^3)$$

大失谐 J-C 模型

$$\hat{H} = \underbrace{\frac{\hbar \Delta}{2} \hat{\sigma}_z}_{\hat{H}_0} + \underbrace{\hbar \lambda (\hat{\sigma}_+ \hat{a} + \hat{\sigma}_- \hat{a}^\dagger)}_{\hat{V}} \quad \Delta \gg \lambda$$

$$[\hat{H}_0, \hat{V}] = \hbar \lambda \Delta (\hat{\sigma}_+ \hat{a} - \hat{a}^\dagger \hat{\sigma}_-) \quad \text{设 } \hat{S} = \eta (\hat{\sigma}_+ \hat{a} - \hat{a}^\dagger \hat{\sigma}_-)$$

$$[\hat{S}, \hat{H}_0] = -\hbar \Delta \eta (\hat{\sigma}_+ \hat{a} + \hat{\sigma}_- \hat{a}^\dagger) = -\hat{V} \quad \Rightarrow \quad \eta = \frac{\lambda}{\Delta}$$

$$[\hat{S}, \hat{V}] = \hbar \frac{\lambda^2}{\Delta} [\hat{\sigma}_+ \hat{a} - \hat{a}^\dagger \hat{\sigma}_-, \hat{\sigma}_+ \hat{a} + \hat{a}^\dagger \hat{\sigma}_-] = 2\hbar \frac{\lambda^2}{\Delta} (\hat{\sigma}_+ \hat{\sigma}_- + \hat{\sigma}_- \hat{a}^\dagger \hat{a})$$

$$\hat{H}_{\text{eff}} = \hat{H}_0 + \frac{1}{2} \varepsilon^2 [\hat{S}, \hat{V}]$$

$$= \frac{\hbar \Delta}{2} \hat{\sigma}_z + \hbar \frac{\lambda^2}{\Delta} (\hat{\sigma}_+ \hat{\sigma}_- + \hat{\sigma}_- \hat{a}^\dagger \hat{a})$$

自发辐射

$$|e\rangle \xrightarrow{\quad} N_e$$

$$|g\rangle \xrightarrow{\quad} N_g$$

$$0 = \frac{dN_e}{dt} = -N_e A_{eg} - N_e B_{eg} \rho(\omega_{eg}) + N_g B_{ge} \rho(\omega_{eg})$$

自发辐射

受激辐射

受激吸收

$\rho(\omega_{eg})$: 外场能量密度

多模式 J-C 模型

$$\hat{H}_I = \hbar \sum_{\lambda} \left[g_{\lambda} |e\rangle\langle g| \hat{a}_{\lambda} e^{i\delta\omega_{\lambda}t} + h.c. \right] \quad \delta\omega_{\lambda} = \omega_{\lambda} - \omega_{eg}$$

$$|\phi(t)\rangle = |\phi(0)\rangle + \frac{1}{i\hbar} \int_0^t d\tau \hat{H}_I(\tau) |\phi(0)\rangle \quad (\text{Born 近似})$$

$$|\phi(0)\rangle = |e\rangle|0\rangle \rightarrow |g\rangle|1_{\lambda}\rangle$$

$$|\phi(t)\rangle^{(1)} = |e\rangle|0\rangle + \underbrace{\sum_{\lambda} g_{\lambda}^* \frac{1}{\delta\omega_{\lambda}} [1 - \exp(i\delta\omega_{\lambda}t)] |g\rangle|1_{\lambda}\rangle}_{\text{自发辐射}}$$

t时刻发射1个光子总概率:

$$\sum_{\lambda} \left| \frac{g_{\lambda}}{\delta\omega_{\lambda}} \right|^2 [1 - \exp(i\delta\omega_{\lambda}t)]^2$$

单位时间内跃迁概率:

$$P = \sum_{\lambda} |g_{\lambda}|^2 \approx \frac{\sin \delta\omega_{\lambda}t}{\delta\omega_{\lambda}} \Rightarrow P = 2\pi \sum_{\lambda} |g_{\lambda}|^2 \delta(\omega_{\lambda} - \omega_{eg})$$

自由空间 $d\Omega$ 内自发辐射跃迁概率

$$\vec{k}_{\lambda} \text{ 方向行波: } u_{\lambda} = \frac{1}{\sqrt{V}} \hat{e} \exp(i\vec{k}_{\lambda} \cdot \vec{r}) \quad V \rightarrow \infty$$

$$dN = \frac{V}{(2\pi)^3} k^2 dk d\Omega$$

$$\text{代入 } g_{\lambda} = -i \frac{\omega_{\lambda}}{2 \epsilon_0 \hbar V} \vec{d}_{eg} \cdot \hat{e}_{\sigma} \quad \text{辐射光子 } k \approx \omega_{eg}/c$$

$$P(d\Omega) = \frac{\omega_{eg}^3}{8\pi^2 \hbar \epsilon_0 c^3} |\hat{e} \cdot \vec{d}_{eg}|^2 d\Omega \quad \vec{d}_{eg} = -e \int \psi_e^*(r) \vec{r} \psi_g(r) dV$$

$$\Rightarrow P = \frac{\omega_{eg}^3}{8\pi^2 \hbar \epsilon_0 c^3} \frac{8\pi}{3} |\vec{d}_{eg}|^2 = \frac{\omega_{eg}^3}{3\pi \hbar \epsilon_0 c^3} |\vec{d}_{eg}|^2$$

受激辐射

$$P = 2\pi |g_{\lambda}|^2 \delta(\omega_{\lambda} - \omega_{eg}) (n+1) + 2\pi \sum_{\lambda \neq \lambda_0} |g_{\lambda}|^2 \delta(\omega_{\lambda} - \omega_{eg})$$

$$= \underbrace{2\pi |g_{\lambda_0}|^2 \delta(\omega_{\lambda_0} - \omega_{eg}) n}_{\text{外界激发}} + 2\pi \sum_{\lambda} |g_{\lambda}|^2 \delta(\omega_{\lambda} - \omega_{eg}) \quad \text{自发}$$

$$P_{st} = 2\pi n |g_{\lambda}|^2 \delta(\omega_{\lambda} - \omega_{eg})$$

$$P_{st} = \frac{n (\epsilon\pi)^3}{V k^2 dk d\Omega} \frac{\omega_{eg}^3}{8\pi^2 \hbar \epsilon_0 c^3} |\hat{e} \cdot \vec{d}_{eg}|^2 d\Omega = \underbrace{\left(\frac{n \hbar \omega_{eg}}{\Delta\omega \cdot d\Omega \cdot V} \right)}_{\rho_e(\omega_{eg}, d\Omega)} \underbrace{\left(\frac{\pi}{\hbar^2 \epsilon_0} \right)}_{B_{eg, \hat{e}}} |\hat{e} \cdot \vec{d}_{eg}|^2 d\Omega$$

$$\frac{A_{eg, \hat{e}}}{B_{eg, \hat{e}}} = \frac{\hbar \omega_{eg}^3}{(2\pi)^3 c^3}$$

Dicke 模型

$$\hat{H} = \hbar \omega \hat{a}^\dagger \hat{a} + \hbar \Delta \sum_{i=1}^N \sigma_z^{(i)} + \frac{\hbar g}{\sqrt{N}} \sum_{i=1}^N (\hat{a}^\dagger + \hat{a}) (\sigma_+^{(i)} + \sigma_-^{(i)})$$

$$\text{集体算符} \quad \hat{J}_\alpha = \frac{1}{2} \sum_{i=1}^N \sigma_\alpha^{(i)} \quad \hat{J}_\pm = \sum_{i=1}^N \sigma_\pm^{(i)}$$

$$|j, m\rangle = \frac{1}{(j+m)!} \binom{2j}{j+m}^{-\frac{1}{2}} \hat{J}_+^{j+m} |j, -j\rangle$$

$$\Rightarrow \hat{H}_I = \frac{\hbar}{\sqrt{N}} g (\hat{a}^\dagger \hat{J}_- + \hat{a} \hat{J}_+)$$

$$\hat{H}_I = \frac{\hbar}{\sqrt{N}} \sum_k \left(g_k \hat{a}_k^\dagger \hat{J}_- + g_k^* \hat{a}_k \hat{J}_+ \right) \quad \text{自发辐射相关}$$

$$P \propto \left| \langle j, m-1 | \hat{J}_- | j, m \rangle \right|^2 = (j-m+1)(j+m)$$

对单个原子 $j=m=\frac{1}{2}$, $P \sim 1$.

$$\text{Dicke态 } j = \frac{N}{2}, \quad P \sim \left(\frac{N}{2} - m + 1 \right) \left(\frac{N}{2} + m \right)$$

$m = \frac{N}{2}$ 全激发, $P \sim N$

$m = 0$ 半激发, $P \sim N^2$ 超辐射

无宏观偶极矩

对比: 自旋相干态 存在宏观偶极矩

$$P \propto \sum_m \left| \left\langle \frac{N}{2}, m \mid \hat{J}_- \mid 0, \phi \right\rangle \right|^2 = \langle 0, \phi \mid \hat{J}_+ \hat{J}_- \mid 0, \phi \rangle$$

$$= \left(\frac{N}{2} \right)^2 \sin^4 \theta + 2 \left(\frac{N}{2} \right) \sin^4 \frac{\theta}{2}, \quad \theta = \frac{\pi}{2} \text{ 时 超辐射}$$

原子集体算符的关联

$$\langle \hat{J}_+ \hat{J}_- \rangle = \left\langle \sum_{i=1}^N \hat{\sigma}_+^{(i)} \hat{\sigma}_-^{(i)} \right\rangle = N \langle \hat{\sigma}_+^{(i)} \hat{\sigma}_-^{(i)} \rangle + \underbrace{N(N-1) \langle \hat{\sigma}_+^{(i)} \hat{\sigma}_-^{(j)} \rangle}_{\text{不同原子关联}}$$

$$N(N-1) \cdot \frac{(N+2m)(N-2m)}{4N(N-1)}$$

自旋相干态关联

$$\langle \hat{J}_+ \hat{J}_- \rangle - \langle \hat{J}_+ \rangle \langle \hat{J}_- \rangle = \sum_{i=1}^N \left(\langle \hat{\sigma}_+^{(i)} \hat{\sigma}_-^{(i)} \rangle - \langle \hat{\sigma}_+^{(i)} \rangle \langle \hat{\sigma}_-^{(i)} \rangle \right) = N \cos \theta$$

超辐射相变

$$\hat{H} = \hbar \omega_a \hat{J}_z + \hbar \omega \hat{a}^\dagger \hat{a} + \frac{\hbar g}{\sqrt{N}} (\hat{a}^\dagger + \hat{a}) (\hat{J}_+ + \hat{J}_-)$$

$N \gg 1$ 时, Holstein-Primakoff 变换

$$\hat{J}_z = \hat{b}^\dagger \hat{b} - N/2$$

$$\hat{J}_+ = \hat{b}^\dagger \sqrt{N - \hat{b}^\dagger \hat{b}}$$

$$\hat{J}_- = \sqrt{N - \hat{b}^\dagger \hat{b}}$$

$$\Rightarrow \hat{H} = \hbar \omega_a \left(\hat{b}^\dagger \hat{b} - \frac{N}{2} \right) + \hbar \omega \hat{a}^\dagger \hat{a} + \frac{\hbar g}{\sqrt{N}} (\hat{a}^\dagger + \hat{a}) \left(\hat{b}^\dagger \sqrt{N - \hat{b}^\dagger \hat{b}} + \sqrt{N - \hat{b}^\dagger \hat{b}} \hat{b} \right)$$

自旋 — 谐振子

$$\hat{J}_+ |J, M\rangle = \sqrt{(J-M)(J+M+1)} |J, M+1\rangle \quad \Rightarrow \quad \hat{b}^\dagger \sqrt{N - \hat{b}^\dagger \hat{b}} |J+M\rangle = \sqrt{(J-M)(J+M+1)} |J+M+1\rangle$$

$$\hat{J}_z |J, M\rangle = M |J, M\rangle \quad \left(\hat{b}^\dagger \hat{b} - \frac{N}{2} \right) |J+M\rangle = M |J+M\rangle$$

$$\hat{H} = \hbar \omega_a \left(\hat{b}^\dagger \hat{b} - \frac{N}{2} \right) + \hbar \omega \hat{a}^\dagger \hat{a} + \frac{\hbar g}{\sqrt{N}} (\hat{a}^\dagger + \hat{a}) \left(\hat{b}^\dagger \sqrt{N - \hat{b}^\dagger \hat{b}} + \sqrt{N - \hat{b}^\dagger \hat{b}} \hat{b} \right)$$

$$\hat{a} = \sqrt{N} \underbrace{\alpha}_\text{平均} + \underbrace{\hat{c}}_\text{涨落}$$

$$\alpha_n = 0, \quad \beta_n = 0 \quad \text{正常态}$$

$$\hat{b} = \sqrt{N} \underbrace{\beta}_\text{平均} + \underbrace{\hat{d}}_\text{涨落}$$

$$\alpha_s = \mp \frac{g}{\omega} \sqrt{1 - \frac{1}{g^2}}, \quad \beta_s = \pm \sqrt{\frac{1}{2} \left(1 - \frac{1}{g^2} \right)} \quad \text{超辐射态}$$

平均 涨落

$$g = g/g_c \quad (g \geq 1)$$

单轴压缩 Hamiltonian

Dicke 模型的 Heff 含 J_z^2 项, 非线性

$$H_{\text{OAT}} = \hbar \eta J_z^2$$

$$U(u) = \exp(-iu J_z^2) \quad u = \eta t$$

$$\hat{J}_+(t) = U(u) \hat{J}_+(0) U(u) = \hat{J}_+(0) e^{iu(2\hat{J}_z+1)}$$

$$\hat{J}_z(t) = \frac{1}{2} (\hat{J}_+(t) + \hat{J}_-(t)) = \frac{1}{2} \left[\hat{J}_+(0) e^{iu(2\hat{J}_z+1)} + e^{-iu(2\hat{J}_z+1)} \hat{J}_-(0) \right]$$

演化依赖 J_z 本征值

... 最小涨落方向随 t 变化

双轴压缩

$$H_{\text{MT}} = \hbar \eta \left(\hat{J}_y \hat{J}_z + \hat{J}_z \hat{J}_y \right) = \hbar \eta \left[\left(\frac{\hat{J}_y + \hat{J}_z}{\sqrt{2}} \right)^2 - \left(\frac{\hat{J}_y - \hat{J}_z}{\sqrt{2}} \right)^2 \right]$$

主方程

$$\hat{H} = \hat{H}_S \otimes \hat{I}_B + \hat{I}_S \otimes \hat{H}_B + \lambda \hat{H}_{SB}$$

$$\hat{H}_I(t) = \hat{U}^\dagger(t) \hat{H}_{SB} \hat{U}(t)$$

Schrödinger Picture

$$\frac{d\rho(t)}{dt} = \frac{1}{i\hbar} [\hat{H}, \rho(t)]$$

Interaction Picture

$$\frac{d\tilde{\rho}(t)}{dt} = \frac{1}{i\hbar} [\hat{H}_I(t), \tilde{\rho}(t)]$$



$$\frac{d\rho(t)}{dt} = \frac{1}{i\hbar} [\hat{H}_0, \rho(t)] + U(t) \frac{d\tilde{\rho}}{dt} U^\dagger(t)$$

Interaction Picture 解为

$$\tilde{\rho}(t) = \tilde{\rho}(0) + \frac{\lambda}{i\hbar} \int_0^t ds [\tilde{H}_I(s), \tilde{\rho}(s)] - \frac{\lambda^2}{\hbar^2} \int_0^t ds_1 \int_0^{s_1} ds_2 [\tilde{H}_I(s_1), [\tilde{H}_I(s_2), \tilde{\rho}(s_2)]] + O(\lambda^3)$$

初始为非关联直积态 $\tilde{\rho}(0) = \tilde{\rho}_S(0) \otimes \tilde{\rho}_B$

$$\dot{\tilde{\rho}}_S(t) = \frac{\lambda}{i\hbar} \text{Tr}_B [\tilde{H}_I(t), \tilde{\rho}(0)] - \frac{\lambda^2}{\hbar^2} \int_0^t ds \text{Tr}_B [\tilde{H}_I(t), [\tilde{H}_I(s), \tilde{\rho}(s)]]$$

可选取 $H_S, H_B, \dots$

$$\xrightarrow{\quad} = - \frac{\lambda^2}{\hbar^2} \int_0^t ds \text{Tr}_B [\tilde{H}_I(t), [\tilde{H}_I(s), \tilde{\rho}(s)]]$$

Markov

$$\xrightarrow{\quad} = - \frac{\lambda^2}{\hbar^2} \int_0^t ds \text{Tr}_B [\tilde{H}_I(t), [\tilde{H}_I(s), \tilde{\rho}(s)]]$$

$\tau_B \ll \tau_R$

时间尺度 弛豫时间

$$\xrightarrow{\quad} = - \frac{\lambda^2}{\hbar^2} \int_0^\infty ds \text{Tr}_B [\tilde{H}_I(t), [\tilde{H}_I(t-s), \tilde{\rho}(t)]]$$

$$\begin{aligned}\tilde{H}_I &= \sum_{\alpha} \tilde{A}_{\alpha}(t) \otimes \tilde{B}_{\alpha}(t) \\ &= \sum_{\alpha, \omega} e^{-i\omega t} \tilde{A}_{\alpha}(\omega) \otimes \tilde{B}_{\alpha}(t) = \sum_{\alpha, \omega} e^{i\omega t} \tilde{A}_{\alpha}^{\dagger}(\omega) \tilde{B}_{\alpha}(t).\end{aligned}$$

代入得

$$\frac{d}{dt} \tilde{\rho}_s(t) = \frac{\lambda^2}{\hbar^2} \int_0^{\infty} ds \operatorname{Tr}_B \left[\tilde{H}_I(t-s) \tilde{\rho}_s(t) \otimes \rho_B \tilde{H}_I(t) - \tilde{H}_I(t) \tilde{H}_I(t-s) \tilde{\rho}_s(t) \otimes \tilde{\rho}_B \right] + h.c.$$

= ...

$$= \frac{1}{i\hbar} [\hat{H}_{LS}, \rho_s(t)] + \sum_{\omega, \alpha, \beta} \gamma_{\alpha\beta}(\omega) \left[\hat{A}_{\beta}(\omega) \tilde{\rho}_s(t) \hat{A}_{\alpha}^{\dagger}(\omega) - \frac{1}{2} \{ \hat{A}_{\alpha}^{\dagger}(\omega) \hat{A}_{\beta}(\omega), \tilde{\rho}_s(t) \} \right]$$

$$\Gamma_{\alpha\beta}(\omega) = \frac{\lambda^2}{\hbar^2} \int_0^{\infty} e^{i\omega s} \operatorname{Tr}_B \left[\tilde{B}_{\alpha}^{\dagger}(t) \tilde{B}_{\beta}(t-s) \tilde{\rho}_B \right] = \frac{\lambda^2}{\hbar^2} \int_0^{\infty} ds e^{i\omega s} \operatorname{Tr}_B \left[\tilde{B}_{\alpha}^{\dagger}(s) \tilde{B}_{\beta}(s) \tilde{\rho}_B \right].$$

$$\gamma(\omega) = \Gamma(\omega) + \Gamma(\omega)^{\dagger}, \quad \Delta(\omega) = \frac{1}{2i} [\Gamma(\omega) - \Gamma(\omega)^{\dagger}].$$

$$\hat{H}_{LS} = \hbar \sum_{\omega} \sum_{\alpha, \beta} \Delta_{\alpha\beta}(\omega) \hat{A}_{\alpha}^{\dagger}(\omega) \hat{A}_{\beta}(\omega).$$

...

$$\gamma_{\alpha\beta} = \Gamma_{\alpha\beta} + \tilde{\Gamma}_{\alpha\beta} = \frac{\lambda^2}{\hbar^2} \int_{-\infty}^{\infty} ds e^{i\omega s} \operatorname{Tr}_B \left[\tilde{B}_{\alpha}^{\dagger}(s) \tilde{B}_{\beta}(s) \tilde{\rho}_B \right].$$

$\gamma(\omega)$ 对角化表象中.

$$\dot{\tilde{\rho}}_s(t) = \frac{1}{i\hbar} [\hat{H}_{LS}, \tilde{\rho}_s(t)] + \sum_{\omega, k} \gamma_k(\omega) \left[\hat{L}_k(\omega) \tilde{\rho}_s(t) \hat{L}_k^{\dagger}(\omega) - \frac{1}{2} \{ \hat{L}_k(\omega) \hat{L}_k(\omega), \tilde{\rho}_s(t) \} \right]$$

$$\hat{L}_k(\omega) = O_{kp} A_p(\omega).$$

单频热序

$$\dot{\tilde{\rho}}_s(t) = \frac{1}{i\hbar} [\hat{H}_{LS}, \tilde{\rho}_s(t)] + \sum_k \gamma_k \left[\hat{L}_k \tilde{\rho}_s(t) \hat{L}_k^{\dagger} - \frac{1}{2} \{ \hat{L}_k^{\dagger} \hat{L}_k, \tilde{\rho}_s(t) \} \right]$$

简谐振子热库

$$\hat{H}_0 = \hbar\omega \hat{a}^\dagger \hat{a} + \hbar \sum_j \omega_j \hat{b}_j^\dagger \hat{b}_j$$

$$\hat{H}_{S8} = \hat{a} \otimes \sum_j g_j^* \hat{b}_j^\dagger + \hat{a}^\dagger \otimes \sum_j g_j \hat{b}_j$$

Interaction Picture $\rightarrow$

$$\tilde{H}_{S8} = U^\dagger H_{S8} U = \hat{a} e^{-i\omega t} \otimes \sum_j \hbar g_j^* \hat{b}_j^\dagger e^{i\omega_j t} + \hat{a}^\dagger e^{i\omega t} \otimes \sum_j \hbar g_j \hat{b}_j e^{-i\omega_j t}$$

对谐振子热库 $\rho_B = \prod_k (1 - e^{-\beta \hbar \omega_k}) e^{-\beta \hbar \omega_k \hat{b}_k^\dagger \hat{b}_k}$ 平衡态

$$\hat{b}_k \equiv \hat{b}(\omega_k) \quad \beta_k = \hbar \omega_k / k_B T$$

$$\langle \hat{b}_k \rangle = \langle \hat{b}_k^\dagger \rangle = 0 \quad \langle \hat{b}_k^\dagger \hat{b}_{k'}^\dagger \rangle = \langle \hat{b}_k \hat{b}_{k'} \rangle = 0$$

$$\langle \hat{b}_k^\dagger \hat{b}_{k'} \rangle = \bar{n}_k \delta_{kk'} \quad \langle \hat{b}_k \hat{b}_{k'}^\dagger \rangle = (\bar{n}_k + 1) \delta_{kk'}$$

$$\text{令 } \tilde{B}(t) = \sum_j g_j \hat{b}_j e^{i\omega_j t}$$

$$\Gamma_{11}(\omega) = \int_0^\infty e^{i\omega s} \text{Tr}_B \{ \tilde{B}(s) \tilde{B}^\dagger(0) \rho_B \}$$

$$= \sum_{jj'} g_j g_{j'}^* \int_0^\infty ds e^{i(\omega - \omega_j)s} \underbrace{\langle \hat{b}_j \hat{b}_{j'}^\dagger \rangle_B}_{\text{Tr}_B \{ \hat{b}_j \hat{b}_{j'}^\dagger \rho_B \}}$$

$$= \int_0^\infty d\tilde{\omega} \underbrace{\sum_j |g_j|^2 \delta(\tilde{\omega} - \omega_j)}_{J(\tilde{\omega}) \text{ 谱函数}} \int_0^\infty ds e^{i(\omega - \tilde{\omega})s} [n(\tilde{\omega}) + 1]$$

$J(\tilde{\omega})$ 谱函数

$$= \pi J(\tilde{\omega}) [n(\tilde{\omega}) + 1] + i \mathcal{P} \int_0^\infty d\tilde{\omega} \frac{J(\tilde{\omega}) [n(\tilde{\omega}) + 1]}{\omega - \tilde{\omega}} \quad \Delta_1$$

$$\Gamma_{22}(-\omega) = \int_0^\infty e^{-i\omega s} \text{Tr}_B \{ \tilde{B}^\dagger(s) \tilde{B}(0) \rho_B \}$$

$$= \pi J(\omega) \bar{n}(\omega) - i \mathcal{P} \int_0^\infty d\tilde{\omega} \frac{J(\tilde{\omega}) \bar{n}(\tilde{\omega})}{\omega - \tilde{\omega}} \quad \Delta_2$$

代入主方程得

$$\frac{d}{dt} \tilde{\rho}(t) = -i \Delta [\hat{a}^\dagger \hat{a}, \tilde{\rho}(t)] + \frac{\gamma}{2} (\bar{n}+1) \underbrace{D(\hat{a})[\tilde{\rho}]} + \frac{\gamma}{2} \bar{n} D(\hat{a}^\dagger)[\tilde{\rho}]$$

$$\int_0^\infty d\tilde{\omega} \frac{\tilde{J}(\tilde{\omega})}{\omega - \tilde{\omega}} \quad 2\hat{a}\tilde{\rho}\hat{a}^\dagger - \{\tilde{\rho}, \hat{a}^\dagger \hat{a}\}$$

→ Schrödinger Picture

$$\frac{d}{dt} \rho(t) = -i \underbrace{\omega + \Delta} [\hat{a}^\dagger \hat{a}, \rho(t)] + \frac{\gamma}{2} (\bar{n}+1) D(\hat{a})[\rho] + \frac{\gamma}{2} \bar{n} D(\hat{a}^\dagger)[\rho]$$

若系统为 TLS

$$\hat{H}_{\text{SB}} = \hat{\sigma}^- \otimes \sum_j g_j^* \hat{b}_j^\dagger + \hat{\sigma}^+ \otimes \sum_j g_j \hat{b}_j$$

Schrödinger Picture:

$$\frac{d}{dt} \rho(t) = \frac{1}{i\hbar} [\hat{H}_{\text{SB}}, \rho(t)] + \frac{\gamma}{2} (\bar{n}+1) D(\hat{\sigma}^-)[\rho] + \frac{\gamma}{2} \bar{n} D(\hat{\sigma}^+)[\rho]$$

$\omega' = 0$ 时, $\rho_n = \langle n | \rho | n \rangle$ 演化为

$$\dot{\rho}_n = \underbrace{\gamma_\downarrow}_{\gamma_\downarrow} (\bar{n}+1) \rho_{n+1} - \gamma_\downarrow (\bar{n}+1) n \rho_n + \underbrace{\gamma_\uparrow}_{\gamma_\uparrow} \bar{n} \rho_{n-1} - \gamma_\uparrow (n+1) \rho_n$$

$$= \gamma_\downarrow (n+1) \rho_{n+1} - \gamma_\downarrow n \rho_n + \gamma_\uparrow n \rho_{n-1} - \gamma_\uparrow (n+1) \rho_n$$

$$= -[\gamma_\downarrow n + \gamma_\uparrow (n+1)] \rho_n + \gamma_\downarrow (n+1) \rho_{n+1} + \gamma_\uparrow n \rho_{n-1}$$

细致平衡

$$\rho_n \gamma_\uparrow (n+1) = \rho_{n+1} \gamma_\downarrow (n+1), \quad \rho_n P_{n \rightarrow n+1} = \rho_{n+1} P_{n+1 \rightarrow n}$$

$$\frac{\gamma_\downarrow}{\gamma_\uparrow} = \frac{\rho_n}{\rho_{n+1}} = \exp\left(\frac{\hbar\omega}{k_B T}\right) = 1 + \frac{1}{\bar{n}}$$

$$\frac{d}{dt} \langle \hat{a} \rangle = \text{Tr}[\dot{\hat{a}} \rho] = -\left(\frac{\gamma}{2} + i\omega\right) \langle \hat{a} \rangle$$

$$\frac{d}{dt} \langle \hat{a}^\dagger \hat{a} \rangle = \text{Tr}[\dot{\hat{a}^\dagger \hat{a}} \rho] = -\gamma (\langle \hat{a}^\dagger \hat{a} \rangle - \bar{n})$$

$$\langle \hat{n}(t) \rangle = \underbrace{[\hat{n}(0) - \bar{n}]}_{\text{最终与环境中同频率激发数相同}} e^{-\gamma t} + \bar{n}$$

$$\text{稳态} \quad \rho = \frac{e^{-\hat{H}_S/k_B T}}{\text{Tr}(e^{-\hat{H}_S/k_B T})} \quad \hat{H}_S = \hbar \omega \hat{a}^\dagger \hat{a}$$

压缩态库 模式存在 quantum correlation

$$\rho_{sq} = \hat{S}(\xi) \rho_s \hat{S}^\dagger(\xi)$$

关联的模式

$$\hat{S}^\dagger(\xi) = \prod_k \exp \left[\xi^* \hat{b}(\Omega - \omega_k) \hat{b}(\Omega + \omega_k) - \text{h.c.} \right]$$

$$\hat{S}^\dagger(\xi) \hat{b}(\Omega \pm \omega_k) \hat{S}(\xi) = \cosh(|\xi|) \hat{b}(\Omega \pm \omega_k) + e^{i\phi} \sinh(|\xi|) \hat{b}^\dagger(\Omega \mp \omega_k)$$

$$\hat{b}_k = \hat{b}(\omega_k) \quad \text{手征子时} \quad \langle \hat{b}_k \rangle = \langle \hat{b}_k^\dagger \rangle = 0 \quad \langle \hat{b}_k^\dagger \hat{b}_{k'} \rangle = N_k \delta_{kk'}$$

$$\langle \hat{b}_k^\dagger \hat{b}_{k'}^\dagger \rangle = (N_k + 1) \delta_{kk'} \quad \langle \hat{b}_k^\dagger \hat{b}_{k'} \rangle = M_k \delta_{\omega_k + \omega_{k'} - 2\Omega}$$

$$N_k = \bar{n} \left[\cosh^2(|\xi|) + \sinh^2(|\xi|) \right] + \sinh^2(|\xi|)$$

$$M_k = (2\bar{n} + 1) \sinh(|\xi|) \cosh(|\xi|) e^{i\phi}$$

$$N_k(N_k + 1) = |M_k|^2 + \bar{n}(\bar{n} + 1)$$

$$\tilde{H}_{SB} = \hat{a} e^{-i\omega t} \otimes \sum_j g_j^* \hat{b}_j^\dagger e^{i\omega_j t} + \hat{a}^\dagger e^{i\omega t} \otimes \sum_j g_j \hat{b}_j e^{-i\omega_j t}$$

$$A_1(\omega)$$

$$\tilde{B}_1(t)$$

$$\tilde{A}_2(-\omega)$$

$$\tilde{B}_2(t)$$

多出 $\langle \hat{b}_k \hat{b}_{k'} \rangle$ 关联项

$$\Gamma_a(-\omega) = \int_0^\infty ds e^{-i\omega s} \text{Tr} \left\{ \tilde{B}_1^\dagger(s) \tilde{B}_2(0) \rho_{sq} \right\}$$

$$= \sum_{jj'} g_j g_{j'} \int_0^\infty ds e^{-i(\omega + \omega_j)s} \text{Tr}_B \{ b_j b_{j'} \rho_B \}$$

$$= \sum_j g(\omega_j) g(2\Omega - \omega_j) \int_0^\infty ds M(\omega_j) e^{-i(\omega + \omega_j)s}$$

$$= \pi \sum_j g(\omega_j) g(2\Omega - \omega_j) M(\omega_j) \delta\omega + \omega_j.$$

$$\frac{d}{dt} \tilde{\rho}(t) \propto \Gamma_{12}(-\omega) e^{i2\omega t} \left[\hat{A}_2(-\omega) \tilde{\rho}(t) \hat{A}_1^\dagger(\omega) - \hat{A}_1^\dagger(\omega) \hat{A}_2(-\omega) \tilde{\rho}(t) \right] + h.c.$$

$$+ \Gamma_{21}(\omega) e^{-i2\omega t} \left[\hat{A}_1(\omega) \tilde{\rho}(t) \hat{A}_2^\dagger(-\omega) - \hat{A}_2^\dagger(-\omega) \hat{A}_1(\omega) \tilde{\rho}(t) \right] + h.c.$$

$$= \frac{\tilde{\gamma}}{2} M e^{i2\omega t} \left\{ \left[\hat{a}^\dagger \tilde{\rho}(t) \hat{a}^\dagger - \hat{a}^\dagger \hat{a} \tilde{\rho}(t) \right] + h.c. \right\} + \frac{\tilde{\gamma}}{2} M^* e^{-i2\omega t} \left\{ \left[\hat{a} \tilde{\rho}(t) \hat{a} - \hat{a} \hat{a} \tilde{\rho}(t) \right] + h.c. \right\}$$

$$\tilde{\gamma} = 2\pi \sum_j g(\omega_j) g(2\Omega - \omega_j) \delta\omega + \omega_j.$$

Schrödinger Picture

$$\frac{d}{dt} \rho = \frac{1}{i\hbar} [\hat{H}_s, \rho] + \frac{\gamma}{2} (N+1) (2\hat{a}\rho\hat{a}^\dagger - \hat{a}^\dagger\hat{a}\rho - \rho\hat{a}^\dagger\hat{a}) + \frac{\gamma}{2} N (2\hat{a}^\dagger\rho\hat{a} - \hat{a}\hat{a}^\dagger\rho - \rho\hat{a}\hat{a}^\dagger)$$

$$+ \frac{\tilde{\gamma}}{2} M^* (2\hat{a}\rho\hat{a} - \hat{a}\hat{a}\rho - \rho\hat{a}\hat{a}) + \frac{\tilde{\gamma}}{2} M (2\hat{a}^\dagger\rho\hat{a} - \hat{a}^\dagger\hat{a}\rho - \rho\hat{a}^\dagger\hat{a})$$

对 TLS.

$$\frac{d}{dt} \rho = \frac{1}{i\hbar} [\hat{H}_s, \rho] + \frac{\gamma}{2} (N+1) (2\hat{\sigma}^- \rho \hat{\sigma}^+ - \hat{\sigma}^+ \hat{\sigma}^- \rho - \rho \hat{\sigma}^+ \hat{\sigma}^-)$$

$$+ \frac{\gamma}{2} N (2\hat{\sigma}^+ \rho \hat{\sigma}^- - \hat{\sigma}^- \hat{\sigma}^+ \rho - \rho \hat{\sigma}^- \hat{\sigma}^+) + \tilde{\gamma} M^* \hat{\sigma}^- \rho \hat{\sigma}^- + \tilde{\gamma} M \hat{\sigma}^+ \rho \hat{\sigma}^+.$$

算符表象中的主方程

Schrödinger Picture

$$\frac{d\rho(t)}{dt} = \mathcal{L}[\rho(t)] = \frac{1}{i\hbar} [\hat{H}, \rho(t)] + \sum_k \gamma_k \left(\hat{L}_k \rho \hat{L}_k^\dagger - \frac{1}{2} \{ \hat{L}_k^\dagger \hat{L}_k, \rho(t) \} \right)$$

Heisenberg Picture

态演化的算子

$$\langle \hat{A} \rangle = \text{Tr} \{ \hat{A} \rho(t) \} = \text{Tr} \{ \hat{A} \underline{V(t)} [\rho(0)] \} = \text{Tr} \{ \hat{A}_{H(t)} \rho(0) \} = \text{Tr} \{ V^\dagger(t) [\hat{A}] \rho(0) \}$$

$$V(t) = T_{\leftarrow} \exp \left[\int_0^t ds \mathcal{L}(s) \right]$$

$$\frac{d}{dt} V(t) = \mathcal{L}(t) V(t)$$

$$V^\dagger(t) = T_{\rightarrow} \exp \left[\int_0^t ds \mathcal{L}^\dagger(s) \right]$$

$$\frac{d}{dt} V^\dagger(t) = V^\dagger(t) \mathcal{L}^\dagger(t)$$

$$\hat{A}_H(t) = \mathcal{U}^\dagger(t) [\hat{A}] \quad \text{有}$$

$$\frac{d}{dt} \hat{A}_H(t) = \frac{d}{dt} \mathcal{U}^\dagger(t) \hat{A} = \mathcal{U}^\dagger(t) [\mathcal{L}^\dagger(t) \hat{A}]$$

$\mathcal{L}$ 不含时时

$$\frac{d}{dt} \hat{A}_H(t) = \mathcal{L}^\dagger(t) [\mathcal{U}^\dagger \hat{A}] = \mathcal{L}^\dagger [\hat{A}_H(t)]$$

$$= \frac{1}{i\hbar} [\hat{A}_H(t), \hat{H}] + \sum_k \gamma_k \left(\hat{\mathcal{L}}_k^\dagger \hat{A}_H(t) \hat{\mathcal{L}}_k - \frac{1}{2} \{ \hat{\mathcal{L}}_k^\dagger \hat{\mathcal{L}}_k, \hat{A}_H(t) \} \right) + \hat{f}_A$$

量子回归定理 quantum regression theorem

$$\langle \hat{O}_1(t) \hat{O}_2(t+\tau) \rangle = \text{Tr}_{SE} [\rho_{SE}(0) \hat{O}_1(t) \hat{O}_2(t+\tau)]$$

系统算符

总 Hamiltonian: H

$$\hat{O}_i(t) = e^{i\hat{H}t/\hbar} \hat{O}_i(0) e^{-i\hat{H}t/\hbar}$$

$$\rho_{SE}(0) = e^{i\hat{H}t/\hbar} \rho_{SE}(t) e^{-i\hat{H}t/\hbar}$$

$$\text{代入得} \quad \langle \hat{O}_1(t) \hat{O}_2(t+\tau) \rangle = \text{Tr}_S \left\{ \hat{O}_2(0) \text{Tr}_E \left[\underbrace{e^{-i\hat{H}\tau/\hbar} \rho_{SE}(t) \hat{O}_1(0) e^{i\hat{H}\tau/\hbar}}_{\chi_{SE}(t)} \right] \right\}$$

$$\frac{d}{dt} \chi_{SE}(t) = \frac{1}{i\hbar} [H, \chi_{SE}(t)]$$

$\chi_{SE}(t)$

$$\text{Markov approx} \rightarrow \rho_{SE}(t) \simeq \rho(t) \otimes R_0 \quad \rho(t) = \text{Tr}_E [\rho_{SE}(t)]$$

$$\chi_{SE}(0) = \rho_{SE}(0) \hat{O}_1(0) \simeq [\rho(0) \hat{O}_1(0)] \otimes R_0$$

$\chi_{SE}(t)$, $\rho_{SE}(t)$ 满足相同动力学演化方程及初条件

$$\Rightarrow \frac{d}{dt} \chi(t) = \mathcal{L} [\chi(t)] \quad \chi(t) = \text{Tr}_E [\chi_{SE}(t)]$$

$$\chi(t) = e^{\mathcal{L}t} [\chi(0)] = e^{\mathcal{L}t} [\rho(t) \hat{O}_1(0)]$$

$$\Rightarrow \langle \hat{O}_1(t) \hat{O}_2(t+\tau) \rangle = \text{Tr}_S \left\{ \hat{O}_2(\circ) \chi(\tau) \right\} = \text{Tr}_S \left\{ \hat{O}_2(\circ) e^{\lambda \tau} [\rho(t) \hat{O}_1(\circ)] \right\}$$

$\hat{O}_2$ 在 $\chi(\tau)$ 下的平均值

$$\langle \hat{O}_1(t+\tau) \hat{O}_2(t) \rangle = \text{Tr}_S \left\{ \hat{O}_1(\circ) e^{\lambda \tau} [\hat{O}_2(\circ) \rho(t)] \right\}$$

$$\langle \hat{O}_1(t) \hat{O}_2(t+\tau) \hat{O}_3(t) \rangle = \text{Tr}_S \left\{ \hat{O}_2(t) e^{\lambda \tau} [\hat{O}_3(\circ) \rho(t) \hat{O}_1(\circ)] \right\}$$

$$\frac{d}{dt} \langle \hat{A}_\mu \rangle = \frac{d}{dt} \text{Tr}_S [\hat{A}_\mu \rho(t)] = \text{Tr}_S \left\{ \hat{A}_\mu \mathcal{L}[\rho(t)] \right\} = \sum_\lambda M_{\mu\lambda} \text{Tr}_S [\hat{A}_\lambda \rho(t)]$$

$$\text{有 } \frac{d}{dt} \langle \hat{O}(t) \hat{A}_\mu(t+\tau) \rangle = \text{Tr}_S \left\{ \hat{A}_\mu(\circ) \mathcal{L}[\rho(t) \hat{O}(\circ)] \right\} = \sum_\lambda M_{\lambda} \langle \hat{O}(t) \hat{A}_\lambda(t+\tau) \rangle$$

以谐振子系统为例

$$\frac{d}{dt} \langle \hat{a}(t) \rangle = -\left(\frac{\gamma}{2} + i\omega_0\right) \langle \hat{a}(t) \rangle$$

$$\frac{d}{dt} \langle \hat{a}^\dagger(t) \hat{a}(t+\tau) \rangle = -\left(\frac{\gamma}{2} + i\omega_0\right) \langle \hat{a}^\dagger(t) \hat{a}(t+\tau) \rangle$$

$$\hat{a}^\dagger(t) \hat{a}(t+\tau) = \langle \hat{a}^\dagger(t) \hat{a}(t) \rangle e^{-\left(\frac{\gamma}{2} + i\omega_0\right)\tau}$$

$$\langle \hat{n}(t) \rangle = \langle \hat{n}(\circ) \rangle - \bar{n} e^{-\gamma t} + \bar{n}$$

$$= \left[\langle \hat{n}(\circ) \rangle e^{-\gamma t} + \bar{n} (1 - e^{-\gamma t}) \right] e^{-\left(\frac{\gamma}{2} + i\omega_0\right)\tau}$$

$$\frac{d}{dt} \langle \hat{n}(t) \rangle = -\gamma \langle \hat{n}(t) \rangle + \gamma \bar{n}$$

$$\xrightarrow{t \rightarrow \infty} \bar{n} e^{-\left(\frac{\gamma}{2} + i\omega_0\right)\tau}$$

$$\frac{d}{dt} \langle \hat{a}^\dagger(t) \hat{n}(t+\tau) \hat{a}(t) \rangle = -\gamma \langle \hat{a}^\dagger(t) \hat{n}(t+\tau) \hat{a}(t) \rangle + \gamma \bar{n} \langle n(t) \rangle$$

$$\langle \hat{a}^\dagger(t) \hat{n}(t+\tau) \hat{a}(t) \rangle = \langle \hat{a}^\dagger(t) \hat{n}(t) \hat{a}(t) \rangle e^{-\gamma \tau} + \bar{n} \langle n(t) \rangle (1 - e^{-\gamma \tau})$$

$$= \langle \hat{a}^\dagger(t) \hat{a}^\dagger(t) \rangle = 2\bar{n} (1 - e^{-\gamma \tau}) \left[2\langle \hat{n}(\circ) \rangle e^{-\gamma \tau} + \bar{n} (1 - e^{-\gamma \tau}) \right] + \left[\langle \hat{n}^2(\circ) \rangle - \langle \hat{n}(\circ) \rangle^2 \right] e^{-2\gamma \tau}$$

$$\lim_{t \rightarrow \infty} \langle \hat{a}^\dagger(t) \hat{n}(t) \hat{a}(t) \rangle = 2\bar{n}^2$$

$$\lim_{t \rightarrow \infty} \langle \hat{a}^\dagger(t) \hat{n}(t+\tau) \hat{a}(t) \rangle = \bar{n}^2 (1 + e^{-\gamma \tau})$$

$$\lim_{t \rightarrow \infty} \langle \hat{n}(t) \rangle = \bar{n}$$

量子轨迹方法

主方程演化 $\rightarrow$ 系列纯态演化

$$\rho_0 = \sum_i p_i |\psi_i\rangle\langle\psi_i|$$

$$|\tilde{\psi}(t+dt)\rangle = (1 - i\hat{H}_c/\hbar) |\psi(t)\rangle$$

$$\text{例 } \hat{H}_{\text{eff}} = \hat{H} - i\frac{\hbar}{2} \sum_k \gamma_k \hat{L}_k^\dagger \hat{L}_k$$

$$\langle\tilde{\psi}(t+dt)|\tilde{\psi}(t+dt)\rangle = \langle\psi(t)|(1+i\hat{H}_c^\dagger\delta t/\hbar)(1-i\hat{H}_c\delta t/\hbar)|\psi(t)\rangle = 1 - \delta p$$

$$\delta p = \delta t \langle\psi(t)|(\hat{H}_c - \hat{H}_c^\dagger)|\psi(t)\rangle = \delta t \sum_m \gamma_m \langle\hat{L}_m^\dagger \hat{L}_m\rangle = \sum p_m$$

耗散通道导致

布居减少比例

$$\psi(t) = |\psi(t)\rangle\langle\psi(t)| \quad \mathcal{R}$$

$$\overline{\psi(t+\delta t)} = \frac{1}{\sqrt{1-\delta p}} |\tilde{\psi}(t+\delta t)\rangle + \delta p \sum_m p_m \gamma_m \frac{\hat{L}_m |\psi(t)\rangle\langle\psi(t)| \hat{L}_m^\dagger}{\sqrt{\delta p_m/\delta t} \sqrt{\delta p_m/\delta t}}$$

$$\text{又 } |\psi(t+\delta t)\rangle = \frac{1}{\sqrt{1-\delta p}} |\tilde{\psi}(t+\delta t)\rangle = \frac{1}{\sqrt{1-\delta p}} (1 - i\hat{H}_c\delta t) |\psi(t)\rangle$$

$$\Rightarrow \overline{\psi(t+\delta t)} = \psi(t) + \frac{\delta t}{i\hbar} [\hat{H}_c \psi(t) - \psi(t) \hat{H}_c^\dagger] + \delta t \sum_m \gamma_m \hat{L}_m \sigma(t) \hat{L}_m^\dagger$$

$$\Rightarrow \dot{\rho}(t) = \frac{1}{i\hbar} [\hat{H}_c, \rho(t)] + \sum_m \gamma_m \hat{L}_m \rho(t) \hat{L}_m^\dagger$$

跃迁时间估计的不确定性?

$$\text{下次跃迁的时间 } \tau \quad P_0(\tau) = \|e^{-i\hat{H}_c\tau/\hbar} |\psi(t)\rangle\|^2$$

$t \in [t_0, t_0 + \tau]$ 间系统波函数

$$|\psi(t)\rangle = \frac{e^{-i\hat{H}_c(t-t_0)/\hbar} |\psi(t_0)\rangle}{\|e^{-i\hat{H}_c(t-t_0)/\hbar} |\psi(t_0)\rangle\|}$$

$$\text{跳变后 } |\psi(t+\tau^+)\rangle = \frac{\hat{L}_m |\psi(t+\tau)\rangle}{\|\hat{L}_m |\psi(t+\tau)\rangle\|} \quad \dots$$

无扰相于子空间 decoherence free space DFS.

$$\hat{H} = \hat{H}_S \otimes \hat{I}_E + \hat{I}_S \otimes \hat{H}_E + \hat{H}_{SE} \quad \hat{H}_{SE} = \sum_{\alpha} \hat{S}_{\alpha} \otimes \hat{B}_{\alpha}$$

记 DFS 为 $\hat{H}_G$. $\hat{H}_S = \hat{H}_G \oplus \hat{H}_W$

$|r_j\rangle$

① $\rho_S(0) = \rho_S(0) \oplus 0 = \sum_{ij} \alpha_{ij} |r_i\rangle \langle r_j|$. 初态落在 $\hat{H}_G$ 中.

② $\hat{S}_{\alpha} |r_j\rangle = C_{\alpha} |r_j\rangle$. C_{α} 不依赖 $|r_j\rangle$. $\hat{H}_G$ 基矢为 $\hat{S}_{\alpha}$ 本征矢.

③ $\hat{H}_S$ 作用于 $|r_j\rangle$ 后仍包含于 $\hat{H}_G$.

$$\Rightarrow \rho_S(t) = \text{Tr}_B \left\{ U(t) [\rho_S(0) \otimes \rho_B(0)] U^\dagger(t) \right\} = U_S(t) \rho_S(0) U_S^\dagger(t) \in \hat{H}_G$$

$e^{-i\hat{H}_S t/\hbar}$ $e^{-i\hat{H}_S t/\hbar}$

$$\begin{aligned} \hat{H} |r_i\rangle \otimes |\beta_j\rangle &= (\hat{H}_S \otimes \hat{I}_E) |r_i\rangle \otimes |\beta_j\rangle + |r_i\rangle \otimes (\hat{H}_E |\beta_j\rangle) + \sum_{\alpha} C_{\alpha} |r_i\rangle \otimes (\hat{B}_{\alpha} |\beta_j\rangle) \\ &= [\hat{H}_S \otimes \hat{I}_E + \hat{I}_S \otimes \hat{H}_E] |r_i\rangle \otimes |\beta_j\rangle \\ &\quad \hat{H}_E + \sum_{\alpha} C_{\alpha} \hat{B}_{\alpha} \end{aligned}$$

$$U(t) |r_i\rangle \otimes |\beta_j\rangle = U_S(t) \otimes U_E(t) (|r_i\rangle \otimes |\beta_j\rangle)$$

等效为 S 与 E 不耦合的系统

动力学退耦

$$\hat{H}_{SE} = \sum_{\alpha=x,y,z} \hat{\sigma}_{\alpha} \otimes \hat{B}_{\alpha}$$

$$\begin{aligned} \hat{X} \hat{H}_{SE} \hat{X} &= (\hat{\sigma}_x \otimes \hat{I}_E) \sum_{\alpha=x,y,z} (\hat{\sigma}_{\alpha} \otimes \hat{B}_{\alpha}) (\hat{\sigma}_x \otimes \hat{I}_E) \\ &= \hat{\sigma}_x \otimes \hat{B}_x - \hat{\sigma}_y \otimes \hat{B}_y - \hat{\sigma}_z \otimes \hat{B}_z \end{aligned}$$

$$\hat{H}_S = \lambda(t) \sigma_x \quad \text{控制 } \lambda(t) \text{ 在 } \delta \ll \tau \text{ 内达高强度. } \lambda \delta = \frac{\pi}{2}$$

$$e^{-i\delta \lambda \hat{\sigma}_x} = \cos(\lambda \delta) - i \sinh(\lambda \delta) \hat{\sigma}_x = -i \hat{\sigma}_x$$

$$\text{又} \hat{H}_{SE} = \hat{\sigma}_z \otimes \hat{B}_z$$

$$\begin{aligned} U_{2\tau} &= \hat{X} \hat{U}_\tau \hat{X} U_\tau = e^{-i\delta\lambda \hat{\sigma}_x} e^{-i\tau \hat{H}_{SE}} e^{-i\delta\lambda \hat{\sigma}_x} e^{-i\tau \hat{H}_{SE}} \\ &= (-i\hat{\sigma}_x \otimes \hat{I}_E) \cdot e^{-i\tau \hat{H}_{SE}} \cdot (-i\hat{\sigma}_x \otimes \hat{I}_E) \cdot e^{-i\tau \hat{H}_{SE}} \\ &= -e^{-i\tau \underbrace{\hat{X} \hat{H}_{SE} \hat{X}}_{-\hat{H}_{SE}}} \cdot e^{-i\tau \hat{H}_{SE}} = -\hat{I}_S \otimes \hat{I}_E \quad \text{抵消了 } \hat{H}_{SE} \end{aligned}$$

$$\hat{H}_{SE} = \sum_{\alpha=x,y,z} \hat{\sigma}_\alpha \otimes \hat{B}_\alpha \quad U'_{2\tau} = \hat{X} \hat{U}_\tau \hat{X} \hat{U}_\tau \simeq e^{-i2\tau \hat{\sigma}_x \otimes \hat{B}_x} + O(\tau)$$

$$U''_{4\tau} = \hat{Y} U'_{2\tau} \hat{Y} U'_{2\tau} = \underbrace{\hat{Z}}_{i\hat{Y}\hat{X}} U_\tau \hat{X} U_\tau \hat{Z} U_\tau \hat{X} U_\tau + O(\tau)$$

$$= (\hat{Z} U_\tau \hat{Z})(\hat{Y} U_\tau \hat{Y})(\hat{X} U_\tau \hat{X})(\hat{I} U_\tau \hat{I})$$

Langevin 方程

$$\langle F_s(t) \rangle = 0$$

$$m \frac{dv}{dt} = -\gamma v + F_s(t)$$

$$\langle F_s(t) F_s(t') \rangle = m^2 \Omega \delta(t-t')$$

$$\frac{dv}{dt} = -\gamma v + F(t)$$

$$v(t) = v_0 \exp(-\gamma t) + \int_0^t \exp[-\gamma(t-\tau)] \underbrace{F(\tau) d\tau}_{dU}$$

$$\Rightarrow d v(t) = -\gamma v(t) dt + dU(t)$$

$$\begin{aligned} v(t) - v(0) &= -\gamma \int_0^t v(s) ds + \underbrace{[U(t) - U(0)]}_{= \int_0^t d\tau F(\tau)} \\ &\simeq \sum_{k=1}^n [U(t_k) - U(t_{k-1})] \end{aligned}$$

$n \rightarrow \infty$ 时, $U(t)$ 对应平均值为 0 的 Gauss 分布.

Wiener 过程 $U(t) = W(t)$.

$$v(t) = v_0 \exp(-\gamma t) + \int_0^t \exp[-\gamma(t-\tau)] dW(\tau)$$

⑦ $\langle v(t) v(t') \rangle = \langle v(t) \rangle \langle v(t') \rangle = 0$ t, t' 相差远

$$\begin{aligned} \langle v(t) v(t') \rangle &= \int_0^t \int_0^{t'} \exp[-\gamma(t+t'-\tau-\tau')] \langle dW(\tau) dW(\tau') \rangle \\ &= \int_0^t \int_0^{t'} \exp[-\gamma(t+t'-\tau-\tau')] \underbrace{\langle F(\tau) F(\tau') \rangle}_{m^2 \Omega \delta(\tau-\tau')} d\tau d\tau' \\ &= \Omega \int_0^t \exp[-\gamma(t+t') + 2\gamma\tau] d\tau \\ &= \frac{\Omega}{2\gamma} \left\{ \exp[-\gamma(t-t)] - \exp[-\gamma(t+t')] \right\} \end{aligned}$$

$$\xrightarrow{t+t' \gg \frac{1}{\gamma}} = \frac{\Omega}{2\gamma} \exp[-\gamma(t-t)] \quad \text{速度关联} \downarrow \quad \text{弛豫时间 } \frac{1}{\gamma}$$

$$\text{平衡时} \quad \frac{1}{2} m \langle v(t) \rangle^2 = \frac{1}{2} k_B T \quad \Rightarrow \quad Q = \frac{2\gamma k_B T}{m}$$

Fokker-Planck 方程

$$\text{满足} \quad \frac{dv}{dt} = -\gamma v + F(t)$$

概率分布 $P(v, t)$. t 时刻找到 $[v, v+dv]$ 内粒子概率.

$$\frac{dP}{dt} = \left(\frac{\partial}{\partial v} \gamma v + \frac{Q}{2} \frac{\partial^2}{\partial v^2} \right) P$$

$$\text{稳态} \quad \frac{\partial P}{\partial t} = 0 \quad P_{st}(v, t) = \int \frac{\gamma}{\pi Q} \exp(-\gamma v/Q) = \sqrt{\frac{m}{2\pi k_B T Q}} \exp\left(-\frac{mv^2}{2k_B T}\right)$$

Ito 积分

一般随机微分方程形式

$$\frac{dx}{dt} = a(x, t) + b(x, t) \xi(t) \quad \begin{aligned} \langle \xi(t) \rangle &= 0 \\ \langle \xi(t) \xi(t') \rangle &= \delta(t-t') \end{aligned}$$

Wiener 过程

$$dW(\tau) = W(\tau+d\tau) - W(\tau) = \xi(\tau) d\tau$$

$$x(t) - x(0) = \int_0^t d\tau a(x, \tau) + \int_0^t \underbrace{dW(\tau)}_{\text{样本函数}} b(x, \tau)$$

关于随机过程积分

$$\int_0^t G(\tau) dW(\tau) \rightarrow S_n = \sum_{i=1}^n G(\tau_i) [W(t_i) - W(t_{i-1})]$$

积分结果随 T 变化.

$\Rightarrow$ 固定 T 为 t_{i-1} . Ito 积分.

$$\int_0^t G(\tau) dW(\tau) = \lim_{n \rightarrow \infty} \sum_{i=1}^n G(t_{i-1}) [W(t_i) - W(t_{i-1})]$$

平均平方极限

光与物质相互作用

共振荧光 weak field

$$\hat{H}_0 = \frac{1}{2} \hbar \omega_{eg} \hat{\sigma}_z$$

$$\hat{V} = -\frac{1}{2} \hbar \left(\Omega_R \hat{\sigma}_+ e^{-i\omega t} + \Omega_R^* \hat{\sigma}_- e^{i\omega t} \right) \quad \Omega_R = \frac{\langle e | \hat{d} | g \rangle \cdot \mathcal{E}}{\hbar}$$

弱光场激发, 自发辐射

$$\frac{d\rho(t)}{dt} = -\frac{i}{\hbar} [\hat{H}_0 + \hat{V}, \rho] + \frac{\Gamma}{2} (\bar{n}+1) D(\hat{\sigma}_-)[\rho] + \frac{\Gamma}{2} \bar{n} D(\hat{\sigma}_+)[\rho]$$

$$D(\hat{\sigma})[\rho] = 2\hat{\sigma}\rho\hat{\sigma}^\dagger - \hat{\sigma}^\dagger\hat{\sigma}\rho - \rho\hat{\sigma}^\dagger\hat{\sigma}$$

环境中 $\bar{n} \approx 0$

$$\Rightarrow \frac{d\rho_{ee}}{dt} = -\Gamma\rho_{ee} - i\frac{1}{2}(\Omega_R e^{-i\omega t} - c.c.)$$

$$\frac{d\rho_{eg}}{dt} = -\left(i\omega_{eg} + \frac{\Gamma}{2}\right)\rho_{eg} + i\frac{\Omega_R}{2}(\rho_{ee} - \rho_{gg}) e^{-i\omega t}$$

$$\rho_{eg}(t) = i\frac{\Omega_R}{2} \int dt' e^{[(\omega_{eg}-\omega) + \frac{\Gamma}{2}](t-t')} [\rho_{ee}(t') - \rho_{gg}(t')]$$

$$t \sim \Gamma^{-1} \Rightarrow i\frac{\Omega_R}{2} \frac{e^{-i\omega t} - e^{i(\omega_{eg}-\omega+\Gamma/2)t}}{i(\omega_{eg}-\omega) + \Gamma/2} [\rho_{ee}(0) - \rho_{gg}(0)]$$

$$t \gg \Gamma^{-1} \Rightarrow i\frac{\Omega_R}{2} \frac{e^{-i\omega t}}{i(\omega_{eg}-\omega) + \Gamma/2} [\rho_{ee}(0) - \rho_{gg}(0)]$$

偶极矩振荡频率与光场频率相同

强场耦合

dressed state dynamic Stark splitting

$$\hat{V}_I = e^{i\hat{H}_0 t/\hbar} \hat{V} e^{-i\hat{H}_0 t/\hbar} = \frac{1}{2} \hbar \Omega_R (\hat{\sigma}_+ + \hat{\sigma}_-)$$

$$\text{非经典纠缠态 } |t\rangle = \frac{1}{\sqrt{2}}(|e\rangle \pm |g\rangle)$$

$$\frac{d\tilde{\rho}(t)}{dt} = -\frac{i}{\hbar} [\hat{V}_I, \tilde{\rho}] - \frac{\Gamma}{2} (\hat{\sigma}_+ \hat{\sigma}_- \tilde{\rho} + \tilde{\rho} \hat{\sigma}_+ \hat{\sigma}_- - 2\hat{\sigma}_- \tilde{\rho} \hat{\sigma}_+)$$

在 $|t\rangle, |t\rangle$ 基矢

$$\dot{\tilde{\ell}}_{++} = -\frac{\Gamma}{2}\tilde{\ell}_{++} + \frac{\Gamma}{4}$$

$$\dot{\tilde{\ell}}_{+-} = -\left(\frac{3\Gamma}{4} + i\Omega_R\right)\tilde{\ell}_{+-} - \frac{\Gamma}{4}\ell_{++} - \frac{\Gamma}{2}$$

$$\dot{\tilde{\ell}}_{-+} = -\left(\frac{3\Gamma}{4} - i\Omega_R\right)\tilde{\ell}_{-+} - \frac{\Gamma}{4}\ell_{+-} - \frac{\Gamma}{2}$$

$$\text{耗散率} > \frac{\Gamma}{2} \text{ (裸原子)} \quad \text{频率移动} \pm \Omega_R$$

远场光场算符

$$E^+(\vec{r}, t) = \frac{\omega_0^2}{4\pi\epsilon_0 c^2 r} \left(\vec{d} \times \frac{\vec{r}}{r} \right) \times \frac{\vec{r}}{r} \hat{\sigma}_-(t - \frac{r}{c})$$

$$\text{功率} \quad S(\vec{r}, \omega) = \lim_{t \rightarrow \infty} \int_{-\infty}^{\infty} d\tau \langle E^-(\vec{r}, t+\tau) E^+(\vec{r}, t) \rangle e^{i\omega\tau}$$

$$= \lim_{t \rightarrow \infty} \int_{-\infty}^{\infty} d\tau \langle E^-(\vec{r}, t) E^+(\vec{r}, t+\tau) \rangle e^{-i\omega\tau}$$

$$= I_0(\vec{r}) \int_{-\infty}^{\infty} d\tau \langle \hat{\sigma}_+(t) \hat{\sigma}_-(t+\tau) \rangle e^{-i\omega\tau}$$

$$\text{Tr} \left\{ \hat{\sigma}_-(\cdot) e^{i\omega\tau} [\rho(\cdot) \hat{\sigma}_+(\cdot)] \right\}$$

...

$$\langle \hat{\sigma}_+(t) \hat{\sigma}_-(t+\tau) \rangle_I \simeq \left(\frac{\Omega_R}{\Gamma} \right)^2$$

$$\widetilde{S}(\vec{r}, \omega) = I_0(\vec{r}) \int_{-\infty}^{\infty} d\tau \langle \widetilde{\sigma}_+(t) \widetilde{\sigma}_-(t+\tau) \rangle_I e^{-i\omega\tau}$$

$$= I_0(\vec{r}) \left(\frac{\Omega_R}{\Gamma} \right)^2 2\pi \delta(\omega)$$

弱光场激发, $\omega = \omega'$, Rayleigh scattering.

强, $\Omega_R \gg \Gamma/4$, 调制边带.

$\Omega_R \uparrow$, Mollow triplet.

反聚束.

electro-magnetically induced transparency (EIT)

$$\hat{H}_0 = \hbar\omega_1 |1\rangle\langle 1| + \hbar(\omega_1 + \omega_p) |3\rangle\langle 3| + \hbar(\omega_1 + \omega_p - \omega_c) |2\rangle\langle 2|$$

$$\hat{H}_{int} = \hbar\Delta_p |3\rangle\langle 3| + \hbar(\Delta_p - \Delta_c) |2\rangle\langle 2| - \frac{\hbar}{2}(\Omega_p \hat{\sigma}_{31} + \Omega_c \hat{\sigma}_{32} + h.c.)$$

$$\Delta_p = \omega_3 - \omega_1 - \omega_p$$

$$\Delta_p = \Delta_c = \Delta \text{ 时}$$

$$\tan\theta = \frac{\Omega_p}{\Omega_c}$$

$$|\psi^+\rangle = \sin\theta(\sin\phi|1\rangle + \cos\phi|2\rangle) + \cos\theta|3\rangle$$

$$|\psi^-\rangle = \cos\theta(\sin\phi|1\rangle + \cos\phi|2\rangle) - \sin\theta|3\rangle$$

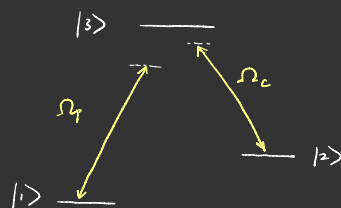
$$|\psi^0\rangle = \cos\phi|1\rangle - \sin\phi|2\rangle$$

dark state

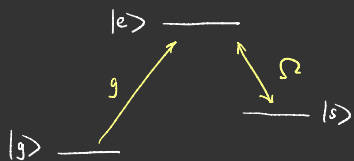
$$\phi = 0, |\psi^0\rangle = |1\rangle, \quad \phi = \frac{\pi}{2}, |\psi^0\rangle = -|2\rangle$$

adiabatic

$\Gamma_2 \rightarrow 0, \chi \rightarrow 0$. 无色散, 无吸收, EIT.



存储



$$|g_a\rangle = |g\rangle_1 |g\rangle_2 \dots |g\rangle_N = \bigotimes_{j=1}^N |g\rangle_j$$

$$|\bar{s}_a\rangle = \frac{1}{\sqrt{N}} \sum_{j=1}^N |g\rangle_1 \dots |s\rangle_j \dots |g\rangle_N$$

$$|\bar{e}_a\rangle = \frac{1}{\sqrt{N}} \sum_{j=1}^N |g\rangle_1 \dots |e\rangle_j \dots |g\rangle_N$$

$$\text{基} \left\{ |g\rangle|1\rangle, |\bar{e}_a\rangle|0\rangle, |\bar{s}_a\rangle|0\rangle \right\}$$

$g\sqrt{N}$

$\Omega(t)$

dark state $|\psi(t)\rangle = \frac{1}{\sqrt{g^2 N + \Omega(t)^2}} \left(\Omega(t) |\bar{g}_a\rangle |1\rangle - g\sqrt{N} |\bar{s}_a\rangle |0\rangle \right)$

初始 $\Omega \gg g\sqrt{N} \longrightarrow \Omega \rightarrow 0$

$|\bar{g}_a\rangle |1\rangle \xrightarrow{\text{转移}} |\bar{s}_a\rangle |0\rangle$

光场耦合原子团

$|\Phi\rangle \propto |\bar{g}_a\rangle |0\rangle + e^{i\phi} \sqrt{p} |\bar{s}_a\rangle |1\rangle + O(p)$
 激发概率 Δ 大时 $p \ll 1$.

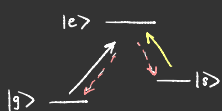
入射光场 $\mathcal{E}_{ge} = u(r, t) e^{i(\vec{k} \cdot \vec{r} - \omega t)}$

出射 $\mathcal{E}_{se} \propto \int \hat{a}_k e^{i(\vec{k} \cdot \vec{r} - \omega t)} d^3k$

$\hat{H}(t) = \underbrace{\Delta \sum_{j=1}^N \hat{\sigma}_{ee}^j}_{\omega_{eg} - \omega_0} + \left\{ g_{eg} \sum_{j=1}^N \hat{\sigma}_{ge}^j u(\vec{r}_j, t) e^{i\vec{k} \cdot \vec{r}_j} + \sum_{j=1}^N \hat{\sigma}_{se}^j \int g_{se}^k \hat{a}_k e^{-i[\vec{k} \cdot \vec{r}_j - (\omega_k - \omega_0 + \omega_{eg})t]} d^3k + h.c. \right\}$

... H_{eff} $|\psi_f\rangle$... 光场 $\longrightarrow \bar{z}$

写 + 读出



$|\phi_s\rangle \propto |0\rangle |0\rangle + e^{i\phi'} |1\rangle |1\rangle + O(p)$

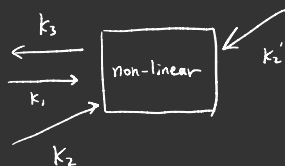
Four-wave mixing

$$\vec{P} = \chi^{(1)} \vec{E} + \chi^{(2)} \cdot \vec{E} \vec{E} + \chi^{(3)} \vec{E} \vec{E} \vec{E} + \dots$$

$$P_i(\omega_4, \vec{r}) = \frac{1}{2} \chi_{ijkl}^{(3)} (-\omega_4, \omega_1, -\omega_2, \omega_3) \varepsilon_j(\omega_1) \varepsilon_k^*(\omega_2) \varepsilon_l(\omega_3) e^{i(\vec{k}_1 - \vec{k}_2 + \vec{k}_3) \cdot \vec{r} - i\omega_4 t} + c.c.$$

phase match $\omega_4 = \omega_1 - \omega_2 + \omega_3$

$$\vec{k}_4 = \vec{k}_1 - \vec{k}_2 + \vec{k}_3$$



$$\vec{k}_1 + \vec{k}_3 = \vec{k}_2 + \vec{k}_2'$$

$$\vec{E}_j = \frac{1}{2} \varepsilon_j e^{i(\vec{k}_j \cdot \vec{r} - \omega t)} + c.c.$$

$$\frac{\partial^2 \varepsilon}{\partial z^2} - \frac{1}{c^2} \frac{\partial^2 \varepsilon}{\partial t^2} = \mu_0 \frac{\partial^2 P}{\partial t^2}$$

$$\varepsilon = \varepsilon_1 + \varepsilon_2 + \varepsilon_2' + \varepsilon_3 \quad P = \chi^{(3)} P^3$$

$$\frac{\partial \varepsilon_i}{\partial z} + \frac{1}{c} \frac{\partial \varepsilon_i}{\partial t} = i \frac{k}{2\varepsilon_0} P_i$$

$$\Rightarrow \frac{\partial \varepsilon_1}{\partial z} = \frac{i\omega}{\varepsilon_0 c} \tilde{P}_1 \quad \tilde{P}_1 = \frac{3\chi^{(3)}}{4} \left(|\varepsilon_2|^2 \varepsilon_1 + |\varepsilon_3|^2 \varepsilon_1 + \varepsilon_2 \varepsilon_2' \varepsilon_3^* \right)$$

$$\frac{\partial \varepsilon_3}{\partial z} = \frac{-i\omega}{\varepsilon_0 c} \tilde{P}_3 \quad \tilde{P}_3 = \frac{3\chi^{(3)}}{4} \left(|\varepsilon_1|^2 \varepsilon_3 + |\varepsilon_2|^2 \varepsilon_3 + \varepsilon_1 \varepsilon_2 \varepsilon_2'^* \right)$$

$$\Rightarrow \frac{\partial \varepsilon_1}{\partial z} = i k_1 \varepsilon_1 + i k_1 \varepsilon_3^* \quad k_1 = \frac{3\omega \chi^{(3)}}{4\varepsilon_0} \left(|\varepsilon_2|^2 + |\varepsilon_3|^2 \right)$$

$$\frac{\partial \varepsilon_3}{\partial z} = -i k_1 \varepsilon_3 - i k_1 \varepsilon_1^* \quad k = \frac{3\omega \chi^{(3)}}{4\varepsilon_0 c} \varepsilon_2 \varepsilon_2'$$

$$\text{define } \tilde{\varepsilon}_1 = \varepsilon_1 e^{-i k_1 z} \quad \tilde{\varepsilon}_3 = \varepsilon_3 e^{i k_1 z}$$

$$\Rightarrow \frac{\partial \tilde{\mathcal{E}}_1}{\partial z} = i k \tilde{\mathcal{E}}_3^*$$

$$\frac{\partial \tilde{\mathcal{E}}_3}{\partial z} = -i k \tilde{\mathcal{E}}_1^*$$

由 $\tilde{\mathcal{E}}_1^*$ 驱动.

相位共轭光场

经典光场与原子的相互作用

设自由原子系统有 $\hat{H}_0 |i\rangle = E_i |i\rangle$.

考察 TLS.

$$E_g = \hbar \omega_g, \quad E_e = \hbar \omega_e.$$

$$|e\rangle \text{ ————— } \hat{H}_0 = (|e\rangle\langle e| + |g\rangle\langle g|) \hat{H}_0 (|e\rangle\langle e| + |g\rangle\langle g|) = \hbar \omega_e |e\rangle\langle e| + \hbar \omega_g |g\rangle\langle g|.$$

$$|g\rangle \text{ ————— }$$

$$\text{定义 } |e\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |g\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

$$\hat{\sigma}_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = |e\rangle\langle g| + |g\rangle\langle e|.$$

$$\hat{\sigma}_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = -i|e\rangle\langle g| + i|g\rangle\langle e|.$$

$$\hat{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = |e\rangle\langle e| - |g\rangle\langle g|.$$

激光照射原子产生电偶极耦合时

$$\hat{H}_I = -\vec{d} \cdot \vec{E}(t)$$

$$= -(|e\rangle\langle e| + |g\rangle\langle g|) \vec{d} (|e\rangle\langle e| + |g\rangle\langle g|) \cdot \vec{E}(t)$$

$$= -(\vec{d}_{eg} |e\rangle\langle g| + \vec{d}_{ge} |g\rangle\langle e|) \cdot \vec{E}(t)$$

$$\vec{d}_{ee} = \vec{d}_{gg} = 0 \quad (\text{稳态})$$

$$\text{假设 } \vec{E}(t) = \hat{e} E \cos \omega t.$$

$\hat{e}$ 电场偏振方向

高频振荡项

$$\hat{H}_I = -\frac{\hbar}{2} \left(\Omega_{eg} |e\rangle\langle g| e^{-i\omega t} + \Omega_{ge} |g\rangle\langle e| e^{i\omega t} + \underbrace{\Omega_{eg} |e\rangle\langle g| e^{i\omega t} + \Omega_{ge} |g\rangle\langle e| e^{-i\omega t}}_{\text{高频振荡项}} \right)$$

相互作用 Hamiltonian

$$\Omega_{eg} = \frac{\vec{d}_{eg} \cdot \hat{e} E}{\hbar}, \quad \Omega_{ge} = \Omega_{eg}^*.$$

转到 interaction picture.

$$\hat{H}_0' = \hbar(\omega_g + \omega) |e\rangle\langle e| + \hbar \omega_g |g\rangle\langle g| - \frac{\hbar(\omega_{eg} - \omega)}{2} (|e\rangle\langle e| + |g\rangle\langle g|).$$

$$\omega_{eg} = \omega_e - \omega_g.$$

$$\tilde{H}_I = e^{i\hat{H}_0' t / \hbar} H_I e^{-i\hat{H}_0' t / \hbar}$$

$$= -\frac{\hbar}{2} \left(\Omega_{eg} |e\rangle\langle g| + \Omega_{ge} |g\rangle\langle e| + \underbrace{\Omega_{eg} |e\rangle\langle g| e^{2i\omega t} + \Omega_{ge} |g\rangle\langle e| e^{-2i\omega t}}_{\text{高频振荡项}} \right)$$

RWA.

$$\text{得 } \tilde{H} = \frac{\hbar\Delta}{2}\sigma_z - \frac{\hbar}{2}(\Omega_{eg}|e\rangle\langle g| + \Omega_{ge}|g\rangle\langle e|)$$

$$\Delta = \omega_{eg} - \omega.$$

$$= \frac{\hbar}{2}[\Delta\sigma_z - \Omega_R(e^{i\phi}\sigma_+ + e^{-i\phi}\sigma_-)].$$

$$|\Omega_{eg}| = |\Omega_{ge}| = \Omega_R.$$

原子波函数 (Interaction Picture)

$$|\tilde{\psi}\rangle = C_e(t)|e\rangle + C_g(t)|g\rangle$$

演化方程

$$\begin{bmatrix} \dot{C}_e(t) \\ \dot{C}_g(t) \end{bmatrix} = -\frac{i}{2} \begin{bmatrix} \Delta & \Omega_R e^{i\phi} \\ \Omega_R e^{-i\phi} & -\Delta \end{bmatrix} \begin{bmatrix} C_e(t) \\ C_g(t) \end{bmatrix}.$$

形式解

$$\begin{bmatrix} C_e(t) \\ C_g(t) \end{bmatrix} = \begin{bmatrix} \cos\left(\frac{\Omega t}{2}\right) - i\frac{\Delta}{\Omega}\sinh\left(\frac{\Omega t}{2}\right) & +i\frac{\Omega_R e^{i\phi}}{\Omega}\sinh\left(\frac{\Omega t}{2}\right) \\ +i\frac{\Omega_R e^{-i\phi}}{\Omega}\sinh\left(\frac{\Omega t}{2}\right) & \cos\left(\frac{\Omega t}{2}\right) + i\frac{\Delta}{\Omega}\sinh\left(\frac{\Omega t}{2}\right) \end{bmatrix} \begin{bmatrix} C_e(0) \\ C_g(0) \end{bmatrix}$$

$$\Omega = \sqrt{\Omega_R^2 + \Delta^2}.$$

$$\text{有 } |C_e(t)|^2 + |C_g(t)|^2 = |C_e(0)|^2 + |C_g(0)|^2 = 1.$$

若假定原子初始处于 $|e\rangle$.

$$\begin{aligned} W(t) &= \langle \hat{\sigma}_z \rangle = |C_e(t)|^2 - |C_g(t)|^2 \\ &= \frac{\Delta^2 - \Omega_R^2}{\Omega^2} \sinh^2\left(\frac{\Omega t}{2}\right) + \cos^2\left(\frac{\Omega t}{2}\right). \end{aligned}$$

共振时 $\Delta = 0$, $\Omega = \Omega_R$.

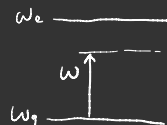
$$W(t) = \cos(\Omega_R t).$$

补充推导

并不是? 应该是 $U_0^\dagger H_0 U_0$

$$\hat{H}_0 = \hbar \omega_e |e\rangle\langle e| + \hbar \omega_g |g\rangle\langle g|$$

$$= \hbar(\omega_g + \omega) |e\rangle\langle e| + \hbar \omega_g |g\rangle\langle g| - \hbar(\omega - \omega_{eg}) |e\rangle\langle e|$$



$$\hat{V} = -(\vec{d}_{eg} |e\rangle\langle g| + \vec{d}_{ge} |g\rangle\langle e|) \cdot \vec{E}(t) \quad \vec{E} \cos \omega t$$

$$= -\vec{d}_{eg} \cdot \vec{E} |e\rangle\langle g| \frac{e^{i\omega t} + e^{-i\omega t}}{2} - \vec{d}_{ge} \cdot \vec{E} |g\rangle\langle e| \frac{e^{i\omega t} + e^{-i\omega t}}{2}$$

$$= -\frac{\hbar}{2} \left(\Omega_{eg} |e\rangle\langle g| e^{-i\omega t} + \Omega_{ge} |g\rangle\langle e| e^{i\omega t} + \Omega_{eg} |e\rangle\langle g| e^{i\omega t} + \Omega_{ge} |g\rangle\langle e| e^{-i\omega t} \right)$$

$$V_I = e^{iH_0 t/\hbar} \hat{V} e^{-iH_0 t/\hbar}$$

验证结果. 若

$$H'_0 = \begin{pmatrix} \hbar(\omega_g + \omega) - \frac{\hbar\Delta}{2} & 0 \\ 0 & \hbar\omega_g - \frac{\hbar\Delta}{2} \end{pmatrix}$$

$$\hbar(\omega_g + \frac{\omega}{2} - \frac{\Delta}{2}) \mathbb{1} + \hbar(\frac{\omega}{2} - \frac{\Delta}{2}) \sigma_z$$

$$e^{A+B} = e^A e^B e^{-\frac{1}{2}[A,B]}$$

$$e^{iH'_0 t/\hbar} = e^{[i(\omega_g + \frac{\omega}{2})t \mathbb{1} + i(\frac{\omega}{2} - \frac{\Delta}{2})t \sigma_z]}$$

$$= e^{i(\omega_g + \frac{\omega}{2})t \mathbb{1}} e^{i(\frac{\omega}{2} - \frac{\Delta}{2})t \sigma_z} = \left[\cos(\omega_g + \frac{\omega}{2})t + i \sinh(\omega_g + \frac{\omega}{2})t \mathbb{1} \right]$$

$$\left[\cos(\frac{\omega}{2} - \frac{\Delta}{2})t + i \sinh(\frac{\omega}{2} - \frac{\Delta}{2})t \sigma_z \right]$$

$$= e^{i(\omega_g + \frac{\omega}{2})t} \left[\mathbb{1} \cos(\frac{\omega}{2} - \frac{\Delta}{2})t + i \sinh(\frac{\omega}{2} - \frac{\Delta}{2})t \sigma_z \right]$$

$$e^{-iH'_0 t/\hbar} = e^{-i(\omega_g + \frac{\omega}{2})t} e^{-i(\frac{\omega}{2} - \frac{\Delta}{2})t \sigma_z}$$

$$e^{-iH_0 t/\hbar} \vee e^{iH_0 t/\hbar} =$$

$$-\frac{\hbar}{2} \left(\Omega_{eg} |e\rangle\langle g| e^{-i\omega t} + \Omega_{ge} |g\rangle\langle e| e^{i\omega t} + \Omega_{eg} |e\rangle\langle g| e^{i\omega t} + \Omega_{ge} |g\rangle\langle e| e^{-i\omega t} \right)$$

$$e^{-i(\frac{\omega-\Delta}{2})t \sigma_z} \left(\right) e^{i(\frac{\omega-\Delta}{2})t \sigma_z}$$

$$\frac{\hbar}{2} \begin{pmatrix} 0 & \Omega_{eg} (e^{i\omega t} + e^{-i(2\omega-\Delta)t}) \\ \Omega_{ge} (e^{i(2\omega-\Delta)t} + e^{-i\omega t}) & 0 \end{pmatrix} \quad \text{不太对?}$$

从头再来一次.

$$H_0 = \hbar\omega_e |e\rangle\langle e| + \hbar\omega_g |g\rangle\langle g| = \begin{pmatrix} \hbar(\Delta + \omega_g + \omega) & \\ & \hbar\omega_g \end{pmatrix}$$

到 interaction picture.

$$H_0' = U_0^\dagger H_0 U_0 = e^{iH_0 t/\hbar} H_0 e^{-iH_0 t/\hbar}$$

$$H_0 = \hbar(\omega_g + \frac{\Delta + \omega}{2}) \mathbb{1} + \hbar \frac{\Delta + \omega}{2} \sigma_z \quad \text{书上的结果}$$

可以直接写?

一般的 Pauli 矩阵

组合可以直接写吗?

$$e^{-iH_0 t/\hbar} = e^{-i(\omega_g + \frac{\Delta + \omega}{2})t} \mathbb{1} \cdot \begin{pmatrix} e^{-i(\frac{\Delta + \omega}{2})t} & 0 \\ 0 & e^{i(\frac{\Delta + \omega}{2})t} \end{pmatrix} = \begin{pmatrix} e^{-i(\omega_g + \Delta + \omega)t} & 0 \\ 0 & e^{-i\omega_g t} \end{pmatrix}$$

$$e^{iH_0 t/\hbar} = \begin{pmatrix} e^{i(\omega_g + \Delta + \omega)t} & 0 \\ 0 & e^{i\omega_g t} \end{pmatrix}$$

$$\begin{pmatrix} e^{i(\omega_g + \Delta + \omega)t} & 0 \\ 0 & e^{i\omega_g t} \end{pmatrix} \cdot -\frac{\hbar}{2} \begin{pmatrix} 0 & \Omega_{eg}(e^{i\omega t} + e^{-i\omega t}) \\ \Omega_{ge}(e^{-i\omega t} + e^{i\omega t}) & 0 \end{pmatrix} \begin{pmatrix} e^{-i(\omega_g + \Delta + \omega)t} & 0 \\ 0 & e^{-i\omega_g t} \end{pmatrix}$$

$$-\frac{\hbar}{2} \begin{pmatrix} 0 & \Omega_{eg}(e^{i(\omega - \omega_g)t} + e^{-i(\omega + \omega_g)t}) \\ \Omega_{ge}(e^{-i(\omega_g + \Delta)t} + e^{-i(\omega_g + \Delta + 2\omega)t}) & 0 \end{pmatrix}$$

$$-\frac{\hbar}{2} \begin{pmatrix} 0 & \Omega_{eg}(e^{i(\omega + \Delta)t} + e^{i\Delta t}) \\ \Omega_{ge}(e^{-i\Delta t} + e^{-i(\Delta + 2\omega)t}) & 0 \end{pmatrix}$$

一样的结果

$$H_0 = \frac{\hbar \omega_{eg}}{2} \sigma_z + \left(\frac{\hbar(\omega_e + \omega_g)}{2} \mathbf{1} \right)$$

没有这一项会怎样?

H'

$$e^{iHt/\hbar} = e^{i\left(\frac{\omega_e + \omega_g}{2}\right)t} \mathbf{1} \cdot \begin{pmatrix} e^{i\frac{\omega_{eg}}{2}t} & 0 \\ 0 & e^{-i\frac{\omega_{eg}}{2}t} \end{pmatrix}$$

差别?

$$= \begin{pmatrix} e^{i\omega_e t} & 0 \\ 0 & e^{i\omega_g t} \end{pmatrix} e^{i\left(\frac{\omega_e + \omega_g}{2}\right)t}$$

差一个相位

$$e^{iH_0 t/\hbar} H_0 e^{-iH_0 t/\hbar} = \begin{pmatrix} e^{i\omega_e t} & 0 \\ 0 & e^{i\omega_g t} \end{pmatrix} \begin{pmatrix} \hbar\omega_e & 0 \\ 0 & \hbar\omega_g \end{pmatrix} \begin{pmatrix} e^{-i\omega_e t} & 0 \\ 0 & e^{-i\omega_g t} \end{pmatrix}$$

$$= \begin{pmatrix} e^{i\omega_e t} & 0 \\ 0 & e^{i\omega_g t} \end{pmatrix} \begin{pmatrix} \hbar\omega_e e^{-i\omega_e t} & 0 \\ 0 & \hbar\omega_g e^{-i\omega_g t} \end{pmatrix}$$

$$V_{qc} = \begin{pmatrix} \hbar\omega_e & 0 \\ 0 & \hbar\omega_g \end{pmatrix} \quad H_0 \text{ 确实不变}$$

$$e^{iH_0 t/\hbar} V e^{-iH_0 t/\hbar} = \begin{pmatrix} e^{i\omega_e t} & 0 \\ 0 & e^{i\omega_g t} \end{pmatrix} \begin{pmatrix} 0 & \Omega_{eg}(e^{i\omega_e t} + e^{-i\omega_e t}) \\ \Omega_{ge}(e^{i\omega_g t} + e^{-i\omega_g t}) & 0 \end{pmatrix} \begin{pmatrix} e^{-i\omega_e t} & 0 \\ 0 & e^{-i\omega_g t} \end{pmatrix}$$

$$W - (\Delta + \omega) = \begin{pmatrix} e^{i\omega_e t} & 0 \\ 0 & e^{i\omega_g t} \end{pmatrix} \begin{pmatrix} 0 & \Omega_{eg}(e^{i(\omega_e - \omega_g)t} + e^{-i(\omega_e - \omega_g)t}) \\ \Omega_{ge}(e^{i(\omega_g - \omega_e)t} + e^{-i(\omega_g - \omega_e)t}) & 0 \end{pmatrix}$$

$$\Delta = \omega_{eg} - \omega = \begin{pmatrix} 0 & \Omega_{eg}(e^{i(2\omega + \Delta)t} + e^{-i(2\omega + \Delta)t}) \\ \Omega_{ge}(e^{i(-\Delta + 2\omega)t} + e^{-i(-\Delta + 2\omega)t}) & 0 \end{pmatrix}$$

$-\Delta \quad 2\omega + \Delta$

$$V_I = \Omega_{eg} e^{i\Delta t} (e^{2i\omega t} + 1) |e\rangle\langle g| + \Omega_{ge} e^{-i\Delta t} (e^{-2i\omega t} + 1) |g\rangle\langle e|.$$

$$|\phi(t)\rangle = \tilde{U}(t) |\psi(t)\rangle$$

$$i\hbar \frac{d\psi}{dt} = H|\psi\rangle$$

$$\begin{aligned} \frac{d|\phi(t)\rangle}{dt} &= \frac{d\tilde{U}(t)}{dt} |\psi(t)\rangle + \tilde{U}(t) \frac{d|\psi(t)\rangle}{dt} \\ &= \frac{d\tilde{U}}{dt} \tilde{U}^\dagger |\phi(t)\rangle + \tilde{U} \cdot \frac{H}{i\hbar} \cdot \tilde{U}^\dagger |\phi(t)\rangle \end{aligned}$$

$$i\hbar \frac{d|\phi(t)\rangle}{dt} = \left(\tilde{U} H \tilde{U}^\dagger + i\hbar \frac{d\tilde{U}}{dt} \tilde{U}^\dagger \right) |\phi(t)\rangle$$

对 TLS. $H_0 = \hbar\omega_e |e\rangle\langle e| + \hbar\omega_g |g\rangle\langle g|$

① $U = e^{-iH_0 t/\hbar}$ $H_{\text{eff}} = H_0 - H_0 = 0$

② $U = e^{-i\frac{\omega t}{2}}$ $H_{\text{eff}} = H_0 + i\hbar \cdot (i\frac{\omega}{2}) e^{-i\frac{\omega t}{2}} \cdot e^{i\frac{\omega t}{2}} = H_0 - \frac{1}{2}\hbar\omega$ ✓

引入相互作用 $V = -\frac{\hbar}{2} \left[\Omega_{eg} (e^{i\omega_e t} + e^{-i\omega_e t}) |e\rangle\langle g| + \Omega_{ge} (e^{i\omega_g t} + e^{-i\omega_g t}) |g\rangle\langle e| \right]$

① $U = e^{iH_0 t/\hbar}$ $H_{\text{eff}} = H_0 + e^{iH_0 t/\hbar} V e^{-iH_0 t/\hbar}$

$$\Rightarrow H_{\text{eff}} = \begin{pmatrix} \hbar\omega_e & \\ & \hbar\omega_g \end{pmatrix} + \begin{pmatrix} e^{i\omega_e t} & \\ & e^{i\omega_g t} \end{pmatrix} \cdot \frac{\hbar}{2} \begin{pmatrix} 0 & \Omega_{eg}(e^{i\omega_e t} + e^{-i\omega_e t}) \\ \Omega_{ge}(e^{i\omega_g t} + e^{-i\omega_g t}) & 0 \end{pmatrix} \begin{pmatrix} e^{-i\omega_e t} & 0 \\ 0 & e^{-i\omega_g t} \end{pmatrix}$$

$$= \begin{pmatrix} \hbar\omega_e & \\ & \hbar\omega_g \end{pmatrix} + \frac{\hbar}{2} \begin{pmatrix} 0 & \Omega_{eg}(e^{i(\omega_e + \omega_e - \omega_g)t} + e^{-i(\omega_e + \omega_g - \omega_e)t}) \\ \Omega_{ge}(e^{i(\omega_e + \omega_g - \omega_e)t} + e^{-i(\omega_e + \omega_e - \omega_g)t}) & 0 \end{pmatrix}$$

$\Delta = \omega_e - \omega_g - \omega$

$$= \begin{pmatrix} \hbar\omega_e & \\ & \hbar\omega_g \end{pmatrix} + \frac{\hbar}{2} \begin{pmatrix} & \Omega_{eg}(e^{i(\omega + \Delta)t} + e^{i\Delta t}) \\ \Omega_{ge}(e^{i\Delta t} + e^{-i(\omega + \Delta)t}) & \end{pmatrix}$$

$$= \hbar \frac{\omega_e + \omega_g}{2} \mathbb{1} + \frac{\hbar}{2} \begin{pmatrix} \Delta + \omega & \Omega_{eg}(e^{i\omega t} + e^{i(2\omega + \omega) t}) \\ \Omega_{ge}(e^{i\omega t} + e^{i(2\omega + \omega) t}) & -(\Delta + \omega) \end{pmatrix}$$

$$\textcircled{2} \quad U = e^{-\frac{i}{\hbar} H t} \quad H_{\text{eff}} = \tilde{U} H \tilde{U}^\dagger - \frac{1}{2} \hbar \omega$$

$$H = \hbar \frac{\omega_e + \omega_g}{2} \mathbb{1} + \frac{\hbar}{2} \begin{pmatrix} \Delta + \omega & \Omega_{eg}(e^{i\omega t} + e^{-i\omega t}) \\ \Omega_{ge}(e^{i\omega t} + e^{-i\omega t}) & -(\Delta + \omega) \end{pmatrix}$$

$$H_{\text{eff}} = \hbar \frac{\omega_e + \omega_g - \omega}{2} \mathbb{1} + \frac{\hbar}{2} \begin{pmatrix} \Delta + \omega & \Omega_{eg}(e^{i\omega t} + e^{-i\omega t}) \\ \Omega_{ge}(e^{i\omega t} + e^{-i\omega t}) & -(\Delta + \omega) \end{pmatrix}$$

$$H_0 = \hbar\omega_e |e\rangle\langle e| + \hbar\omega_g |g\rangle\langle g| = \frac{\hbar(\omega_e + \omega_g)}{2} \mathbb{1} + \frac{\hbar}{2}(\Delta + \omega) \sigma_z$$

$$e^{iH_0 t/\hbar} = \underbrace{e^{i\frac{(\omega_e + \omega_g)}{2}t}}_{\tilde{U}H\tilde{U}^\dagger + i\hbar\frac{d\tilde{U}}{dt}\tilde{U}^\dagger} e^{i\frac{(\Delta + \omega)}{2}t\sigma_z}$$

$$e^{i\frac{(\Delta + \omega)}{2}t\sigma_z} H e^{-i\frac{(\Delta + \omega)}{2}t\sigma_z} + i\hbar\left(\frac{\Delta + \omega}{2} \cancel{e^{i\frac{(\Delta + \omega)}{2}t\sigma_z}} - \frac{(\Delta + \omega)}{2} \cancel{e^{-i\frac{(\Delta + \omega)}{2}t\sigma_z}}\right)$$

$$\frac{\hbar(\omega_e + \omega_g)}{2} \mathbb{1} + \frac{\hbar}{2} \begin{pmatrix} \Delta + \omega & \Omega_{eg}(e^{i\omega_e t} + e^{-i\omega_e t}) \\ \Omega_{ge}(e^{i\omega_e t} + e^{-i\omega_e t}) & -(\Delta + \omega) \end{pmatrix} \quad e^{-i\frac{\omega_e t}{2}\sigma_z}$$

$$\frac{\hbar}{2} \begin{pmatrix} \Delta + \omega & \Omega_{eg}(e^{i\omega_e t} + e^{-i\omega_e t}) \\ \Omega_{ge}(e^{i\omega_e t} + e^{-i\omega_e t}) & -(\Delta + \omega) \end{pmatrix} \begin{pmatrix} e^{-i\frac{(\Delta + \omega)}{2}t} & \\ & e^{i\frac{(\Delta + \omega)}{2}t} \end{pmatrix}$$

$$\begin{pmatrix} e^{i\frac{(\Delta + \omega)}{2}t} & \\ & e^{-i\frac{(\Delta + \omega)}{2}t} \end{pmatrix} \begin{pmatrix} (\Delta + \omega) e^{-i\frac{(\Delta + \omega)}{2}t} & \Omega_{eg}(e^{i(\omega_e + \frac{\Delta + \omega}{2})t} + e^{i\frac{(\Delta - \omega)}{2}t}) \\ \Omega_{ge}(e^{i(\frac{\omega_e - \Delta}{2})t} + e^{-i(\omega_e + \frac{\Delta + \omega}{2})t}) & -(\Delta + \omega) e^{i\frac{(\Delta + \omega)}{2}t} \end{pmatrix}$$

$$\begin{pmatrix} \Delta + \omega & \Omega_{eg}(e^{i(2\omega_e + \Delta)t} + e^{i\Delta t}) \\ \Omega_{ge}(e^{-i\Delta t} + e^{-i(\omega_e + \Delta)t}) & -(\Delta + \omega) \end{pmatrix}$$

$$H_0 = \hbar\omega_e |e\rangle\langle e| + \hbar\omega_g |g\rangle\langle g| = \frac{\hbar(\omega_e + \omega_g)}{2} \mathbb{1} + \frac{\hbar(\Delta + \omega)}{2} \sigma_z$$

$$\underbrace{e^{-\frac{i\omega t}{2} \sigma_z}} \rightarrow \tilde{U} H \tilde{U}^\dagger + i\hbar \frac{d\tilde{U}}{dt} \tilde{U}^\dagger$$

$$\frac{\hbar}{2} \begin{pmatrix} \Delta + \omega & \Omega_{eg}(e^{i\omega t} + e^{-i\omega t}) \\ \Omega_{ge}(e^{i\omega t} + e^{-i\omega t}) & -(\Delta + \omega) \end{pmatrix} \begin{pmatrix} e^{\frac{i\omega t}{2}} & \\ & e^{-\frac{i\omega t}{2}} \end{pmatrix}$$

$$\begin{pmatrix} e^{-\frac{i\omega t}{2}} & \\ & e^{\frac{i\omega t}{2}} \end{pmatrix} \begin{pmatrix} (\Delta + \omega) e^{\frac{i\omega t}{2}} & \Omega_{eg} \left(e^{\frac{i\omega t}{2}} + e^{-\frac{3i\omega t}{2}} \right) \\ \Omega_{ge} \left(e^{\frac{3i\omega t}{2}} + e^{-\frac{i\omega t}{2}} \right) & -(\Delta + \omega) e^{\frac{i\omega t}{2}} \end{pmatrix}$$

$$\frac{\hbar}{2} \begin{pmatrix} \Delta & \Omega_{eg}(1 + e^{-2i\omega t}) \\ \Omega_{ge}(e^{2i\omega t} + 1) & -\Delta \end{pmatrix}$$

$$|\phi(t)\rangle = \tilde{U}(t) |\psi(t)\rangle$$

$$H|\psi(t)\rangle = i\hbar \frac{d|\psi(t)\rangle}{dt}$$

$$\frac{d|\phi(t)\rangle}{dt} = \frac{d\tilde{U}}{dt} |\psi(t)\rangle + \tilde{U} \frac{d|\psi(t)\rangle}{dt}$$

$$= \frac{d\tilde{U}}{dt} \tilde{U}^\dagger |\phi(t)\rangle + \tilde{U} \cdot \frac{H}{i\hbar} \tilde{U}^\dagger |\phi(t)\rangle$$

$$i\hbar \frac{d|\phi(t)\rangle}{dt} = \left(\tilde{U} H \tilde{U}^\dagger + i\hbar \frac{d\tilde{U}}{dt} \tilde{U}^\dagger \right) |\phi(t)\rangle$$

对 TLS. $H_0 = \hbar\omega_e |e\rangle\langle e| + \hbar\omega_g |g\rangle\langle g|$

$$\textcircled{1} \tilde{U} = e^{iH_0 t/\hbar} \quad H_{\text{eff}} = H_0 + i\hbar \cdot \frac{iH_0}{\hbar} = 0$$

$$\textcircled{2} \tilde{U} = e^{\frac{i\omega t}{2} \sigma_z} \quad H_{\text{eff}} = H_0 + i\hbar \cdot \frac{i\omega \sigma_z}{2} = H_0 - \frac{\hbar\omega}{2} \sigma_z$$

引入相互作用 $V = -\frac{\hbar}{2} \left[\Omega_{eg}(e^{i\omega t} + e^{-i\omega t}) |e\rangle\langle g| + \Omega_{ge}(e^{i\omega t} + e^{-i\omega t}) |g\rangle\langle e| \right]$

$$\textcircled{1} \tilde{U} = e^{iH_0 t/\hbar} \quad H_{\text{eff}} = H_0 + e^{iH_0 t/\hbar} V e^{-iH_0 t/\hbar} - H_0 = e^{iH_0 t/\hbar} V e^{-iH_0 t/\hbar}$$

$$H_0 = \hbar \frac{\omega_e + \omega_g}{2} \mathbf{1} + \frac{\hbar}{2} (\Delta + \omega) \sigma_z \quad H_{\text{eff}} = e^{i\frac{\hbar}{2}(\Delta + \omega) \sigma_z} V e^{-i\frac{\hbar}{2}(\Delta + \omega) \sigma_z}$$

$$H_{\text{eff}} = \begin{pmatrix} e^{\frac{i(\Delta + \omega)}{2} t} & 0 \\ 0 & e^{-\frac{i(\Delta + \omega)}{2} t} \end{pmatrix} \cdot \left(\frac{\hbar}{2} \right) \begin{pmatrix} \Omega_{eg}(e^{i\omega t} + e^{-i\omega t}) \\ \Omega_{ge}(e^{i\omega t} + e^{-i\omega t}) \end{pmatrix} \begin{pmatrix} e^{-\frac{i(\Delta + \omega)}{2} t} & 0 \\ 0 & e^{\frac{i(\Delta + \omega)}{2} t} \end{pmatrix}$$

$$= -\frac{\hbar}{2} \begin{pmatrix} 0 & \Omega_{eg}(e^{i2\omega t + i\Delta t} + e^{i\Delta t}) \\ \Omega_{ge}(e^{-i\Delta t} + e^{-i2\omega t - \Delta t}) & 0 \end{pmatrix}$$

$$(2) U = e^{\frac{i\omega t}{2}\sigma_z} \quad H_{\text{eff}} = e^{\frac{i\omega t}{2}\sigma_z} H e^{-\frac{i\omega t}{2}\sigma_z} - \frac{\hbar\omega}{2}\sigma_z$$

$$H = \hbar \frac{\omega_e + \omega_g}{2} \mathbb{1} + \frac{\hbar}{2}(\Delta + \omega)\sigma_z - \frac{\hbar}{2} \begin{pmatrix} \Omega_{eg}(e^{i\omega t} + e^{-i\omega t}) \\ \Omega_{ge}(e^{i\omega t} + e^{-i\omega t}) \end{pmatrix}$$

$$= \hbar \frac{\omega_e + \omega_g}{2} \mathbb{1} + \frac{\hbar}{2} \begin{pmatrix} (\Delta + \omega) & -\Omega_{eg}(e^{i\omega t} + e^{-i\omega t}) \\ \Omega_{ge}(e^{i\omega t} + e^{-i\omega t}) & -(\Delta + \omega) \end{pmatrix}$$

$$H_{\text{eff}} = \hbar \frac{\omega_e + \omega_g - \omega}{2} \mathbb{1} + \frac{\hbar}{2} \begin{pmatrix} (\Delta + \omega) & -\Omega_{eg}(e^{2i\omega t} + 1) \\ \Omega_{ge}(1 + e^{-2i\omega t}) & -(\Delta + \omega) \end{pmatrix}$$

$$= \frac{\hbar}{2}(\omega_e + \omega_g) \mathbb{1} + \frac{\hbar}{2} \begin{pmatrix} \Delta & -\Omega_{eg}(1 + e^{2i\omega t}) \\ \Omega_{ge}(1 + e^{-2i\omega t}) & -\Delta \end{pmatrix}$$

H_0 确实不变。只有 V_I 决定态矢的演化。

态矢会变吗? $|\psi\rangle = U|\psi\rangle$ $U = e^{-iH_0 t/\hbar}$

$$\hbar\omega_e |e\rangle\langle e| + \hbar\omega_g |g\rangle\langle g| \quad \text{与} \quad \frac{\hbar\omega_g}{2} \sigma_z \quad \text{区别?}$$

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$$\frac{\hbar\omega_g}{2} |e\rangle\langle e| + \frac{\hbar\omega_g}{2} |g\rangle\langle g| = \underbrace{\left(\frac{\hbar\omega_g}{2} \sigma_z\right)}_{\text{差了能量参考点}} + \frac{\hbar(\omega_e + \omega_g)}{2} (|e\rangle\langle e| + |g\rangle\langle g|)$$

$$\hbar \frac{\omega_e + \omega_g + \omega_g}{2} |e\rangle\langle e| + \hbar \frac{\omega_e + \omega_g - \omega_g}{2} |g\rangle\langle g|$$

关于绘景的巩固与补充

$$|\psi\rangle, \quad \bar{A} = \langle \psi | A | \psi \rangle$$

$$|\psi\rangle \rightarrow |\tilde{\psi}\rangle = U|\psi\rangle$$

$$A \rightarrow \tilde{A} = UAU^\dagger$$

$$\bar{A}' = \langle \psi | U^\dagger U A U^\dagger U | \psi \rangle = \bar{A} \text{ 不变}$$

Heisenberg Picture

$$A_H(t) = U^\dagger(t) A_S U(t), \quad |\psi\rangle_H = U^\dagger(t) |\psi_S(t)\rangle = |\psi_S(0)\rangle$$

$$\frac{dA_H}{dt} = \frac{dU^\dagger}{dt} A_S U + U^\dagger \frac{\partial A_S}{\partial t} U + U^\dagger A_S \frac{dU}{dt}$$

$$= i\hbar \underbrace{U^\dagger A_S U}_{\text{A in H picture}} + U^\dagger \frac{\partial A_S}{\partial t} U + \underbrace{U^\dagger A_S}_{\text{A in H picture}} \cdot \underbrace{(i\hbar U)}_{\text{H picture}}$$

$$= i\hbar [H, A_H] + \left(\frac{\partial A}{\partial t} \right)_H$$

$$\frac{d\langle A_H \rangle}{dt} = 0$$

$$H \text{ 不显含时间时, } U(t) = e^{-iHt} \text{ 且}$$

$$A_H(t) = e^{iHt} A_S e^{-iHt}$$

Interaction Picture

$$H = H_0 + V$$

U_0 : H_0 下的演化

$$A_I(t) = U_0^\dagger(t) A_S U_0(t)$$

$$|\psi\rangle_I = U_0^\dagger |\psi_S(t)\rangle = U_0^\dagger(t) U(t) |\psi_0\rangle$$

$$\frac{dA_I}{dt} = \frac{dU_0^\dagger}{dt} A_S U_0 + U_0^\dagger \frac{\partial A_S}{\partial t} U_0 + U_0^\dagger A_S \frac{dU_0}{dt}$$

$$= i\hbar U_0^\dagger A_S U_0 + \frac{\partial A_I}{\partial t} + U_0^\dagger A_S \cdot (-i\hbar) U_0$$

$$= i\hbar [H_0, A_I] + \frac{\partial A_I}{\partial t}$$

$$\begin{aligned}
\frac{d|\psi_I\rangle}{dt} &= \frac{dU_0^\dagger}{dt} U_0 |\psi_0\rangle + U_0^\dagger \frac{dU}{dt} |\psi_0\rangle \\
&= iH_0 U_0^\dagger U_0 |\psi_0\rangle + U_0^\dagger \cdot (-iH U) |\psi_0\rangle \\
&= iH_0 U_0^\dagger U |\psi_0\rangle - iU_0^\dagger H_0 U |\psi_0\rangle - \underline{iU_0^\dagger V U |\psi_0\rangle} \\
&= -i \underline{U_0^\dagger V U U_0^\dagger U} |\psi_0\rangle = -iV_I |\psi_I\rangle.
\end{aligned}$$

更一般的情形.

$$\text{令 } |\phi(t)\rangle = \tilde{U}(t) |\psi(t)\rangle$$

$$i\hbar \frac{d|\psi\rangle}{dt} = H|\psi\rangle.$$

$$\begin{aligned}
i\hbar \frac{d|\phi(t)\rangle}{dt} &= i\hbar \left(\frac{d\tilde{U}(t)}{dt} |\psi(t)\rangle + \tilde{U}(t) \frac{d|\psi(t)\rangle}{dt} \right) \\
&= i\hbar \frac{d\tilde{U}(t)}{dt} \tilde{U}^\dagger |\phi\rangle + \tilde{U}(t) H \tilde{U}^\dagger |\phi\rangle
\end{aligned}$$

$$H_{\text{eff}} = \left(\tilde{U} H \tilde{U}^\dagger + i\hbar \frac{d\tilde{U}}{dt} \tilde{U}^\dagger \right) |\phi(t)\rangle$$

$$\begin{aligned}
\tilde{U}(t) &\sim \hat{T} \exp \left[-\frac{i}{\hbar} \int_0^t d\tau \hat{H}(\tau) \right] && \text{Heisenberg Picture} \\
&\sim \exp(iH_0 t / \hbar) && \text{Interaction Picture}
\end{aligned}$$

$$H_0 + V_I + i\hbar \cdot iH$$

Schrodinger 方程的密度矩阵形式?

$$i\hbar \frac{d|\psi\rangle}{dt} = H|\psi\rangle.$$

$$\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|.$$

$$i\hbar \frac{d\rho}{dt} = ?$$

$$p_i = \langle\psi|A_i|\psi\rangle = \langle\psi|a_i\rangle\langle a_i|\psi\rangle.$$

$$i\hbar \frac{d|\psi\rangle}{dt} \langle\psi| + i\hbar |\psi\rangle \frac{d\langle\psi|}{dt}$$

$$-i\hbar \frac{d\langle\psi|}{dt} = \langle\psi|H.$$

$$H|\psi\rangle\langle\psi| + i\hbar |\psi\rangle \frac{d\langle\psi|}{dt} \Rightarrow H|\psi\rangle\langle\psi| - |\psi\rangle\langle\psi|H.$$

$$= [\hat{H}, \rho] = \hat{H}\rho - \rho\hat{H}.$$

$$i\hbar \frac{d\rho}{dt} = i\hbar \cdot \frac{-i\hbar}{\hbar} e^{-iHt/\hbar} = H e^{-iHt/\hbar} = H\rho.$$

算符演化

$$\langle\hat{A}\rangle = \langle\psi(t)|\hat{A}|\psi(t)\rangle = \langle\psi(0)|\underbrace{U^\dagger(t)\hat{A}U(t)}_{\hat{A}(t)}|\psi(0)\rangle.$$

$$\left(i\hbar \frac{dU^\dagger}{dt} A + i\hbar U^\dagger \frac{dA}{dt}\right)|\psi\rangle$$

$$\hat{A}(t).$$

$$i\hbar \frac{d|\psi\rangle}{dt} = H|\psi\rangle.$$

$$U(t) = e^{-iHt/\hbar}.$$

$$i\hbar \frac{d\hat{A}(t)}{dt} |\psi(0)\rangle = i\hbar \frac{d[U^\dagger(t)\hat{A}U(t)]}{dt} |\psi(0)\rangle$$

$$i\hbar \frac{dU}{dt} = HU.$$

$$= i\hbar \frac{d(U^\dagger(t)A)}{dt} U(t)|\psi(0)\rangle + i\hbar U^\dagger(t)A \frac{dU(t)}{dt} |\psi(0)\rangle \quad U^\dagger A H |\psi\rangle$$

$$[\hat{A}(t), H] |\psi(0)\rangle = [U^\dagger A U, H] |\psi(0)\rangle = (U^\dagger A U H - H U^\dagger A U) |\psi(0)\rangle$$

$$= \underline{U^\dagger A U H} |\psi(0)\rangle - H U^\dagger A |\psi\rangle$$

Interaction Picture

$$\hat{H} = \hat{H}_0 + \hat{V}(t)$$

$$(\hat{H}_0 + \hat{V})|\psi\rangle = i\hbar \frac{d|\psi\rangle}{dt}$$

$$|\psi_I(t)\rangle = U^\dagger(t)|\psi(t)\rangle = \exp(-\hat{H}_0 t / i\hbar) |\psi(t)\rangle$$

$$\begin{aligned} \frac{d|\psi_I(t)\rangle}{dt} &= \left(-\frac{\hat{H}_0}{i\hbar}\right) \exp(-\hat{H}_0 t / i\hbar) |\psi(t)\rangle + \exp(-\frac{\hat{H}_0 t}{i\hbar}) \frac{d|\psi(t)\rangle}{dt} \\ &= -\frac{\hat{H}_0}{i\hbar} |\psi_I(t)\rangle + \exp(-\frac{\hat{H}_0 t}{i\hbar}) \cdot \frac{1}{i\hbar} (\hat{H}_0 + \hat{V}) \underbrace{|\psi\rangle}_{\exp(\frac{\hat{H}_0 t}{i\hbar}) |\psi_I\rangle} \\ &= \underbrace{-\frac{\hat{H}_0}{i\hbar} |\psi_I\rangle + \exp(-\frac{\hat{H}_0 t}{i\hbar}) \cdot \frac{\hat{H}_0}{i\hbar} \exp(\frac{\hat{H}_0 t}{i\hbar}) |\psi_I\rangle}_{=0} + \exp(-\frac{\hat{H}_0 t}{i\hbar}) \cdot \frac{\hat{V}}{i\hbar} \exp(\frac{\hat{H}_0 t}{i\hbar}) |\psi_I\rangle \\ &= \hat{H}_I(t) |\psi_I\rangle \end{aligned}$$