数理方程

$$\frac{\partial^2 u}{\partial t^2} = a^2 \Delta u + f(t,\vec{x}) \qquad \vec{x} = (x_1, x_2, \cdots x_n)$$

波动方程 $\quad \Delta = \sum_j \frac{\partial^2}{\partial x_j^2}$

$$\frac{\partial u}{\partial t} = a^2 \Delta u + f(t,x,y,z) \qquad \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

热传导方程.

$\frac{\partial u}{\partial t} = 0 \rightarrow$ Poisson 方程

$f(x,y,z) \equiv 0 \rightarrow$ Laplace 方程

$$\Delta u = -f(\vec{x}) \qquad \vec{x} = (x_1, \cdots x_n)$$

场位方程

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} + \alpha\frac{\partial^3 u}{\partial x^3} + \beta u\frac{\partial u}{\partial x} = 0$$

$\tau = \alpha t$

$\xi = x - at \quad \Rightarrow \quad \frac{\partial \eta}{\partial \tau} + \frac{\partial^3 \eta}{\partial \xi^3} + \eta\frac{\partial \eta}{\partial \xi} = 0 \qquad$ KdV eqn

$\eta = \frac{\beta}{\alpha} u$

u.	$\frac{\partial u}{\partial x}$.	$u, \frac{\partial u}{\partial x}$.	0	齐次
I类	II	III		

$$\left[\alpha_i u + \beta_i \frac{\partial u}{\partial n}\right]_{x=x_i} = F_i(t)$$

$$a(x,y)\frac{\partial u}{\partial x} + b(x,y)\frac{\partial u}{\partial y} + c(x,y)u = f(x,y)$$

a, b, c 为 $D \subset \mathbb{R}^2$ 上的连续函数. f 在 D 上连续.

若在 D 上 $a(x,y) \equiv 0$, $b(x,y) \neq 0$ 则

$$\frac{\partial u}{\partial y} + \frac{c}{b} u = \frac{f}{b}.$$

$$u(x,y) = \exp\left(-\int \frac{c}{b} dy\right)\left[\int \exp\left(\int \frac{c}{b} dy\right)\frac{f}{b} dy + \underbrace{g(x)}_{\forall C^1 \text{函数}}\right]$$

若在 D 上 $a(x,y)\,b(x,y) \neq 0$. 作代换

$$\begin{cases} \xi = \varphi(x,y) \\ \eta = \psi(x,y) \end{cases} \qquad J(\varphi,\psi) = \frac{\partial(\varphi,\psi)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial \varphi}{\partial x} & \frac{\partial \varphi}{\partial y} \\ \frac{\partial \psi}{\partial x} & \frac{\partial \psi}{\partial y} \end{vmatrix} \neq 0$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \xi}\frac{\partial \varphi}{\partial x} + \frac{\partial u}{\partial \eta}\frac{\partial \psi}{\partial x}, \quad \frac{\partial u}{\partial y} = \frac{\partial u}{\partial \xi}\frac{\partial \varphi}{\partial y} + \frac{\partial u}{\partial \eta}\frac{\partial \psi}{\partial y}$$

$$a\left(\frac{\partial u}{\partial \xi}\frac{\partial \varphi}{\partial x} + \frac{\partial u}{\partial \eta}\frac{\partial \psi}{\partial x}\right) + b\left(\frac{\partial u}{\partial \xi}\frac{\partial \varphi}{\partial y} + \frac{\partial u}{\partial \eta}\frac{\partial \psi}{\partial y}\right) + cu = f$$

$$\underbrace{\left(a\frac{\partial \varphi}{\partial x} + b\frac{\partial \varphi}{\partial y}\right)}_{\text{set to } 0}\frac{\partial u}{\partial \xi} + \left(a\frac{\partial \psi}{\partial x} + b\frac{\partial \psi}{\partial y}\right)\frac{\partial u}{\partial \eta} + cu = f$$

$a(x,y)\,dy - b(x,y)\,dx = 0$. 在 D 内存在仅存在一族独立的积分曲线

特征方程

$\varphi(x,y) = h$. 再取 $\psi(x,y)$ 使 $J \neq 0$.

$$\sum_{j=1}^{n} b_j \frac{\partial u}{\partial x_j} + cu = f.$$

找到 $\frac{dx_1}{b_1} = \cdots \frac{dx_n}{b_n}$ 的 $n-1$ 个首次积分

$$\varphi_j(x_1, \cdots x_n) = h_j \qquad (j = 1, 2, \cdots n-1)$$

作代换 $\begin{cases} \xi_j = \varphi_j(x_1, \cdots x_n) \\ \xi_n = \varphi_n(x_1, \cdots x_n) \end{cases}$ $j = 1, 2, \cdots n-1$.

$J \neq 0$

$$\sum_{j=1}^{n} b_j \frac{\partial u}{\partial x_j} = \sum_{j=1}^{n} b_j\left(\sum_{k=1}^{n}\frac{\partial u}{\partial \xi_k}\frac{\partial \varphi_k}{\partial x_j}\right) = \sum_{k=1}^{n}\left(\sum_{j=1}^{n} b_j \frac{\partial \varphi_k}{\partial x_j}\right)\frac{\partial u}{\partial \xi_k}$$

$$\sum_{k=1}^{n}\left(\sum_{j=1}^{n} b_j \frac{\partial \varphi_k}{\partial x_j}\right)\frac{\partial u}{\partial \xi_k} + cu = f.$$ 对 ξ_k 积分得通解.

quasi-linear $\sum_{j=1}^{n} b_j(x_1,x_2,\cdots x_n,u)\frac{\partial u}{\partial x_j} = c(x_1,x_2,\cdots x_n,u)$.

$$\sum_{j=1}^{n} b_j(x_1,\cdots x_n,u)\frac{\partial \varphi}{\partial x_j} + c(x_1,\cdots x_n)\frac{\partial \varphi}{\partial u} = 0$$

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} = a^2\dfrac{\partial^2 u}{\partial x^2}, & t>0,\ -\infty<x<+\infty \\ u|_{t=0} = \varphi(x), & \dfrac{\partial u}{\partial t}\Big|_{t=0} = \psi(x). \end{cases}$$

d'Alembert eqn

代入 $u = f(x-at) + g(x+at)$.

$$f(x)+g(x) = \varphi(x)$$

$$-af'(x) + ag'(x) = \psi(x)$$

$$u(t,x) = \frac{1}{2}\left[\varphi(x-at)+\varphi(x+at)\right] + \frac{1}{2a}\int_{x-at}^{x+at}\psi(\xi)\,d\xi$$

二阶线性偏微分方程

$$\sum_{ij}^{n} a_{ij}(x_1,\cdots x_n)\frac{\partial^2 u}{\partial x_i\partial x_j} + \sum_{j}^{n} b_j(x_1,\cdots x_n)\frac{\partial u}{\partial x_j} + c(x_1,\cdots x_n)\,u = f(x_1,\cdots x_n)$$

非齐次项

a_{ij}, b_j, c 为常数：常系数

$n=2$. $$a_{11}\frac{\partial^2 u}{\partial x^2} + 2a_{12}\frac{\partial^2 u}{\partial x\partial y} + a_{22}\frac{\partial^2 u}{\partial y^2} + b_1\frac{\partial u}{\partial x} + b_2\frac{\partial u}{\partial y} + cu = 0$$

作代换 $\xi(x,y)$, $\eta(x,y)$

$$A_{11}\frac{\partial^2 u}{\partial \xi^2} + 2A_{12}\frac{\partial^2 u}{\partial \xi\partial\eta} + A_{22}\frac{\partial^2 u}{\partial \eta^2} + B_1\frac{\partial u}{\partial \xi} + B_2\frac{\partial u}{\partial \eta} + Cu = 0$$

$$A_{11} = \underline{a_{11}\left(\frac{\partial\xi}{\partial x}\right)^2 + 2a_{12}\frac{\partial\xi}{\partial x}\frac{\partial\xi}{\partial y} + a_{22}\left(\frac{\partial\xi}{\partial y}\right)^2}$$

$$A_{12} = a_{11}\frac{\partial\xi}{\partial x}\frac{\partial\eta}{\partial x} + a_{12}\left(\frac{\partial\xi}{\partial x}\frac{\partial\eta}{\partial y} + \frac{\partial\xi}{\partial y}\frac{\partial\eta}{\partial x}\right) + a_{22}\frac{\partial\xi}{\partial y}\frac{\partial\eta}{\partial y}$$

$$A_{22} = \underline{a_{11}\left(\frac{\partial\eta}{\partial x}\right)^2 + 2a_{12}\frac{\partial\eta}{\partial x}\frac{\partial\eta}{\partial y} + a_{22}\left(\frac{\partial\eta}{\partial y}\right)^2}.$$

归结为 $a_{11}(dy)^2 - 2a_{12}\,dx\,dy + a_{22}(dx)^2 = 0$ 求解

$$\Delta = a_{12}^2 - a_{11}a_{22}$$

$$A_{11}\frac{\partial^2 u}{\partial \xi^2} + 2A_{12}\frac{\partial^2 u}{\partial \xi \partial \eta} + A_{22}\frac{\partial^2 u}{\partial \eta^2} + B_1\frac{\partial u}{\partial \xi} + B_2\frac{\partial u}{\partial \eta} + Cu = 0$$

$$A_{12}^2 - A_{11}A_{22} = J^2\Delta.$$

$\Delta > 0$ 双曲型

$\Delta = 0$ 抛物型

$\Delta < 0$ 椭圆型

例

$$\frac{\partial^2 u}{\partial t^2} = a^2\frac{\partial^2 u}{\partial x^2}$$

$$\frac{\partial u}{\partial t} = a^2\frac{\partial^2 u}{\partial x^2}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$\Delta > 0.$ $\quad \dfrac{dy}{dx} = \dfrac{a_{12} + \sqrt{\Delta}}{a_{11}}.$ $\quad \varphi(x,y) = h_1,\ \psi(x,y) = h_2.$

$$\frac{\partial \varphi}{\partial x}dx + \frac{\partial \varphi}{\partial y}dy = 0$$

$\psi \qquad \psi$

$$\frac{\partial^2 u}{\partial \xi \partial \eta} + \frac{1}{2A_{12}}\left(B_1\frac{\partial u}{\partial \xi} + B_2\frac{\partial u}{\partial \eta} + Cu\right) = 0$$

再代换 $s = \frac{1}{2}(\xi+\eta)$, $t = \frac{1}{2}(\xi-\eta)$ 得

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial s^2} + \frac{1}{A_{11}}\left(\tilde{B}_1\frac{\partial u}{\partial t} + \tilde{B}_2\frac{\partial u}{\partial s} + \tilde{C}u\right) = 0$$

所以在某坐标架下对时空依次求导相当于?

$\Delta = 0.$ $\quad \dfrac{dy}{dx} = \dfrac{a_{12}}{a_{11}}.$ $\quad \cdots$

$$\frac{\partial^2 u}{\partial \eta^2} + \frac{1}{A_{22}}\left(B_1\frac{\partial u}{\partial \xi} + B_2\frac{\partial u}{\partial \eta} + Cu\right) = 0$$

$\Delta < 0.$ $\quad \dfrac{dy}{dx} = \dfrac{a_{12} \pm i\sqrt{-\Delta}}{a_{11}}.$ $\quad \varphi(x,y) + i\psi(x,y) = h.$ $\cdots$

$$\frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \eta^2} + \frac{1}{A_{11}}\left(B_1\frac{\partial u}{\partial \xi} + B_2\frac{\partial u}{\partial \eta} + Cu\right) = 0$$

$$\sum_{ij}^{n} a_{ij}(x_1,\dots x_n)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{j}^{n} b_j(x_1,\dots x_n)\frac{\partial u}{\partial x_j} + c(x_1,\dots x_n)\,u = f(x_1,\dots x_n)$$

相应的二次型 $\quad q(\vec{\lambda}) = \sum_{ij} a_{ij}(\vec{x}^0)\lambda_i\lambda_j$

存在变换 $\quad (\lambda_1,\cdots\lambda_n)^T = \left(d_{ij}(\vec{x}^0)\right)_{n\times n}(\mu_1,\cdots\mu_n)^T$

使 $\quad q(\vec{\lambda}) = Q(\vec{\mu}) = \sum_{j=1}^{k}\mu_j^2 - \sum_{j=k+1}^{k+l}\mu_j^2,\quad k+l \le m$

若为常系数 $d_{ij}(\vec{x}) = d_{ij}$. $\vec{\xi}^T = D\vec{x}^T$, $D = (d_{ij})$.

$$\sum_{j=1}^{k} \frac{\partial^2 u}{\partial \xi_j^2} - \sum_{j=k+1}^{k+l} \frac{\partial^2 u}{\partial \xi_j^2} + \sum_{j=1}^{n} B_j \frac{\partial u}{\partial \xi_j} + Cu = 0$$

齐次化原理（冲量原理）

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} = a^2 \dfrac{\partial^2 u}{\partial x^2} + f(t,x), & t>0,\ -\infty<x<+\infty \\ u|_{t=0} = 0, \quad \dfrac{\partial u}{\partial t}\Big|_{t=0} = 0 \end{cases}$$

$\varphi(x)$ $\qquad$ $\psi(x)$

$$u(t,x) = \int_0^t w(t,x;\tau)\,d\tau$$

$$\begin{cases} \dfrac{\partial^2 w}{\partial t^2} = a^2 \dfrac{\partial^2 w}{\partial t^2}, & t>\tau,\ -\infty<x<+\infty \\ w|_{t=\tau} = 0, \quad \dfrac{\partial w}{\partial t}\Big|_{t=\tau} = f(\tau,x). \end{cases}$$

$$\Rightarrow u(t,x) = \frac{1}{2a}\int_0^t d\tau \int_{x-a(t-\tau)}^{x+a(t-\tau)} f(\tau,\xi)\,d\xi$$

$$= \frac{1}{2a^2}\int_0^{at} dr \int_{x-r}^{x+r} f\left(t-\frac{r}{a},\xi\right) d\xi$$

$$+\frac{1}{2}\left[\varphi(x-at)+\varphi(x+at)\right] + \frac{1}{2a}\int_{x-at}^{x+at}\psi(\xi)\,d\xi$$

一般地，若 $w(t,\vec{x};\tau)$ 满足齐次方程

$$\begin{cases} \dfrac{\partial^m w}{\partial t^m} = Lw, & t>\tau>0,\ \vec{x}\in\mathbb{R}^n \\ w|_{t=\tau} = \dfrac{\partial w}{\partial t}\Big|_{t=\tau} = \cdots \dfrac{\partial^{m-2} w}{\partial t^{m-2}}\Big|_{t=\tau} = 0 \\ \dfrac{\partial^{m-1} w}{\partial t^{m-1}}\Big|_{t=\tau} = f(\tau,\vec{x}) \end{cases}$$

则 $u(t,\vec{x}) = \displaystyle\int_0^t w(t,x;\tau)\,d\tau$ 为非齐次初值问题

$$\begin{cases} \dfrac{\partial^m u}{\partial t^m} = Lu + f(t,\vec{x}), & t>0,\ \vec{x}\in\mathbb{R}^n \\ u|_{t=0} = \cdots \dfrac{\partial^{m-1} u}{\partial t^{m-1}}\Big|_{t=0} = 0 \end{cases}$$

的解.

边值问题．分离变量法

(固有值问题)

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}, \quad t>0,\ 0<x<l \\ u|_{x=0} = u|_{x=l} = 0 \\ u|_{t=0} = \varphi(x),\ \frac{\partial u}{\partial t}\Big|_{t=0} = \psi(x) \end{cases}$$

设 $u(t,x) = T(t)\,X(x)$.

$$\frac{T''(t)}{a^2 T(t)} = \frac{X''(x)}{X(x)} = -\lambda.$$

$$X_n(x) = \sin\frac{n\pi}{l}x. \quad \lambda_n = \left(\frac{n\pi}{l}\right)^2.$$

$$T_n(t) = C_n\cos\frac{an\pi}{l}t + D_n\sin\frac{an\pi}{l}t.$$

$$u(t,x) = \sum_{n=1}^{+\infty}\left(C_n\cos\frac{an\pi}{l}t + D_n\sin\frac{an\pi}{l}t\right)\sin\frac{n\pi}{l}x$$

$$\sum_{n=1}^{+\infty} C_n\sin\frac{n\pi}{l}x = \varphi(x). \qquad \sum_{n=1}^{+\infty}\frac{an\pi}{l}D_n\sin\frac{n\pi}{l}x = \psi(x).$$

$$C_n = \frac{2}{l}\int_0^l \varphi(x)\sin\frac{n\pi}{l}x\,dx. \qquad D_n = \frac{2}{an\pi}\int_0^l \psi(x)\sin\frac{n\pi}{l}x\,dx.$$

…收敛时．古典解．

更一般的定解问题

$$\begin{cases} L_t u + L_x u = 0, \quad t\in I,\ a<x<b \\ \left(\alpha_1 u - \beta_1\frac{\partial u}{\partial x}\right)\Big|_{x=a} = 0. \quad \left(\alpha_2 u - \beta_2\frac{\partial u}{\partial x}\right)\Big|_{x=b} = 0. \\ \text{关于 } t \text{ 的定解条件} \end{cases}$$

$f(x)$ 关于正交基 $\{X_j(x),\ j=1,2\cdots\}$ 的广义 Fourier 展开．

$L^2[a,b]$ 中的正交函数 $\langle X_i(x), X_j(x)\rangle = \|X_j(x)\|^2\delta_{ij}$ 使 $\forall f(x)\in L^2[a,b]$ 有

$$f(x) = \sum_{j=1}^{+\infty} c_j X_j(x). \qquad c_j = \frac{\langle f(x), X_j(x)\rangle}{\|X_j(x)\|^2} = \frac{\int_a^b f(x)X_j(x)\,dx}{\int_a^b |X_j(x)|^2\,dx}.$$

$$[k(x)X'(x)]' - q(x)X(x) + \lambda\rho(x)X(x) = 0. \qquad \text{Sturm–Liouville (S-L) 型方程}$$

…对应的固有值问题是自共轭算子的固有值问题

对应的 eigenvalue, eigenfunction 满足 $\lambda\geq 0$．可数性．正交性．完备性．

$$\frac{\partial^2 u}{\partial t^2} = a^2 \Delta_3 u \qquad \xrightarrow{\text{分离变量}} \qquad T'' + a^2k^2T = 0$$

$$\frac{\partial u}{\partial t} = a^2 \Delta_3 u \qquad\qquad T' + a^2k^2T = 0$$

$$\Delta_3 v + k^2 v = 0 \qquad \text{Helmholtz eqn}$$

$k=0$. Laplace eqn

直角坐标系下.

$$X'' + \lambda X = 0$$

$$Y'' + \mu Y = 0 \qquad \lambda+\mu+\nu = k^2.$$

$$Z'' + \nu Z = 0.$$

Fourier 展开. $X_n(x)$. $[0,a]$ 上带权 $\rho(x)$ 的一元完备正交系

固定 n. $Y_{nm}(y)$ $[0,b]$ 上 $\sigma(y)$

则 $X_n(x)Y_{nm}(y)$ 是 $[0,a]\times[0,b]$ 上加权 $\rho(x)\cdot\sigma(y)$ 的完备正交函数系

$$\forall f(x,y) \in L^2_{\rho\sigma}\big[[0,a]\times[0,b]\big] = \left\{ f(x,y) \,\Big|\, \int_0^b\int_0^a |f(x,y)|^2 \rho(x)\sigma(y)\,dx\,dy < +\infty \right\} \text{ 有}$$

$$f(x,y) = \sum_{n,m=1}^{\infty} C_{nm} X_n(x) Y_{nm}(y). \qquad C_{nm} = \frac{\int_0^a\int_0^b f(x,y)\, X_n(x)\, Y_{nm}(y)\, \rho(x)\,\sigma(y)\, dy\,dx}{\|X_n(x)\|^2\, \|Y_{nm}(y)\|^2}$$

柱坐标下

$$\Delta_3 v + k^2 v = 0. \qquad \Delta_3 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}.$$

$R\,\Theta\,Z$

$$\frac{\frac{1}{r}(rR')'}{R} + \frac{1}{r^2}\frac{\Theta''}{\Theta} + \frac{Z''}{Z} + k^2 = 0.$$

$$Z'' + \mu Z = 0. \qquad \frac{1}{r}(rR')' + \left(k^2-\mu-\frac{\sigma}{r^2}\right) R = 0.$$

$$\Theta'' + \sigma\Theta = 0.$$

$$(rR')' + \Big(\underbrace{(k^2-\mu)}_{\lambda} r - \frac{\sigma}{r}\Big) R = 0$$

$$\left(r\left(\frac{y}{r}\right)'\right)' + \left(\lambda r - \frac{\sigma}{r}\right)\frac{y}{r} = 0 \qquad \left(\frac{ry'-y}{r}\right)' = \frac{r(y'+ry''-y') - (ry'-y)r'}{r^2} = \frac{r^2y''-rr'y'+yr'}{r^2}$$

$$\left(\frac{ry'-yr'}{r}\right)' + \lambda y - \frac{\sigma y}{r^2} = 0$$

$$y'' - y'\frac{r'}{r} - y\frac{rr''-r'^2}{r^2} + \lambda y - \frac{\sigma y}{r^2} = 0 \qquad x^2y'' - \sqrt{\lambda}\,x y' + \lambda y + \lambda y(x^2-\sigma) = 0$$

$$r^2y'' - ry'r' - yrr'' + yr'^2 + \lambda y r^2 - \sigma y = 0$$

$$\frac{x^2}{\lambda}y'' - \frac{x}{\sqrt{\lambda}}y' + y + \lambda y\frac{x^2}{\lambda} - \sigma y = 0$$

$$x = \sqrt{\lambda}\, r.$$

$$y = R\left(\frac{x}{\sqrt{\lambda}}\right) \qquad x^2y'' + xy' + (x^2-\nu^2)\,y = 0$$

球坐标下

$$\Delta_3 v + k^2 v = 0. \qquad \Delta_3 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}$$

$R\,\Theta\,\Phi$

$$\frac{1}{r^2R}(r^2R')' + \frac{1}{\Theta r^2\sin\theta}(\sin\theta\,\Theta')' + \frac{1}{r^2\sin^2\theta}\frac{\Phi''}{\Phi} + k^2 = 0$$

$$\Phi'' + \mu\Phi = 0$$

$$\frac{1}{\sin\theta}(\sin\theta\,\Theta')' + \left(\lambda - \frac{\mu}{\sin^2\theta}\right)\Theta = 0$$

$$\frac{1}{r^2}(r^2R')' + \left(k^2 - \frac{\lambda}{r^2}\right)R = 0 \quad \text{球 Bessel eqn}$$

$$k=0 \Rightarrow r^2R'' + 2rR' - \lambda R = 0 \quad \text{Euler eqn}$$

$x = \cos\theta$

$y(x) = \Theta(\arccos x)$ $\qquad$ $$\left[(1-x^2)y'\right]' + \left(\lambda - \frac{m^2}{1-x^2}\right)y = 0 \quad \text{Legendre eqn}$$

$\mu = m^2$

二阶线性常微分方程的解析理论

$$w''(z) + p(z)w'(z) + q(z)w(z) = 0$$

① 若 z_0 是 $p(z)$. $q(z)$ 的解析点. 常点. $\qquad$ Cauchy

设 $p(z)$. $q(z)$ 在 $|z-z_0|<R$ 内解析. 则初值问题

$$\begin{cases} w'' + pw' + qw = 0 \\ w(z_0) = a_0, \quad w'(z_0) = a_1 \end{cases}$$

在 $|z-z_0|<R$ 内的解存在唯一且解析

选取初值 (a_0, a_1), (b_0, b_1) 得解析解

$$w_1(z) = \sum_{n=0}^{+\infty} a_n (z-z_0)^n. \qquad w_2(z) = \sum_{n=0}^{+\infty} b_n(z-z_0)^n. \qquad \text{通解 } C_1w_1 + C_2w_2$$

② 若 z_0 是 $p(z)$ 的至多一级极点, $q(z)$ 的至多二级极点. 正则极点 $\qquad$ Fuchs

$(z-z_0)p(z)$, $(z-z_0)^2q(z)$ 在 $|z-z_0|<R$ 内解析. 去心圆域上 $0<|z-z_0|<R$

有 $$w_1(z) = (z-z_0)^{\rho_1} \sum_{n=0}^{+\infty} a_n (z-z_0)^n$$

$$w_2(z) = \alpha\, w_1(z) \ln(z-z_0) + (z-z_0)^{\rho_2} \sum_{n=0}^{+\infty} b_n (z-z_0)^n$$

③ 若 z_0 是 $p(z)$. $q(z)$ 的超过一、二级的极点或本性奇点. 非正则奇点

此时若 $p(z)$. $q(z)$ 在 $0<|z-z_0|<R$ 内解析.

$$w_1(z) = (z-z_0)^{\rho_1} \sum_{n=-\infty}^{+\infty} a_n (z-z_0)^n$$

$$w_2(z) = \alpha\, w_1(z) \ln(z-z_0) + (z-z_0)^{\rho_2} \sum_{n=-\infty}^{+\infty} b_n (z-z_0)^n$$

Legendre eqn $\quad \left[(1-x^2)\, y'\right]' + \lambda y = 0$ $\qquad x=\pm 1$ 正则奇点

其他皆为常点

$|x|<1$ 内可求解析解

$$y'' - \frac{2x}{1-x^2} y' + \frac{\lambda}{1-x^2} y = 0$$

设 $y(x) = \sum_{n=0}^{+\infty} a_n x^n$.

$$(1-x^2)\, y'' = \sum_{n=0}^{+\infty} (n+2)(n+1)\, a_{n+2}\, x^n - \sum_{n=0}^{+\infty} n(n-1)\, a_n x^n$$

$$-2xy' = -\sum_{n=0}^{+\infty} 2n\, a_n x^n. \qquad \lambda y = \sum_{n=0}^{+\infty} \lambda a_n x^n.$$

$$\sum_{n=0}^{+\infty} \left\{ (n+1)(n+2)\, a_{n+2} - \left[n(n+1)-\lambda\right] a_n \right\} x^n = 0.$$

取 $a_0 \neq 0$, $a_1 = 0$.

$$y_1(x) = a_0 \sum_{k=0}^{+\infty} \frac{2^{2k}\, \Gamma(k-\frac{l}{2})\, \Gamma(k+\frac{l+1}{2})}{(2k)!\, \Gamma(-\frac{l}{2})\, \Gamma(\frac{l+1}{2})} x^{2k}.$$

取 $a_0 = 0$, $a_1 \neq 0$.

$$y_2(x) = a_1 \sum_{k=0}^{+\infty} \frac{2^{2k}\, \Gamma(k+\frac{l-1}{2})\, \Gamma(k+\frac{2+l}{2})}{(2k+1)!\, \Gamma(\frac{1-l}{2})\, \Gamma(1+\frac{l}{2})} x^{2k+1}.$$

$l \neq n$ 时. $n\to\infty$. $\frac{a_{n+2}}{a_n} \to 1$. 在 $x=\pm 1$ 发散.

$l = n$ 时. 取 $a_n = \frac{(2n)!}{2^n (n!)^2}$. 得

$$P_n(x) \overset{d}{=} \sum_{k=0}^{[\frac{n}{2}]} \frac{(-1)^k (2n-2k)!}{2^n k! (n-k)! (n-2k)!} x^{n-2k}.$$ Legendre 多项式

$$Q_n(x) \overset{d}{=} P_n(x) \int \frac{1}{(P_n(x))^2} \exp\left\{-\int \frac{-2x}{1-x^2} dx\right\} dx$$

$$= P_n(x) \int \frac{dx}{(1-x^2)(P_n(x))^2}.$$ 第二类 Legendre 函数

Bessel 方程的广义幂级数解及 Bessel 函数

$$x^2 y'' + xy' + (x^2-\nu^2) y = 0, \qquad \nu^2 \geq 0.$$

除正则奇点 $x=0$ 外解析. $0<|x|<+\infty$ 有广义幂级数解.

设 $y(x) = x^{\rho} \sum_{n=0}^{+\infty} a_n x^n$.

$$a_0(\rho^2-\nu^2) x^{\rho} + a_1\left[(1+\rho)^2-\nu^2\right] x^{1+\rho} + \sum_{n=2}^{+\infty} \left\{a_n\left[(n+\rho)^2-\nu^2\right] + a_{n-2}\right\} x^{n+\rho} = 0$$

$\rho = \pm\nu$. $\qquad (1\pm 2\nu) \qquad n(n\pm 2\nu)$

取 $\rho_1 = \nu \geq 0$. $\qquad a_n = -\frac{a_{n-2}}{n(n+2\nu)}, \quad n \geq 2$

$a_{2k+1} = 0$. $\qquad a_{2k} = \frac{(-1)^k \Gamma(\nu+1)}{2^{2k} k! \Gamma(k+\nu+1)} a_0$. $\qquad$ 取 $a_0 = \frac{1}{2^{\nu}\Gamma(\nu+1)}$.

得 $y_1(x) = \sum_{k=0}^{+\infty} \frac{(-1)^k}{k! \Gamma(k+\nu+1)} \left(\frac{x}{2}\right)^{2k+\nu}$. $\qquad J_\nu(x)$

第一类 Bessel 函数

$$\lim_{x\to 0} J_\nu(x) = \begin{cases} 0. & \nu>0 \\ 1. & \nu=0 \end{cases}$$

取 $\rho_2 = -\nu$. $\qquad y_2(x) = \sum_{k=0}^{+\infty} \frac{(-1)^k}{k! \Gamma(k-\nu+1)} \left(\frac{x}{2}\right)^{2k-\nu}$. $\qquad J_{-\nu}(x)$

$\lim_{x\to 0} J_{-\nu}(x) = +\infty$. $\quad \nu>0$. $\nu \neq$ 整数

$\nu = m$ 时. $\quad \cdots \qquad N_\nu(x) \overset{d}{=} \frac{\cos\nu\pi}{\sin\nu\pi} J_\nu(x) - \frac{1}{\sin\nu\pi} J_{-\nu}(x)$

Legendre 多项式 $\qquad P_n(x) = \sum_{k=0}^{[\frac{n}{2}]} \frac{(-1)^k (2n-2k)!}{2^n k! (n-k)! (n-2k)!} x^{n-2k}$

$$(x^2-1)^n = \sum_{k=0}^{n} \frac{(-1)^k n!}{k!(n-k)!} x^{2(n-k)}.$$

$$\frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n = \frac{1}{2^n n!}\sum_{k=0}^{[\frac{n}{2}]} \frac{(-1)^k n!}{k!(n-k)!}\, 2(n-k)\,(2(n-k)-1)\cdots(2(n-k)-n+1)\, x^{2n-2k-n}$$

$$= \frac{1}{2^n}\sum_{k=0}^{[\frac{n}{2}]} \frac{(-1)^k (2n-2k)!}{k!(n-k)!(n-2k)!} x^{n-2k} = P_n(x)$$ Rodrigues eqn

$$P_n(x) = \frac{1}{2\pi i}\frac{1}{2^n}\oint_c \frac{(z^2-1)^n}{(z-x)^{n+1}}\,dz$$ Schlafli eqn

$$P_n(x) = \frac{1}{\pi}\int_0^{\pi}\left(x+\sqrt{1-x^2}\, i\cos\theta\right)^n d\theta$$ Laplace eqn

记 $w(t,x) = (1-2xt+t^2)^{-\frac{1}{2}}$, $w(0,x)=1$. $|t|<1$ 内解析

$$w(t,x) = \sum_{n=0}^{+\infty} C_n(x)\, t^n. \qquad C_n(x) = \frac{1}{2\pi i}\oint_c \frac{(1-2xt+t^2)^{-\frac{1}{2}}}{t^{n+1}}\,dt$$

$$\xrightarrow{(1-2xt+t^2)^{\frac{1}{2}} = 1-tz.} \qquad C_n(x) = \frac{1}{2\pi i}\oint_{c'} \frac{(z^2-1)^n}{2^n(z-x)^{n+1}}\,dz$$ n阶 Legendre 多项式 $P_n(x)$

有 $$(1-2xt+t^2)^{-\frac{1}{2}} = \begin{cases} \sum_{n=0}^{+\infty} P_n(x)\, t^n. & |t|<1 \\ \frac{1}{t}\sum_{n=0}^{+\infty} P_n(x)\left(\frac{1}{t}\right)^n. & |t|>1. \end{cases}$$

$w(t,x)$. 母函数/生成函数

$f(x)\in L^2[-1,1]$ 广义 Fourier 展开 $f(x) = \sum_{n=0}^{+\infty} C_n P_n(x)$. $\qquad C_n = \frac{1}{\|P_n(x)\|^2}\int_{-1}^{1} f(x)P_n(x)\,dx$

$$(1-2xt+t^2)^{-1} = \sum_{n=0}^{+\infty}\sum_{m=0}^{+\infty} P_n(x)P_m(x)\, t^{n+m}, \quad |t|<1.$$

$$\int_{-1}^{1}\frac{1}{1-2xt+t^2}\,dx = \sum_{n=0}^{+\infty}\sum_{m=0}^{+\infty}\left(\int_{-1}^{1}P_n(x)P_m(x)\,dx\right)t^{n+m}.$$

$$-\frac{1}{2t}\ln(1-2xt+t^2)\Big|_{-1}^{1} = \sum_{n=0}^{+\infty}\|P_n(x)\|^2 t^{2n} = \frac{1}{t}\ln\frac{1+t}{1-t} = \sum_{n=0}^{+\infty}\frac{2}{2n+1}t^{2n}$$

轴对称 Laplace eqn 边值问题

$$\Delta_3 u = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right) = 0$$

$R\Theta$

$$\frac{1}{\sin\theta}\left(\sin\theta\,\Theta'\right)' + \lambda\Theta = 0 \qquad x = \cos\theta \qquad y = \Theta(\arccos x) \qquad \begin{cases} \left[(1-x^2)y'\right]' + \lambda y = 0 \\ |y(\pm 1)| < +\infty \end{cases}$$

$$(r^2R')' - \lambda R = 0 \qquad \text{Euler}$$

$$\lambda_n = n(n+1). \quad \Theta_n(\theta) = P_n(\cos\theta).$$

$$R_n(r) = C_n r^n + D_n r^{-(n+1)}. \qquad u(r,\theta) = \sum_{n=0}^{+\infty}\left[C_n r^n + D_n r^{-(n+1)}\right] P_n(\cos\theta)$$

伴随 Legendre eqn

$\mu = m^2$. m阶伴随 Legendre eqn

$$\frac{1}{\sin\theta}\left(\sin\theta\,\Theta'\right)' + \left(\lambda - \frac{\mu}{\sin^2\theta}\right)\Theta = 0 \qquad x = \cos\theta \qquad y(x) = \Theta(\arccos x)$$

$$\left[(1-x^2)y'\right]' + \left(\lambda - \frac{m^2}{1-x^2}\right)y = 0$$

$$y(x) = (1-x^2)^{\frac{m}{2}} v^{(m)}(x).$$

$$P_n^m(x) = \frac{1}{2^n n!}(1-x^2)^{\frac{m}{2}}\left[(x^2-1)^n\right]^{(n+m)}$$

Laplace 球面边值问题

$$\Delta_3 u = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 u}{\partial\varphi^2}\right] = 0$$

$RY \qquad (r^2R')' - \lambda R = 0$

$$\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right]Y = -\lambda Y.$$

$$Y_{nm}(\theta,\varphi) = \begin{Bmatrix}\cos m\varphi \\ \sin m\varphi\end{Bmatrix} P_n^m(\cos\theta)$$

Bessel 函数

$$J_\nu(x) = \sum_{k=0}^{+\infty}\frac{(-1)^k}{k!\,\Gamma(k+\nu+1)}\left(\frac{x}{2}\right)^{2k+\nu}$$

$$(x^\nu J_\nu)' = x^\nu J_{\nu-1}$$

$$(x^{-\nu} J_\nu)' = x^{-\nu} J_{\nu+1}$$

$$2J_\nu' = J_{\nu-1} - J_{\nu+1}$$

$$2\nu x^{-1} J_\nu = J_{\nu-1} + J_{\nu+1}$$

$$J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}}\sin x, \quad J_{-\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}}\cos x$$

...

$e^{\frac{x}{2}(\zeta-\zeta^{-1})}$ 在 $\zeta=0$ 外解析. 在 $0<|\zeta|<+\infty$ 作 Laurent 展开

母函数

$$e^{\frac{x}{2}(\zeta-\zeta^{-1})} = e^{\frac{x}{2}\zeta}e^{-\frac{x}{2}\zeta^{-1}} = \sum_{l=0}^{+\infty}\frac{1}{l!}\left(\frac{x}{2}\zeta\right)^l \cdot \sum_{k=0}^{+\infty}\frac{1}{k!}\left(-\frac{x}{2}\zeta^{-1}\right)^k$$

$$= \sum_{n=-\infty}^{+\infty}\underbrace{\left[\sum_{k=0}^{+\infty}\frac{(-1)^k}{k!(k+n)!}\left(\frac{x}{2}\right)^{2k+n}\right]}_{J_n(x)}\zeta^n$$

$$J_n(x) = \frac{1}{2\pi i}\oint_c \frac{e^{\frac{x}{2}(\zeta-\zeta^{-1})}}{\zeta^{n+1}}\,d\zeta \xrightarrow{\zeta = e^{i\theta}} \frac{1}{\pi}\int_0^{\pi} \cos(x\sin\theta - n\theta)\,d\theta, \quad n \geq 0$$

$$x^2y'' + xy' + (x^2-\nu^2)y = 0.$$

$$x^2y'' + xy' - (x^2+\nu^2)y = 0$$ ⇒ 虚变量 Bessel eqn

$$y(x) = C J_\nu(ix) + D N_\nu(ix)$$

球 Bessel eqn $$(r^2R')' + [k^2r^2 - l(l+1)]R = 0$$

$k=0$, Euler eqn $$r^2R'' + 2rR' - l(l+1)R = 0$$

$$R(r) = Cr^l + Dr^{-(l+1)}$$

$x = kr$,

$y = R(\frac{x}{k})$.

$$x^2y'' + 2xy' + [x^2 - l(l+1)]y = 0$$

$$y(x) = \frac{C}{\sqrt{x}} J_{l+\frac{1}{2}}(x) + \frac{D}{\sqrt{x}} N_{l+\frac{1}{2}}(x)$$

积分变换法

$$\hat{f}(\lambda) = \int_{-\infty}^{+\infty} f(x)\,e^{-i\lambda x}\,dx$$

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \hat{f}(\lambda)\,e^{i\lambda x}\,d\lambda$$

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(\xi)\,e^{-i(\xi-x)}\,d\xi\,d\lambda$$

半无界区间可用

$$\hat{f}_s(\lambda) = \int_0^{+\infty} f(x)\sin\lambda x\,dx$$

$$\hat{f}_c(\lambda) = \int_0^{+\infty} f(x)\cos\lambda x\,dx$$

高维卷积 $$f * g = \iiint_{-\infty}^{+\infty} f(x-\xi, y-\eta, z-\zeta)\,g(\xi,\eta,\zeta)\,d\xi\,d\eta\,d\zeta.$$

一般积分变换 $\hat{f}(\lambda) = \int_a^b f(x) \underset{}{\overset{\text{核}}{K(x;\lambda)}} dx$

基本解方法 $\int_a^b \delta(x-\xi)\varphi(x)dx = \begin{cases} \varphi(\xi), & \xi \in [a,b] \\ 0, & \xi \notin [a,b] \end{cases}$

$$\begin{array}{ccc} \varphi(x) & \longrightarrow & \varphi(\xi) \\ C(\mathbb{R}) & & \mathbb{R} \end{array}$$

线性泛函. 广义函数

$u(x) \in C^1(\mathbb{R})$. 在实轴上只有单零点 x_k. 则

$$\delta[u(x)] = \sum_{k=1}^{N} \frac{\delta(x-x_k)}{|u'(x_k)|}.$$

有 $\delta(ax) = \frac{\delta(x)}{|a|}$. $\delta(x^2-a^2) = \frac{\delta(x-a)}{|2a|} + \frac{\delta(x+a)}{|2a|} = \frac{\delta(x-a)}{2|x|} + \frac{\delta(x+a)}{2|x|}$

$\delta(\sin x) = \sum_{k=-\infty}^{+\infty} \delta(x-k\pi)$. $|x|\,\delta(x^2) = \lim_{a\to 0} |x|\,\delta(x^2-a^2) = \delta(x)$

广义函数

引入 $C_0^\infty(\mathbb{R})$ 或记为 $\mathcal{D}(\mathbb{R})$ 基本函数空间.

$\varphi(x)$ 无穷次连续可导且在有界闭区间外为0.

$\mathcal{D}(\mathbb{R})$ 上线性泛函全体构成广义函数空间 $\mathcal{D}'(\mathbb{R})$.

$$\langle f^{(n)}(x), \varphi(x)\rangle \overset{d}{=} (-1)^n \langle f(x), \varphi^{(n)}(x)\rangle. \quad \forall \varphi(x) \in C_0^\infty(\mathbb{R}).$$

$$\langle h(x), \varphi(x)\rangle \overset{d}{=} \langle f(x), \langle g(y), \varphi(x+y)\rangle\rangle \qquad h(x) = f(x) * g(x)$$

另一个基本函数空间 $\varphi(\mathbb{R})$.

$\varphi(x)$ 无穷次连续可导且速降. --- $\varphi'(\mathbb{R})$

$f(x)\in\varphi'(\mathbb{R})$, 则 $F[f]$, $F^{-1}[f]$ 也是广义函数.

$$\langle F[f],\varphi\rangle \overset{d}{=} \langle f, F[\varphi]\rangle$$

$$\langle F^{-1}[f],\varphi\rangle \overset{d}{=} \langle f, F^{-1}[\varphi]\rangle$$

$$F[\delta(x)] = \int_{-\infty}^{+\infty}\delta(x)e^{-i\lambda x}\,dx = 1.$$

有 $x \rightleftharpoons 2\pi i\,\delta'(\lambda)$ 　　即 $\int_{-\infty}^{+\infty} e^{\pm i\lambda x}\,dx = 2\pi\delta(\lambda)$

$1 \rightleftharpoons 2\pi\delta(\lambda)$

$\cos ax \rightleftharpoons \pi[\delta(\lambda+a)+\delta(\lambda-a)]$ 　　$\int_{-\infty}^{+\infty}\cos\lambda x\,dx = 2\pi\delta(\lambda)$

$\sin ax \rightleftharpoons i\pi[\delta(\lambda+a)-\delta(\lambda-a)]$ 　　$\int_{-\infty}^{+\infty} x e^{-i\lambda x}\,dx = 2\pi i\,\delta'(\lambda)$

$x,\xi\in(-\pi,\pi)$ 时. $\delta(x-\xi) = \frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx)$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi}\delta(x-\xi)\cos nx\,dx = \frac{1}{\pi}\cos n\xi.$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi}\delta(x-\xi)\sin nx\,dx = \frac{1}{\pi}\sin n\xi.$$

$f_n(x)$ 为广义函数序列. $f(x)$ 为广义函数. 若对基本函数空间上任意 $\varphi(x)$ 均有

$$\lim_{n\to\infty}\int_{-\infty}^{+\infty} f_n(x)\varphi(x)\,dx = \int_{-\infty}^{+\infty} f(x)\varphi(x)\,dx.$$ 则 $\lim_{n\to\infty} f_n(x) \overset{w}{=} f(x)$

弱收敛

$$\lim_{\varepsilon\to 0^+}\frac{\varepsilon}{\pi(x^2+\varepsilon^2)} \overset{w}{=} \delta(x), \qquad \lim_{t\to +0}\frac{1}{\sqrt{2\pi t}}e^{-\frac{x^2}{2t^2}} \overset{w}{=} \delta(x)$$

$$\lim_{N\to+\infty} \frac{\sin Nx}{\pi x} \overset{w}{=} \delta(x).$$

$$F^{-1}[1] = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{i\lambda x} d\lambda$$

$$= \lim_{N\to+\infty} \frac{1}{2\pi}\int_{-N}^{N} e^{i\lambda x} d\lambda = \lim_{N\to+\infty} \frac{\sin Nx}{\pi x} \overset{w}{=} \delta(x)$$

$$\langle \delta(M-M_0), \varphi(M)\rangle = \iiint_{\mathbb{R}^3} \delta(M-M_0)\varphi(M)\, dM = \varphi(M_0)$$

$$\Delta \frac{1}{r} = -4\pi\delta(x,y,z)$$

$$Lu = f(x,y,z) \qquad -\infty < x,y,z < +\infty$$

u, f 作为广义函数. 广义解. $\longrightarrow$ 正则且光滑：古典解

$$\Delta_3 u = f(M). \qquad \Delta_3 U = \delta(M).$$ 基本解. 无定解条件. 非唯一.

$$u(M) = \int_{\mathbb{R}^3} U(M-M_0) f(M_0)\, dM_0 = U(M) * f(M).$$

$u_t = Lu$ 型初值问题

$$\begin{cases} \dfrac{\partial U}{\partial t} = LU, \quad t>0, \ M\in\mathbb{R}^n, \ n=1,2,3 \\ U|_{t=0} = \delta(M) \end{cases}$$

$U(t,M)$ 基本解.

$$u(t,M) = U(t,M) * \varphi(M) + \int_0^t U(t-\tau,M) * f(\tau,M)\, d\tau$$

$u_{tt} = Lu$ 型

$$\begin{cases} \dfrac{\partial^2 U}{\partial t^2} = LU \\ U|_{t=0} = 0, \ \dfrac{\partial U}{\partial t}\Big|_{t=0} = \delta(M) \end{cases}$$

$U(t,M)$ 基本解 奇异性与维数?

$$u(t,M) = U(t,M) * \psi(M) + \frac{\partial}{\partial t}\left[U(t,M) * \varphi(M)\right] + \int_0^t U(t-\tau,M) * f(\tau,M)\, d\tau$$

广义函数

$x = (x_1, \cdots x_n)$ $\mathbb{R}^n$上的变量

$M(\Omega)$ 定义在Ω上的某类函数构成的函数空间

$$\lim_{k\to\infty} \varphi_k(x) = \varphi(x)$$ 基本空间

定义在$M(\Omega)$上的线性连续泛函f为一个广义函数

$$M(\Omega) \longrightarrow \mathbb{R}$$

$$\varphi(x) \longrightarrow \langle f, \varphi(x)\rangle$$

形式上用积分的符号表示配对的值

试验函数

$$\langle f(x), \varphi(x)\rangle = \int_\Omega f(x)\varphi(x)\,dx. \qquad \forall \varphi(x) \in M(\Omega)$$

$$\lim_{k\to\infty} \langle f_k(x), \varphi(x)\rangle = \langle f(x), \varphi(x)\rangle.$$ $M'(\Omega)$中 $\lim_{k\to\infty} f_k(x) = f(x)$. 弱极限

$\mathcal{E}(\mathbb{R}^n)$ $\mathbb{R}^n$上无穷阶连续可微 C^∞ 无穷阶连续可微函数类

$\varphi(\mathbb{R}^n)$ $\cdots$且速降 C_0^∞ $\cdots$有界支集

$D(\mathbb{R}^n)$ $\cdots$且有界的支集

$\operatorname{supp} f(x) = \{$在$\mathbb{R}^1$中$f(x)$为0的小区间的并集的余集$\}$

$$\langle f(x), \varphi(x)\rangle = \int_{\mathbb{R}^n} f(x)\varphi(x)\,dx = \int_{\Omega\cap\Omega_0} f(x)\varphi(x)\,dx = \int_{\Omega_0} f(x)\varphi(x)\,dx$$

乘子. $\alpha(x) \in C^\infty(\mathbb{R})$, $f(x) \in D'(\mathbb{R}^n)$. 直积 $f(x), g(y) \in D'(\mathbb{R}^1)$. $f(x)\otimes g(y)$

自变量代换 $\quad \langle f(u(x)),\ \varphi(x)\rangle \stackrel{d}{=} \langle f(u),\ \varphi(x(u))\left|\frac{dx}{du}\right|\rangle$

微分方程的变分方法

$f(x)$ 在 $[a,b]$ 连续，若对两个端点为0的 C^2 类函数 $\eta(x)$ 均有

$$\int_a^b f(x)\eta(x)\,dx = 0, \quad \text{则必有 } f(x)=0 \qquad (\text{变分法基本引理})$$

$$J[y(x)] = \int_a^b F(x,y,y')\,dx.$$

$$\frac{d}{d\alpha} J\left[y(x)+\alpha\delta y(x)\right]\Big|_{\alpha=0} \longrightarrow \delta J[y(x)]$$

$$\delta J[y(x)] = \int_a^b \left[\frac{\partial F}{\partial y}\delta y(x) + \frac{\delta F}{\partial y'}\delta y'(x)\right]dx = \int_a^b \left(F_y - \frac{d}{dx}F_{y'}\right)\delta y(x)\,dx + \frac{\partial F}{\partial y'}\delta y(x)\Big|_a^b.$$

$y(x)$ 为极值曲线时必满足

$$\begin{cases} F_y - \dfrac{d}{dx}F_{y'} = 0 \\ \dfrac{\partial F}{\partial y'}\delta y\Big|_a^b = \dfrac{\partial F}{\partial y'}\Big|_{x=b}\delta y(b) - \dfrac{\partial F}{\partial y'}\Big|_{x=a}\delta y(a) = 0 \end{cases}$$

$$\underline{F_y - \frac{d}{dx}F_{y'}} = \underline{F_y - F_{y'x} - F_{y'y}\,y' - F_{y'y'}\,y'' = 0}. \qquad \text{Euler eqn}$$

Euler 导数 $[F]_y$ $\qquad$ 稳定元

$$J[u(x,y)] = \iint_\Omega F(x,y,u,p,q)\,dx\,dy.$$

$$\delta J[u(x,y)] = \iint_\Omega \left[F_u\delta u + F_p(\delta u)_x + F_q(\delta u)_y\right]dx\,dy.$$

$$= \iint_\Omega \underbrace{\left[F_u - \frac{\partial}{\partial x}F_p - \frac{\partial}{\partial y}F_q\right]}_{=0}\delta u\,dx\,dy + \int_{\partial\Omega} \underbrace{\left[F_p\cos(\vec{n},x) + F_q\cos(\vec{n},y)\right]\delta u\,ds}_{=0}$$

$J[u(x,y)]$ 的 Euler eqn

$$J[u(x,y)] = \iint_\Omega F[x,y,u,p,q]\,dx\,dy + \int_{\partial\Omega} g(x,y,u)\,ds \quad \text{附加项}$$

$$\delta J[u(x,y)] = \iint_{\Omega} \left(F_u - \frac{\partial}{\partial x}F_p - \frac{\partial}{\partial y}F_q\right)\delta u\, dxdy + \int_{\partial\Omega}\left[F_p\cos(\vec{n},x) + F_q\cos(\vec{n},y) + g_u\right]\delta u\, ds$$

$$\delta J[y(x), z(x)] = \int_a^b \left(F_y - \frac{d}{dx}F_{y'}\right)\delta y\, dx + \int_a^b \left(F_z - \frac{d}{dx}F_{z'}\right)\delta z\, dx + \left(F_{y'}\delta y + F_{z'}\delta z\right)\Big|_a^b$$

$$J[y(x)] = \int_a^b F(x, y, y', y'')\, dx.$$

$$[F]_y = F_y - \frac{d}{dx}F_{y'} + \frac{d^2}{dx^2}F_{y''} = 0. \quad \cdots$$

活动区间问题

$$J[y(x)] = \int_{x_1}^{x_2} F(x, y, y')\, dx$$

(x_1, y_1) 在 $\varphi(x,y) = 0$ 上

(x_2, y_2) 在 $\psi(x,y) = 0$ 上. 变动

$$\delta J[y(x)] = \frac{d}{d\alpha}\int_{x_1+\alpha\delta x_1}^{x_2+\alpha\delta x_2} F(x, y+\alpha\delta y, y'+\alpha\delta y')\, dx\Big|_{\alpha=0}$$

$$= \int_{x_1}^{x_2} [F]_y\, \delta y\, dx + F(x,y,y')\,\delta x\Big|_{x_1}^{x_2} + F_{y'}\,\delta y\Big|_{x_1}^{x_2}$$

$(\delta y(x_2) = \delta y_2 - y'(x_2)\,\delta x_2,$

$\delta y(x_1) = \delta y_1 - y'(x_1)\,\delta x_1.)$

$$= \int_{x_1}^{x_2} [F]_y\, \delta y\, dx + (F - y'F_{y'})\,\delta x\Big|_{x_1}^{x_2} + F_{y'}\big|_{x_2}\delta y_2 - F_{y'}\big|_{x_1}\delta y_1$$

直接法 $\quad J[y(x)] = \int_a^b F(x, y, y')\, dx \qquad$ 容许函数类 $D(J)$.

$$y_n(x) = \sum_{i=1}^{n} a_i \varphi_i(x) \qquad \frac{\partial J(a_1, \cdots a_n)}{\partial a_i} = 0.$$

$$J[y_n(x)] = \int_a^b F(x, y_n, y_n')\, dx$$

n级近似解

微分方程的变分方法

$$\frac{d}{dx}\left(k(x)\frac{dy}{dx}\right) - q(x)y + f(x) = 0$$

$$J[y(x)] = \int_a^b \left[ky'^2 + qy^2 - 2f(x)y\right] dx$$

$$J[y(x)] = -\left\langle \frac{d}{dx}k(x)y' - q(x)y, y\right\rangle - 2\langle f(x), y\rangle$$

$$= \int_a^b \left(ky'^2 + q(x)y^2 - 2f(x)y\right) dx$$

等周问题 $J_1[y(x)] = \int_a^b G(x,y,y')\,dx = L$ 下 求

$$J[y(x)] = \int_a^b F(x,y,y')\,dx$$ 极值.

固有值问题
$$\begin{cases} \dfrac{d}{dx}\left(k\dfrac{dy}{dx}\right) - q(x)y + \lambda\rho(x)y = 0 \\ y(a) = 0,\quad y(b) = 0 \end{cases}$$

构造 $J[y(x)] = \left\langle -\dfrac{d}{dx}\left(k\dfrac{dy}{dx}\right) + q(x)y,\ y \right\rangle$

$$= \int_a^b \left[k(x)y'^2 + q(x)y^2\right]dx$$

$LX = \lambda X$. $\rightleftarrows$ $\|X\|^2 = 1$ 下 $J[X] = \langle LX, X\rangle$ 极值问题