Title

Name

1 Introduction

title: entanglement and instability in tripartite optomechanical systems with detuned drives

abstract: We consider optomechanical realization of such bosonic system and use the well-established approach of quantum Langevin equations(QLEs)

It's common practice in modern quantum information science to create entanglement between multiple parties via quantum channel mediated with ancilla, and this process could be described by a corresponding quantum operation, or inversely, a global quantum operation is enabled via multipartite entanglement and LOCC(local operation classical communication). Fidelity of LOCC-based communication protocols is enhanced by either quantum error correction techniques or pre-established entanglement in physical systems, we are focusing on the latter. Fundamentally, the only way of creating entanglement is through either direct or indirect interactions mediated by gauge bosons. Aside from entangling objects at atomic levels, it is of broader interest to explore the possibility of entangling macroscopic, massive bodies, and radiation pressure force(as one type of direct interaction) is found to work effectively for this goal[1].

Establishing entanglement in hybrid systems could be achieved via a quantum interface capable of transferring quantum state between different degrees of freedom. In this context, an intermediate mechanical resonator between target electromagnetic modes within cavities well perform this role, and we get a typical three-mode system which may well serve the goal of quantum communication[2], mechanical transduction, and entanglement generation. In generic optomechanical systems, where photons are confined within an optical cavity coupled to a small-mass mechanical resonator, by setting one mirror much lighter than the other for the conventional Fabry-Perot setup, the accumulated radiation pressure force from photon reflection leads to observable optomechanical effects. However, the energy scale difference(THz vs MHz) forbids us from efficently coupling photon and phonon excitations. This problem could be solved by supplying extra drive to optical cavities. While bare cavity could be treated as harmonic trapping potential and operator of single mode a possesses a discrete set of eigenstates(Fock states), the quantum state within a driven cavity is approximately coherent state, which is a superposition of Fock states. Physically, when optomechanical coupling is not strong and perturbation picture applies, the driven cavity plus vacuum noise from supply port acts as engineered reservoir for the mechanical resonator. A red(blue)-detuned input photon absorb(emit) a phonon from mechanical resonator, in order to excite the resonant optical cavity mode. The red-shifted, and asymmetric noise spectrum of reservoir results in a net damping rate (reduction of occupation) for the mechanical resonator[3]. Known as cavity cooling, this techique is successfully applied to cool mechanical resonators to its motional ground state in the far-detuned regime, and is favorable for generating entanglement robust against thermal fluctuations. Based on this idea, one could deliberately engineer the coupling of given quantum system to a cold disspative reservoir and design unique steady-state entanglement after time evolution[4].

As a type of system(two or multi-mode optomechanical system with drive) brought to attention in the early 20th century, since the development of experimental technologies make regimes of theoretical interest realizable in labs, there have seen a significant number of related studies in the past decades. A classical treatment[8] verified the presence of multipartite entanglement in radiation pressure-coupled mirror and driven cavities. Three-mode system with symmetric side drives that are both red-detuned allows for efficient transfer of quantum states between cavities[5]. In contrast, anti-symmetric drives that offset oppositely from resonance frequency provides practical means of creating steady state intra-cavity entanglement. The effect of non-resonant drive detuning along with other system parameters were discussed in [6], [7], and they reached the same conclusion that anti-symmetric(opposite) drive detunings at exact mechanical frequency(motional sidebands) $\Delta = \pm\omega_{\mathrm{M}}$ provides the condition for optimal entanglement, as is the case we will be going through in following discussions. A counterpart system using Lindblad equation approach is discussed in [9], and they propose to achieve a highly pure two-mode squeezed state by optimizing two or four drive tunes. From above mentioned studies and vast studies that were not mentioned, three-mode bosonic system with one intermediate mode and two target side modes with anti-symmetric drive stands out for the goal of creating macroscopic entanglement. Specially, [4] pointed out this type of three-mode system enables us to achieve an amount of steady-state entanglement that surpass the limit allowed by a coherent parametric interaction. We are going to look into this type of system and discuss its possible mechanism and implications.

2 Model

2.1

Continuous variable entanglement gained attention as opposed to discrete ones (e.g. qubits) for its convenient access to continuous quadratures of target mode. Another outstanding feature is unconditionalness, which results in imperfection of obtained entanglement and leaves room for optimization[10]. Entanglement in our context is defined as the correlation between fluctuation operators around steady state, achieved at infinite time in a mean-field spirit. As will be followed by a set of approximations we discuss later, the Gaussian nature of quantum states(along with noises) are preserved following the evolution of linearized system Hamiltonian, and we get a c-number steady-state(eigenstate of a) plus fluctuation with zero-mean value. This could also be represented by a displacement transformation, with $D(\alpha) = \exp(\alpha a^\dagger - \alpha^* a)$, $D^\dagger(\alpha) a D(\alpha) = a + \alpha$. Around critical points fluctuation could become significant and this type of operator expansion $\hat{a} = \langle \hat{a} \rangle + \hat{d}$ might no longer be a good approximation. While pure state entanglement is readily characterized by von-Neumann entropy, establishing an entanglement monotone which does not increase under LOCC for mixed states is still an ongoing attempt. We consider the widely accepted notion, logarithmic negativity E_N. Entanglement of bipartite mixed states(in our case gaussian states that could be represented by Wigner functions in phase space) is easily characterized by this computable measure[11].

As a measure of bipartite entanglement for two-mode gaussian states that satisfy bosonic commutation relations $[\hat{a}_i, \hat{a}_j^\dagger] = \delta_{ij}$, we introduce logarithmic negativity $E_N = \max[0, -\text{In}2\eta^-]$, with η the lowest symplectic eigenvalue of covariance matrix V (see Appendix.C). We expect to transform the covariance matrix V to its standard form[10] via local linear canonical symplectic operation which preserves the uncertainty relation. To be more specific, Heisenberg uncertainty principle place restrictions on fluctuations of observables that do not commute. For Hermitian operators A and B that satisfy commutation relation $[\hat{A}, \hat{B}] = [\Delta\hat{A}, \Delta\hat{B}]$, we have $\langle(\Delta\hat{A})^2\rangle\langle(\Delta\hat{B})^2\rangle = 1/4\left(\langle[\Delta\hat{A}, \Delta\hat{B}]\rangle^2 + \langle\{\Delta\hat{A}, \Delta\hat{B}\}\rangle^2\right) \geq 1/4|\langle[\Delta\hat{A}, \Delta\hat{B}]\rangle|^2$. We soon find that $\langle(\Delta\hat{x})^2\rangle\langle(\Delta\hat{p})^2\rangle \geq 1/4$, as in below matrix, this is always satisfied with $n_1, n_2 \geq 0$. For coherent state, $\langle(\Delta\hat{x})^2\rangle = \langle(\Delta\hat{p})^2\rangle = 1/2$, and n_1, n_2 actually quantifies the deviation of one certain state(steady state within cavity in our case) from coherent state (one special case of minimum uncertainty state where the equation sign holds).

The special type of covariance matrix we will be dealing with in the $\{\hat{x_1}, \hat{p_1}, \hat{x_2}, \hat{p_2}\}$ basis is

$$V = \begin{pmatrix} A & C \\ C^T & B \end{pmatrix}, \quad A = \begin{pmatrix} n_1 + \frac{1}{2} & \\ & n_1 + \frac{1}{2} \end{pmatrix}, \quad B = \begin{pmatrix} n_2 + \frac{1}{2} & \\ & n_2 + \frac{1}{2} \end{pmatrix}, \quad C = \begin{pmatrix} n_{xx} & n_{xp} \\ n_{px} & n_{pp} \end{pmatrix}$$

Under a special case $n_{xx} = n_{pp} = 0, n_{xp} = n_{px}$ as will be shown in following calculation, the condition for non-zero E_N is $\sqrt{n_1 n_2} \leq |n_{xp}| \leq \sqrt{(n_1+1)(n_2+1)}$, which is consistent with Simon's separability criterion[12] $4(ab - c_1^2)(ab - c_2^2) \geq (a^2 + b^2) + 2|c_1 c_2| - 1/4$. Within this range, and for fixed n_1, n_2, there's optimal $|n_{px}|$ to maximize E_N.

2.2

canonical commutation relation

We provide analytical solution of steady state occupation and correlation terms of a generic dissipative three-mode bosonic system with anti-symmetric drives. We confine our discussion to an optomechanical realization, but the physical process applies to general bosonic systems coupled with parametric interaction. Consider a three-mode system with two optical modes coupled to one mechanical mode via radiation pressure, and each optical mode is subject to a coherent drive $\omega_{1,2} = \omega_c \pm \Delta$. We keep our discussion within the condition of exact drive detuning $\Delta = \pm\omega_{\rm M}$, which could be fulfilled readily with contemporary fine-tuned signal sources. We also restrict attention to only one/two of the many possible mechanical/optical modes from system. We have

$$H = \omega_{\rm M}\hat{b}^{\dagger}\hat{b} + \sum_{i=1,2}\left(\omega_i \hat{a}_i^{\dagger}\hat{a}_i + g_i \hat{a}_i^{\dagger}\hat{a}_i(\hat{b}^{\dagger} + \hat{b})\right) + H_{\rm diss}$$

Dynamics of this Hamiltonian with nonlinear optomechanical interaction could not be solved analytically in principle, but we are able to linearize around steady state via operator expansion. To gain the linear QLEs we assumed large coherent amplitude α, which depends on the input power(circulating photon number) and is made real by appropriately choosing a phase reference, so that to drop the non-linear quadratic photon terms. This is not in contrast with the weak coupling assumption for operator expansion, since this linearize treatment still holds for $g \ll \kappa$. Move to a rotating frame(interaction picture) with respect to cavity drive to get effective Hamiltonian

$$H_{\rm eff} = \omega_{\rm M}(b^{\dagger}b + d_1^{\dagger}d_1 - d_2^{\dagger}d_2) + H_{\rm int} + H_{\rm CR} + H_{\rm diss}$$

$$H_{\rm int} = g_1\alpha_1(\hat{b}\hat{d}_1^{\dagger} + \hat{d}_1\hat{b}^{\dagger}) + g_2\alpha_2(\hat{b}\hat{d}_2 + \hat{d}_2^{\dagger}\hat{b}^{\dagger})$$

$$H_{\rm CR} = g_1\alpha_1(\hat{b}\hat{d}_1 + \hat{d}_1^{\dagger}\hat{b}^{\dagger}) + g_2\alpha_2(\hat{b}\hat{d}_2^{\dagger} + \hat{d}_2\hat{b}^{\dagger})$$

In this picture evolution of quantum states is governed by $H_{\rm eff}$ and operators rotate respect to cavity drives. By working in resolved-sideband regime $\omega_{\rm M} \gg \kappa_1, \kappa_2$ (period of mechanical oscillator is much smaller than photon lifetime), we could further drop fast-oscillating(counter-rotating) terms emerged, known as rotating wave approximation (RWA). The bare optomechanical coupling g is enhanced by the average coherent amplitude α, which could be large under strong drive, and is controllable, for example, by applying modulated/pulsed optical drives, we then introduce effective coupling strength $G_i = g_i\alpha_i$. After above treatment we get below equations of motion. This QLE treatment allows us to reach small damping as well as strong coupling regimes beyond Lindblad master equation approach[13].

$$\frac{d}{dt}\hat{b} = (-i\omega_{\rm M} - \frac{\gamma}{2})\hat{b} - i(G_1\hat{d}_1 + G_2\hat{d}_2^{\dagger}) - \sqrt{\gamma}\hat{b}_{\rm in} \tag{1}$$

$$\frac{d}{dt}\hat{d}_1 = (-i\omega_{\rm M} - \frac{\kappa_1}{2})\hat{d}_1 - iG_1\hat{b} - \sqrt{\kappa_1}\hat{d}_{1,\rm in} \tag{2}$$

$$\frac{d}{dt}\hat{d}_2^{\dagger} = (-i\omega_{\rm M} - \frac{\kappa_2}{2})\hat{d}_2^{\dagger} + iG_2\hat{b} - \sqrt{\kappa_2}\hat{d}_{2,\rm in}^{\dagger} \tag{3}$$

The noise terms obey commutation relation $\langle f_{\rm in}(t)f_{\rm in}^{\dagger}(t')\rangle = (n_{\rm f,in} + 1)\delta(t - t')$, $n_{\rm f,in}$ denotes thermal occupation of corresponding bath. While each driving port of optical cavity is associated with a zero-mean Gaussian vacuum noise, the mechanical resonator suffers from Brownian noise, which obeys a different statistics and leads to the vulnerability of entanglement against environment. Thus we assumed relatively large quality factor of mechanical resonator so that to restore the Markovian feature of its noise terms[14]. Correlation of the resulting mechanical "input noise" term (mean thermal excitation number) $\langle\hat{b}_{\rm in}^{\dagger}\hat{b}_{\rm in}\rangle = n_{\rm b,in}$ then has temperature dependence and obeys Boltzmann statistics.

Detailed expressions of occupation is available in appendix. Specially, setting $G_2 = 0$ the tripartite system degrades to a bipartite optomechanical system with red-detuned drive, and it mediates an effective state-transfer(beam-splitter) interaction $\hat{b}\hat{d}_1^{\dagger} + \hat{d}_1\hat{b}^{\dagger}$. We get steady state occupation of mechanical mode n_b that shows dependence on mechanical bath temperatures and cavity vacuum noise, and these thermal occupation numbers could be further reduced by tuning enhanced optomechanical coupling strength G_1, as is the well-known result of cavity cooling[3]. A higher-order occupation expression beyond RWA treated with perturbation expansion is available in [15].

$$n_b = \frac{4G_1^2\kappa_1}{(\gamma+\kappa_1)(\gamma\kappa_1 + 4G_1^2)}n_{1,\rm in} + \left(\frac{\gamma}{\gamma+\kappa_1} + \frac{\gamma\kappa_1^2}{(\gamma+\kappa_1)(\gamma\kappa_1+4G_1^2)}\right)n_{\rm b,in} \tag{4}$$

$$n_{b1xp} = -n_{b1px} = \frac{2G_1\gamma\kappa_1}{(\gamma+\kappa_1)(\gamma\kappa_1 - 4G_1^2)}(n_{1,\rm in} - n_{\rm b,in})$$

Setting $G_1 = 0$ we get a blue-detuned driven bipartite system governed by parametric (two-mode squeezing) interaction $\hat{b}\hat{d}_2 + \hat{d}_2^\dagger\hat{b}^\dagger$. Below equations are consistent with [16],[17](note the convention difference $G \to 2G_2, \kappa \to \kappa_2/2$). We also notice the stability condition manifests itself in occupancy expression, ensuring positivity, and is consistent with Routh-Hurwitz Criterion[18]. The occupation numbers become significant around critical points implying anti-damping, and the associated noise terms are amplified as well.

$$n_b = \frac{4G_2^2\kappa_2}{(\gamma+\kappa_2)(\gamma\kappa_2-4G_2^2)}(n_{2,\mathrm{in}}+1) + \left(\frac{\gamma}{\gamma+\kappa_2} + \frac{\gamma\kappa_2^2}{(\gamma+\kappa_2)(\gamma\kappa_2-4G_2^2)}\right)n_{\mathrm{b,in}}$$

$$n_2 = \frac{4G_2^2\gamma}{(\gamma+\kappa_2)(\gamma\kappa_2-4G_2^2)}(n_{\mathrm{b,in}}+1) + \left(\frac{\kappa_2}{\gamma+\kappa_2} + \frac{\kappa_2\gamma_2^2}{(\gamma+\kappa_2)(\gamma\kappa_2-4G_2^2)}\right)n_{2,\mathrm{in}}$$

$$n_{b2xp} = n_{b2px} = \frac{2G_2\gamma\kappa_2}{(\gamma+\kappa_2)(\gamma\kappa_2-4G_2^2)}(n_{2,\mathrm{in}}+n_{\mathrm{b,in}}+1)$$

The calculated logarithmic negativity E_N of this blue-detuned driven two-mode bosonic system is upper-bounded by the stability condition applied on G_2, showing the limitation of amount of produced entanglement governed by two-mode squeezing Hamiltonian. (how to get tms vacuum from here?) Not only is E_N limited by the maximum G_2 that could be achieved, but the E_N it self converges to a constant value. The steady state only exists for a usually weak coupling strength $G_2 \leq \sqrt{\gamma\kappa_2}/2$, we will see the three-mode system makes a difference.

$$E_N \approx \ln\left(1+\frac{4G_2}{\gamma+\kappa_2}\right) \leq \ln(2)$$

The three-mode system is a combination of above mentioned state transfer(cavity cooling) and parametric amplification(two-mode squeezing) process. At zero temperature vanish noise terms we get occupation and correlation expressions

$$n_b = \frac{4G_2^2\kappa_2(4G_1^2\kappa_2 - 4G_2^2\kappa_1 + \kappa_1(\gamma+\kappa_1)(\kappa_1+\kappa_2))}{((4G_1^2+\gamma\kappa_1)\kappa_2 - 4G_2^2\kappa_1)\left(4G_1^2(\gamma+\kappa_1) - (\gamma+\kappa_2)(4G_2^2 - (\gamma+\kappa_1)(\kappa_1+\kappa_2))\right)}$$

$$n_1 = \frac{16G_1^2G_2^2\kappa_2(\gamma+\kappa_1+\kappa_2)}{((4G_1^2+\gamma\kappa_1)\kappa_2 - 4G_2^2\kappa_1)\left(4G_1^2(\gamma+\kappa_1) - (\gamma+\kappa_2)(4G_2^2 - (\gamma+\kappa_1)(\kappa_1+\kappa_2))\right)}$$

$$n_2 = \frac{4G_2^2(4G_1^2(\gamma+\kappa_1)(\kappa_1+\kappa_2) - 4G_2^2\gamma\kappa_1 + \gamma\kappa_1(\gamma+\kappa_1)(\kappa_1+\kappa_2))}{((4G_1^2+\gamma\kappa_1)\kappa_2 - 4G_2^2\kappa_1)\left(4G_1^2(\gamma+\kappa_1) - (\gamma+\kappa_2)(4G_2^2 - (\gamma+\kappa_1)(\kappa_1+\kappa_2))\right)}$$

correlation between mechanical mode and each optical mode:

$$V_{b1} = \begin{pmatrix} n_b+\frac{1}{2} & & & n_{b1xp} \\ & n_b+\frac{1}{2} & n_{b1px} & \\ & n_{b1px} & n_1+\frac{1}{2} & \\ n_{b1xp} & & & n_1+\frac{1}{2} \end{pmatrix}, V_{b2} = \begin{pmatrix} n_b+\frac{1}{2} & & & n_{b2xp} \\ & n_b+\frac{1}{2} & n_{b2px} & \\ & n_{b2px} & n_2+\frac{1}{2} & \\ n_{b2xp} & & & n_2+\frac{1}{2} \end{pmatrix}$$

$$n_{b1xp} = -n_{b1px} = \frac{8G_1G_2^2\kappa_2(\gamma+\kappa_1+\kappa_2)}{(4G_1^2\kappa_2 - 4G_2^2\kappa_1 + \gamma\kappa_1\kappa_2)\left(4G_1^2(\gamma+\kappa_1) - (\gamma+\kappa_2)(4G_2^2 - (\gamma+\kappa_1)(\kappa_1+\kappa_2))\right)}$$

$$n_{b2xp} = n_{b2px} = \frac{2G_2\kappa_2\left(4G_1^2(\gamma+\kappa_1) - (\gamma+\kappa_2)(4G_2^2 - (\gamma+\kappa_1)(\kappa_1+\kappa_2))\right)}{(4G_1^2\kappa_2 - 4G_2^2\kappa_1 + \gamma\kappa_1\kappa_2)\left(4G_1^2(\gamma+\kappa_1) - (\gamma+\kappa_2)(4G_2^2 - (\gamma+\kappa_1)(\kappa_1+\kappa_2))\right)}$$

correlation between two optical modes:

$$V_{12} = \begin{pmatrix} n_1+\frac{1}{2} & & n_{12xx} & \\ & n_1+\frac{1}{2} & & n_{12pp} \\ n_{12xx} & & n_2+\frac{1}{2} & \\ & n_{12pp} & & n_2+\frac{1}{2} \end{pmatrix}$$

$$n_{12xx} = -n_{12pp} = \frac{-4G_1G_2\kappa_2\left(4G_2^2\kappa_1 + 4G_1^2(\gamma+\kappa_1) + \gamma\kappa_1(\gamma+\kappa_1)\right)}{(4G_1^2\kappa_2 - 4G_2^2\kappa_1 + \gamma\kappa_1\kappa_2)\left(4G_1^2(\gamma+\kappa_1) - (\gamma+\kappa_2)(4G_2^2 - (\gamma+\kappa_1)(\kappa_1+\kappa_2))\right)}$$

We notice there's non-zero displacement-momentum(x-p) correlation between fluctuation of the mechanical and optical modes, and for the target optical modes there exist x-x and p-p type correlations, as is usually observed in EPR-type or Stern-Gerlach-like experiments. We also notice the x-p correlation in V_{b1} enabled by beam-splitter type interaction does not lead to contribution to entanglement. The intra-cavity entanglement we get roots from parametric amplification between mechanical mode and blue-detuned driven optical mode 2, and the generated correlation is further transferred to red-detuned driven optical mode 1.

From above relations we are able to plot analytically intra-cavity entanglement E_N dependence on varying optical damping terms. There's optimal coupling strength to maximize entanglement, and smaller damping is more favorable for larger absolute value of entanglement peak. (expression?) The optimal value easily corresponds to a strong coupling regime. Purity is given by symplectic invariant $\mu = \mathrm{Tr}[\rho^2] = 1/4\sqrt{\det V}$. With provided set of parameters there exists a range of κ_2 that we get considerable entanglement without sacrificing purity. (should we go further on this?)

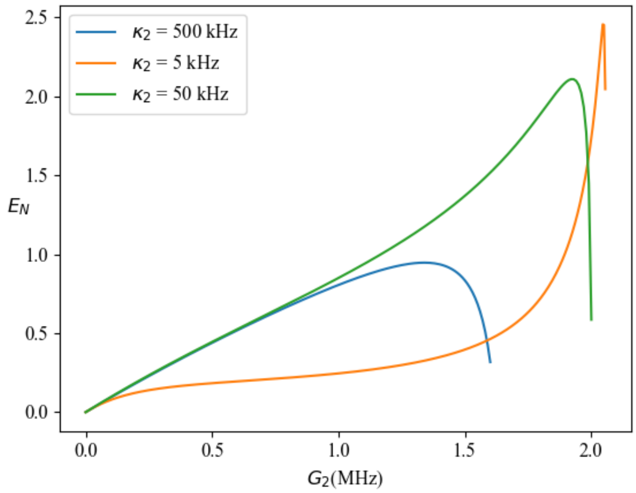

Figure 1: Entanglement E_N dependence on G_2 with fixed G_1 at zero bath temperature. $\kappa_1 = 2\pi \times 50 \times 10^3$ Hz; $\gamma = 2\pi \times 0.8 \times 10^6$ Hz; $G_1 = 2\pi \times 2 \times 10^6$ Hz; κ_2 values: $2\pi \times 50 \times 10^3$ Hz, $2\pi \times 500 \times 10^3$ Hz, $2\pi \times 5 \times 10^3$ Hz;

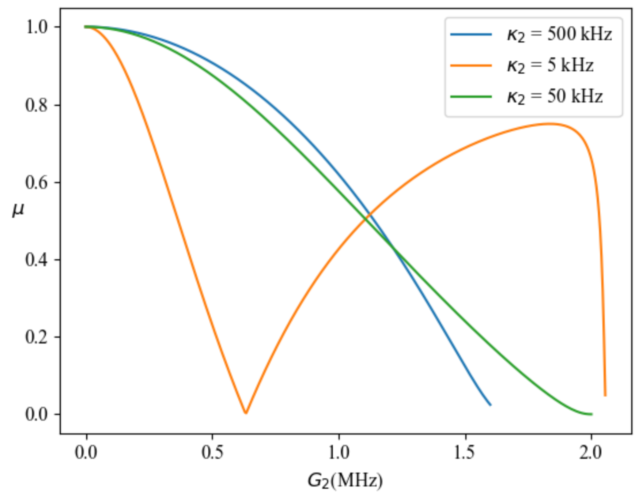

Figure 2: purity μ dependence on G_2 with fixed G_1 at zero bath temperature. $\kappa_1 = 2\pi \times 50 \times 10^3$ Hz; $\gamma = 2\pi \times 0.8 \times 10^6$ Hz; $G_1 = 2\pi \times 2 \times 10^6$ Hz; κ_2 values: $2\pi \times 50 \times 10^3$ Hz, $2\pi \times 500 \times 10^3$ Hz, $2\pi \times 5 \times 10^3$ Hz;

As could be seen from Appendix.C,E, covariance matrix of the two target optical modes has the form of a two-mode squeezed thermal state. (what if we treat the squeezing operation/Bogliubov transformation as representation transformation? how to get this state from given Hamiltonian?)

$$
\begin{aligned}
U &= \exp(rd_1d_2 - h.c.) \\
H &= g_1\alpha_1(\hat{b}\hat{d}_1^\dagger + \hat{d}_1\hat{b}^\dagger) + g_2\alpha_2(\hat{b}\hat{d}_2 + \hat{d}_2^\dagger\hat{b}^\dagger) \\
H' &= UHU^\dagger + i\hbar\frac{dU}{dt}U^\dagger
\end{aligned}
$$

what is the relationship between squeezing parameter and Gt?

As is discussed in [6], the mechanical mode mediates a time-dependent retarded parametric two-mode squeezing interaction between two target modes, which is resonantly large at exact detuning $\Delta = \pm\omega_{\mathrm{M}}$. The thermal robustness of the Bogoliubov-mode-based schemes hinges on the achievement of a large multiphoton optomechanical coupling rate that far exceeds the damping rates of the relevant optical and mechanical modes.

Under a special condition of symmetric damping, i.e. setting $\kappa_1 = \kappa_2 = \kappa$ the above equations reduces to.

$$
\begin{aligned}
n_1 &= \frac{8G_1^2G_2^2(\gamma+2\kappa)}{(\gamma+\kappa)(4G_1^2-4G_2^2+\gamma\kappa)(2G_1^2-2G_2^2+\kappa(\gamma+\kappa))} \\
n_2 &= \frac{4G_2^2\left(-2G_2^2\gamma+4G_1^2(\gamma+\kappa)+\gamma\kappa(\gamma+\kappa)\right)}{(\gamma+\kappa)(4G_1^2-4G_2^2+\gamma\kappa)(2G_1^2-2G_2^2+\kappa(\gamma+\kappa))} \\
n_{12} &= -\frac{2G_1G_2\left(4G_2^2\kappa+4G_1^2(\gamma+\kappa)+\gamma\kappa(\gamma+\kappa)\right)}{(\gamma+\kappa)(4G_1^2-4G_2^2+\gamma\kappa)(2G_1^2-2G_2^2+\kappa(\gamma+\kappa))}
\end{aligned}
$$

By applying Bogliubov transformation(two-mode squeezing as unitary operation?) to field operators d_1, d_2, we could recover expression of occupancy and correlation of β_A, β_B provided by previous work [4]. We should note that the diagonalization of given Hamiltonian is dependent on non-Hermitian dissipation terms $-i\hbar\gamma\hat{b}^\dagger\hat{b} - i\hbar\kappa_1\hat{d}_1^\dagger\hat{d}_1 - i\hbar\kappa_2\hat{d}_2^\dagger\hat{d}_2$ (to be confirmed, the Bogliubov mode should have something to do with symmetric damping), and the existence of mechanical dark mode β_B is only valid under symmetric damping $\kappa_1 = \kappa_2$. The decoupled, thus unchanged occupancy of $n_B = \sinh^2(r) = G_1^2/(G_1^2 - G_2^2)$ ensures $n_{12} = -\frac{1}{2}\left(n_1\frac{G_2}{G_1} + n_2\frac{G_1}{G_2}\right)$,

dark mode: is it decoherence free subspace? what is the dark mode?

Intuitively, when the detuning $|\Delta| = \omega_{\mathrm{M}}$ is much larger than optical and mechanical damping rates, we could adiabatically eliminate the mechanical mode and get an effective parametric interaction between the two target modes.

automatically induces an effective interaction between optical modes

2.3

Before discussing non-equilibrium dynamics of provided system, let's make clear its stable regime and locate critical points. The stability condition could be obtained either from applying Routh-Hurwitz criterion to dynamical matrix gained from the QLEs, or ensuring positivity of the occupation expressions we derived. The system become unstable under conditions

$$
G_2 \geq \sqrt{\frac{\kappa_2}{\kappa_1}\left(G_1^2 + \frac{\gamma\kappa_1}{4}\right)}
$$

or

$$
G_2 \leq \min\left[\sqrt{\frac{\gamma+\kappa_1}{\gamma+\kappa_2}\left(G_1^2 + \frac{(\gamma+\kappa_2)(\kappa_1+\kappa_2)}{4}\right)}, \sqrt{G_1^2 + \frac{\gamma}{2}\frac{\kappa_1}{2} + \frac{\gamma}{2}\frac{\kappa_2}{2} + \frac{\kappa_1}{2}\frac{\kappa_2}{2}}\right]
$$

The "turning point" we saw in previous plots is . We should also notice that G_i is associated with coherent amplitude α_i, and this amplitude itself might reveal multi-stability.

antidamping can lead to amplification of thermal fluctuations, and finally to an instability if the full damping rate becomes negative (good for entanglement but thermal fluctuation is also amplified?) In that case, any small initial (e.g., thermal) fluctuation will at first grow exponentially in time. Later, nonlinear effects will saturate the growth of the mechanical oscillation amplitude.

In order to understand stability behavior of our system and locate critical points, we revisit the simple two-mode optomechanical system model with blue-detuned drive. We get equations (no need? directly start with three-mode)

$$\frac{d}{dt}\hat{b} = (-i\omega_{\mathrm{M}} - \frac{\gamma}{2})\hat{b} - iG_2\hat{d}_2^\dagger - \sqrt{\gamma}\hat{b}_{\mathrm{in}} \tag{5}$$

$$\frac{d}{dt}\hat{d}_2^\dagger = (-i\omega_{\mathrm{M}} - \frac{\kappa_2}{2})\hat{d}_2^\dagger + iG_2\hat{b} - \sqrt{\kappa_2}\hat{d}_{2,\mathrm{in}}^\dagger \tag{6}$$

we get steady state solutions

for a fixed set of parameters, A can in general take on multiple stable values, corresponding to several stable attractors of this dynamical system. This effect is known as dynamical multistability, and in experiments it may lead to hysteretic behavior

In the quantum regime, the parametric instability threshold is broadened due to quantum fluctuations, with strong amplification of fluctuations below threshold

phase transition?

2.4

pulsed scheme: eliminate the driving term by going into a time-dependent displaced picture, PhysRevA.84.052327

the verification of entanglement between the intracavity field and the moving mirror has to be performed via measurements on the outcoupled light leaving the optomechanical system. Ultimately only correlations between modes of the light field are measured, from which any entanglement involving the mechanical oscillator has to be inferred

time-dependence, oscillation of entanglement

In contrast to schemes that work in a steady-state regime under a continuous-wave drive, this protocol is not subject to stability requirements that normally limit the strength of achievable entanglement

The effect of time-dependent counter-rotating terms is discussed in[9], and they found a degradation of entanglement and potential instability.

Entanglement dynamics based on a pulsed scheme (using Sørensen-Mølmer approach) is discussed in [19] [20],it is similar to the three-mode setup with detuned drives we discussed, but use optical driving pulses. works in weak-coupling regime. They showed noise-resilient entanglement achieved at time frames where the mechanical mode is eliminated.

2.5

to be figured out:
https://doi.org/10.1364/OE.25.017237 implies variance in intensity could possibly surpass the bound of stable regime. They also claim the dynamical approach they used could be applied beyond dynamical stable regime

https://journals.aps.org/pra/abstract/10.1103/PhysRevA.87.063846 Here, we point out that it is possible to exactly null the deleterious effects of asymmetric optomechanical couplings by introducing additional parametric drives on each mechanical resonator.

3 Discussion

Provided above solutions, we might still hope to discuss about several aspects of this scheme. Why this scheme allows larger entanglement than two-mode squeezing coupling? Why the intermediate mechanical mode mediates coupling between the two target modes, and the final entanglement contains no information of the mechanical mode? What is the role of extra cooling on mechanical mode? What it possibly imply by the stability classes?

non-Hermitian

The optomechanical system of two coupled optical modes can also be viewed as a photonic version of the Josephson effect and its classical dynamics can give rise to chaos

instability/bi-stability and entanglement

Entanglement generation by parametric driving[21] (Its robustness against thermal influence is enhanced, since it employs parametric instabilities for entanglement generation?)

quantum optomechanics in the bistable regime[22], tristability[23]

It is remarkable that independent of the particular form of the initial state of the system, the mechanical oscillator periodically returns to its initial state, and leaves the optical modes increasingly entangled upon each return. The entanglement is generated through the mechanical motion of the system. However, the final entangled optical state contains no information of the mechanical system, and can thus be robust against thermal Brownian noise that enters the system through the mechanical oscillator. Note that in the limit that far exceeds, the mechanical degrees of freedom can be adiabatically eliminated. This is interesting.

4

Some of above mentioned content is inspired from a textbook "Quantum Optics" by Guangcan Guo.

Appendix A. treatment(delete, but might be useful for the thermal state)

For a generic unitary transformation $\widetilde{U}(t)$, $|\phi(t)\rangle = \widetilde{U}(t)|\psi(t)\rangle$

$$\mathrm{i}\frac{d|\phi(t)\rangle}{dt} = \mathrm{i}\frac{d\widetilde{U}(t)}{dt}\widetilde{U}^{\dagger}|\phi(t)\rangle + \widetilde{U}(t)H\widetilde{U}^{\dagger}|\phi(t)\rangle$$
$$= (\widetilde{U}H\widetilde{U}^{\dagger} + \mathrm{i}\frac{d\widetilde{U}}{dt}\widetilde{U}^{\dagger})|\phi(t)\rangle = H_{\mathrm{eff}}|\phi(t)\rangle$$

used formula

$$e^{A}Be^{-A} = B + [A,B] + \frac{1}{2!}[A,[A,B]] + ...$$

$$e^{\mathrm{i}\omega a^{\dagger}at}ae^{-\mathrm{i}\omega a^{\dagger}at} = ae^{-\mathrm{i}\omega t}$$
$$e^{\mathrm{i}\omega a^{\dagger}at}a^{\dagger}e^{-\mathrm{i}\omega a^{\dagger}at} = a^{\dagger}e^{\mathrm{i}\omega t}$$

Apply to the considered system we get effective Hamiltonian

$$H = \omega_{\mathrm{M}}\hat{b}^{\dagger}\hat{b} + \sum_{i=1,2}\left(\omega_i\hat{a}_i^{\dagger}\hat{a}_i + g_i(\hat{b}^{\dagger}+\hat{b})\hat{a}_i^{\dagger}\hat{a}_i\right) + H_{\mathrm{diss}}$$

$$H_0 = (\omega_1 - \omega_{\mathrm{M}})a_1^{\dagger}a_1 + (\omega_2 + \omega_{\mathrm{M}})a_2^{\dagger}a_2$$

$$H_{\mathrm{eff}} = \widetilde{U}H\widetilde{U}^{\dagger} + \mathrm{i}\frac{d\widetilde{U}}{dt}\widetilde{U}^{\dagger}, \widetilde{U} = e^{\mathrm{i}H_0 t}$$

$$H_{\mathrm{eff}} = \omega_{\mathrm{M}}(b^{\dagger}b + a_1^{\dagger}a_1 - a_2^{\dagger}a_2) + g_1(b^{\dagger}+b)a_1^{\dagger}a_1 + g_2(b^{\dagger}+b)a_2^{\dagger}a_2 + H_{\mathrm{diss}}$$

apply operator expansion $\hat{a} = \alpha + \hat{d}$ to get

$$H_{\mathrm{eff}} \approx \omega_{\mathrm{M}}(b^{\dagger}b + d_1^{\dagger}d_1 - d_2^{\dagger}d_2) + g_1\alpha_1(\hat{b}\hat{d}_1^{\dagger} + \hat{d}_1\hat{b}^{\dagger}) + g_2\alpha_2(\hat{b}\hat{d}_2 + \hat{d}_2^{\dagger}\hat{b}^{\dagger})$$
$$+ g_1\alpha_1(\hat{b}\hat{d}_1 + \hat{d}_1^{\dagger}\hat{b}^{\dagger}) + g_2\alpha_2(\hat{b}\hat{d}_2^{\dagger} + \hat{d}_2\hat{b}^{\dagger}) + H_{\mathrm{diss}}$$

Appendix B. occupation

Denote $\omega - \omega_{\mathrm{M}}$ as ω (choose ω_{M} as reference) we get solutions in frequency space:

$$b(\omega) = \frac{-\mathrm{i}(\omega+\mathrm{i}\frac{\kappa_1}{2})(\omega+\mathrm{i}\frac{\kappa_2}{2})\sqrt{\gamma}b_{\mathrm{in}} - \mathrm{i}(\omega+\mathrm{i}\frac{\kappa_2}{2})\sqrt{\kappa_1}G_1 d_{1,\mathrm{in}} - \mathrm{i}(\omega+\mathrm{i}\frac{\kappa_1}{2})\sqrt{\kappa_2}G_2 d_{2,\mathrm{in}}^{\dagger}}{(\omega+\mathrm{i}\frac{\kappa_1}{2})\left((\omega+\mathrm{i}\frac{\gamma}{2})(\omega+\mathrm{i}\frac{\kappa_2}{2}) + G_2^2\right) - G_1^2(\omega+\mathrm{i}\frac{\kappa_2}{2})} = \chi_{bb}b_{\mathrm{in}} + \chi_{b1}d_{1,\mathrm{in}} + \chi_{b2}d_{2,\mathrm{in}}^{\dagger}$$

$$d_1(\omega) = \frac{-\mathrm{i}(\omega+\mathrm{i}\frac{\kappa_2}{2})\sqrt{\gamma}G_1 b_{\mathrm{in}} - \mathrm{i}\left((\omega+i\frac{\gamma}{2})(\omega+\mathrm{i}\frac{\kappa_2}{2}) + G_2^2\right)\sqrt{\kappa_1}d_{1,\mathrm{in}} - \mathrm{i}\sqrt{\kappa_2}G_1G_2 d_{2,\mathrm{in}}^{\dagger}}{(\omega+\mathrm{i}\frac{\kappa_1}{2})\left((\omega+\mathrm{i}\frac{\gamma}{2})(\omega+\mathrm{i}\frac{\kappa_2}{2}) + G_2^2\right) - G_1^2(\omega+\mathrm{i}\frac{\kappa_2}{2})} = \chi_{1b}b_{\mathrm{in}} + \chi_{11}d_{1,\mathrm{in}} + \chi_{12}d_{2,\mathrm{in}}^{\dagger}$$

$$d_2^{\dagger}(\omega) = \frac{\mathrm{i}(\omega+\mathrm{i}\frac{\kappa_1}{2})\sqrt{\gamma}G_2 b_{\mathrm{in}} + \mathrm{i}\sqrt{\kappa_1}G_1G_2 d_{1,\mathrm{in}} - \mathrm{i}\left((\omega+\mathrm{i}\frac{\gamma}{2})(\omega+\mathrm{i}\frac{\kappa_1}{2}) - G_1^2\right)\sqrt{\kappa_2}d_{2,\mathrm{in}}^{\dagger}}{(\omega+\mathrm{i}\frac{\kappa_1}{2})\left((\omega+\mathrm{i}\frac{\gamma}{2})(\omega+\mathrm{i}\frac{\kappa_2}{2}) + G_2^2\right) - G_1^2(\omega+\mathrm{i}\frac{\kappa_2}{2})} = \chi_{2b}b_{\mathrm{in}} + \chi_{21}d_{1,\mathrm{in}} + \chi_{22}d_{2,\mathrm{in}}^{\dagger}$$

We notice that Onsager reciprocal relation is satisfied for the generalized susceptibilities $\chi_{kj} = -\chi_{jk}$, we also expect the Kramers-Kronig dispersion relation holds. We will then be able to calculate the correlators in covariance matrix by integrating along real axis in complex Fourier plane(frequency space). e.g.

$$\langle\hat{b}^{\dagger}\hat{b}\rangle = \frac{1}{2\pi}\int d\omega\left[|\chi_{bb}(\omega)|^2 n_{\mathrm{b,in}} + |\chi_{b1}(\omega)|^2 n_{1,\mathrm{in}} + |\chi_{b2}(\omega)|^2(n_{2,\mathrm{in}}+1)\right]$$

We get following expressions of steady-state occupation.

$$n_b = \left(I_3 + (\frac{\kappa_1^2}{4} + \frac{\kappa_2^2}{4})I_2 + \frac{\kappa_1^2\kappa_2^2}{16}I_1\right)\gamma n_{\mathrm{b,in}} + (I_2 + \frac{\kappa_2^2}{4}I_1)\kappa_1 G_1^2 n_{1,\mathrm{in}} + (I_2 + \frac{\kappa_1^2}{4}I_1)\kappa_2 G_2^2(n_{2,\mathrm{in}}+1)$$

$$n_1 = (I_2 + \frac{\kappa_2^2}{4}I_1)G_1^2\gamma n_{\mathrm{b,in}} + \left(I_3 + (2G_2^2 + \frac{\gamma^2}{4} + \frac{\kappa_2^2}{4})I_2 + (G_2^2 - \frac{\gamma\kappa_2}{4})^2 I_1\right)\kappa_1 n_{1,\mathrm{in}} + I_1\kappa_2 G_1^2 G_2^2(n_{2,\mathrm{in}}+1)$$

$$n_2 = (I_2 + \frac{\kappa_1^2}{4} I_1) G_2^2 \gamma (n_{\mathrm{b,in}}+1) + \left(I_3 + (-2G_1^2 + \frac{\gamma^2}{4} + \frac{\kappa_1^2}{4}) I_2 + (G_1^2 + \frac{\gamma\kappa_1}{4})^2 I_1 \right) \kappa_2 n_{2,\mathrm{in}} + I_1 \kappa_1 G_1^2 G_2^2 (n_{1,\mathrm{in}}+1)$$

with coefficients

$$\begin{aligned} I_1 &= \int d\omega \frac{1}{|(\omega-\omega_1)(\omega-\omega_2)(\omega-\omega_3)|^2} \\ &= \frac{16(\gamma+\kappa_1+\kappa_2)}{((4G_1^2+\gamma\kappa_1)\kappa_2 - 4G_2^2\kappa_1)(4G_1^2(\gamma+\kappa_1)+(\gamma+\kappa_2)((\gamma+\kappa_1)(\kappa_1+\kappa_2)-4G_2^2))} \end{aligned}$$

$$I_2 = \int d\omega \frac{\omega^2}{|(\omega-\omega_1)(\omega-\omega_2)(\omega-\omega_3)|^2} = \frac{4}{4G_1^2(\gamma+\kappa_1)+(\gamma+\kappa_2)((\gamma+\kappa_1)(\kappa_1+\kappa_2)-4G_2^2)}$$

$$I_3 = \int d\omega \frac{\omega^4}{|(\omega-\omega_1)(\omega-\omega_2)(\omega-\omega_3)|^2} = \frac{4G_1^2 - 4G_2^2 + \gamma\kappa_1 + \gamma\kappa_2 + \kappa_1\kappa_2}{4G_1^2(\gamma+\kappa_1)+(\gamma+\kappa_2)((\gamma+\kappa_1)(\kappa_1+\kappa_2)-4G_2^2)}$$

Appendix C. correlation

V is the 4 × 4 covariance matrix of the two modes of interest,

$$V = \begin{pmatrix} A & C \\ C^T & B \end{pmatrix}, \Sigma = \det A + \det B - 2\det C$$

$$\eta^- = \frac{1}{\sqrt{2}} \sqrt{\Sigma - \sqrt{\Sigma^2 - 4\det V}}$$

There's non-zero entanglement when $\eta^- < \frac{1}{2}$, or equivalently, with $\Sigma > 4\det V + \frac{1}{4}$, we have $E_N > 0$.

We consider expression for covariance matrix V and entanglement E_N in terms of bosonic operators. The fluctuation associated with a given operator $\hat{a_i} = \langle \hat{a_i} \rangle + \hat{d_i}$ in phase space could be described by Hermitian opertors(dimensionless quadrature variables) $\hat{x_i} = \frac{\hat{d_i} + \hat{d_i^\dagger}}{\sqrt{2}}$, $\hat{p_i} = -i\frac{\hat{d_i} - \hat{d_i^\dagger}}{\sqrt{2}}$. The special type of covariance matrix we will be dealing with in the $\{\hat{x_1}, \hat{p_1}, \hat{x_2}, \hat{p_2}\}$ basis is

$$V = \begin{pmatrix} A & C \\ C^T & B \end{pmatrix}, \quad A = \begin{pmatrix} n_1 + \frac{1}{2} & \\ & n_1 + \frac{1}{2} \end{pmatrix}, \quad B = \begin{pmatrix} n_2 + \frac{1}{2} & \\ & n_2 + \frac{1}{2} \end{pmatrix}, \quad C = \begin{pmatrix} n_{xx} & n_{xp} \\ n_{px} & n_{pp} \end{pmatrix}$$

where the occupation terms are calculated upon operator expectation over steady states

$$\begin{aligned} n_{xx} &= \frac{1}{2}\langle \hat{x_1}\hat{x_2} + \hat{x_2}\hat{x_1} \rangle = \frac{1}{2}\langle \hat{d}_1\hat{d}_2 + \hat{d}_1^\dagger \hat{d}_2 + \hat{d}_1 \hat{d}_2^\dagger + \hat{d}_1^\dagger \hat{d}_2^\dagger \rangle \\ n_{pp} &= \frac{1}{2}\langle \hat{p_1}\hat{p_2} + \hat{p_2}\hat{p_1} \rangle = -\frac{1}{2}\langle \hat{d}_1\hat{d}_2 - \hat{d}_1^\dagger \hat{d}_2 - \hat{d}_1 \hat{d}_2^\dagger + \hat{d}_1^\dagger \hat{d}_2^\dagger \rangle \\ n_{xp} &= \frac{1}{2}\langle \hat{x_1}\hat{p_2} + \hat{p_2}\hat{x_1} \rangle = \frac{-i}{2}\langle \hat{d}_1\hat{d}_2 + \hat{d}_1^\dagger \hat{d}_2 - \hat{d}_1 \hat{d}_2^\dagger - \hat{d}_1^\dagger \hat{d}_2^\dagger \rangle \\ n_{px} &= \frac{1}{2}\langle \hat{p_1}\hat{x_2} + \hat{x_2}\hat{p_1} \rangle = \frac{-i}{2}\langle \hat{d}_1\hat{d}_2 - \hat{d}_1^\dagger \hat{d}_2 + \hat{d}_1 \hat{d}_2^\dagger - \hat{d}_1^\dagger \hat{d}_2^\dagger \rangle \end{aligned}$$

$$V_{bi} = \begin{pmatrix} n_b + \frac{1}{2} & & n_{bixx} & n_{bixp} \\ & n_b + \frac{1}{2} & n_{bipx} & n_{bipp} \\ n_{bixx} & n_{bipx} & n_i + \frac{1}{2} & \\ n_{bixp} & n_{bipp} & & n_i + \frac{1}{2} \end{pmatrix}, i = 1,2; V_{12} = \begin{pmatrix} n_1 + \frac{1}{2} & & n_{12xx} & n_{12xp} \\ & n_1 + \frac{1}{2} & n_{12px} & n_{12pp} \\ n_{12xx} & n_{12px} & n_2 + \frac{1}{2} & \\ n_{12xp} & n_{12pp} & & n_2 + \frac{1}{2} \end{pmatrix}$$

$$\begin{aligned} n_{b1xx} &= n_{b1pp} = 0 \\ n_{b1xp} &= -n_{b1px} = -\left(\frac{\kappa_1}{2} I_2 + \frac{\kappa_1\kappa_2^2}{8} I_1 \right) \gamma G_1 (n_{\mathrm{b,in}} + \frac{1}{2}) \\ &\quad - \frac{\kappa_1\kappa_2}{2} I_1 G_1 G_2^2 (n_{2,\mathrm{in}} + \frac{1}{2}) + \left(-(\frac{G_2^2\kappa_2}{2} - \frac{\gamma\kappa_2^2}{8}) I_1 + \frac{\gamma}{2} I_2 \right) \kappa_1 G_1 (n_{1,\mathrm{in}} + \frac{1}{2}) \end{aligned}$$

$$
\begin{aligned}
n_{b2xx} &= n_{b2pp} = 0 \\
n_{b2xp} &= n_{b2px} = \left(\frac{\kappa_2}{2} I_2 + \frac{\kappa_2\kappa_1^2}{8} I_1\right) \gamma G_2 (n_{\mathrm{b,in}} + \frac{1}{2}) \\
&\quad + \frac{\kappa_1\kappa_2}{2} G_1^2 G_2 (n_{1,\mathrm{in}} + \frac{1}{2}) + \left((\frac{G_1^2\kappa_1}{2} + \frac{\gamma\kappa_1^2}{8}) I_1 + \frac{\gamma}{2} I_2 \right) \kappa_2 G_2 (n_{2,\mathrm{in}} + \frac{1}{2})
\end{aligned}
$$

$$
\begin{aligned}
n_{12xx} &= -n_{12pp} = -(I_2 + \frac{\kappa_1\kappa_2}{4} I_1) G_1 G_2 \gamma (n_{\mathrm{b,in}} + \frac{1}{2}) \\
&\quad - \left(I_2 + (G_2^2 - \frac{\gamma\kappa_2}{4}) I_1\right) \kappa_1 G_1 G_2 (n_{1,\mathrm{in}} + \frac{1}{2}) + \left(I_2 - (G_1^2 + \frac{\gamma\kappa_1}{4}) I_1\right) \kappa_2 G_1 G_2 (n_{2,\mathrm{in}} + \frac{1}{2}) \\
n_{12xp} &= n_{12px} = 0
\end{aligned}
$$

EPR type

Appendix D. Bogliubov transformation

(simplify with commutation relation)

$$
\beta_A = d_1 \cosh r + d_2^\dagger \sinh r, \;\; \beta_B = d_1^\dagger \sinh r + d_2 \cosh r
$$
$$
\beta_A^\dagger = d_1^\dagger \cosh r + d_2 \sinh r, \;\; \beta_B^\dagger = d_1 \sinh r + d_2^\dagger \cosh r
$$

$$
\begin{aligned}
n_1 = d_1^\dagger d_1 &= \beta_A^\dagger \beta_A \cosh^2 r + (\beta_B^\dagger \beta_B + 1) \sinh^2 r - (\beta_A^\dagger \beta_B^\dagger + \beta_B \beta_A) \cosh r \sinh r \\
n_2 = d_2^\dagger d_2 &= \beta_B^\dagger \beta_B \cosh^2 r + (\beta_A^\dagger \beta_A + 1) \sinh^2 r - (\beta_B^\dagger \beta_A^\dagger + \beta_A \beta_B) \cosh r \sinh r \\
d_1 d_2 &= \beta_A \beta_B \cosh^2 r + \beta_B^\dagger \beta_A^\dagger \sinh^2 r - (\beta_A^\dagger \beta_A + \beta_B^\dagger \beta_B + 1) \sinh r \cosh r \\
d_1^\dagger d_2^\dagger &= \beta_A^\dagger \beta_B^\dagger \cosh^2 r + \beta_B \beta_A \sinh^2 r - (\beta_A^\dagger \beta_A + \beta_B^\dagger \beta_B + 1) \sinh r \cosh r \\
d_1^\dagger d_2 &= \beta_A^\dagger \beta_B \cosh^2 r + \beta_B \beta_A^\dagger \sinh^2 r - \beta_B \beta_B \cosh r \sinh r - \beta_A^\dagger \beta_A^\dagger \cosh r \sinh r \\
d_1 d_2^\dagger &= \beta_A \beta_B^\dagger \cosh^2 r + \beta_B^\dagger \beta_A \sinh^2 r - \beta_B^\dagger \beta_B^\dagger \cosh r \sinh r - \beta_A \beta_A \cosh r \sinh r
\end{aligned}
$$

if $d_1 d_2$ is real, as is in our case, $d_2^\dagger d_1^\dagger = d_1 d_2$, $d_1^\dagger d_2^\dagger = d_2 d_1$.

$$
\beta_A(\omega) = \left(i\omega_M - i\omega + \frac{\kappa}{2} + \frac{\widetilde{G}^2}{i\omega_M - i\omega + \frac{\gamma}{2}} \right)^{-1} \left(\frac{-i\widetilde{G}\sqrt{\gamma}}{i\omega_M - i\omega + \frac{\gamma}{2}} b_{in}(\omega) - \sqrt{\kappa} d_{in}(\omega) \right)
$$
$$
\beta_B(\omega) = \frac{-\sqrt{\kappa} d'_{in}(\omega)}{i\omega - i\omega_M + \frac{\kappa}{2}}
$$
$$
\beta_A = d_1 \cosh r + d_2^\dagger \sinh r, \;\; \beta_B = d_1^\dagger \sinh r + d_2 \cosh r
$$
$$
d_{in} = d_{1,in} \cosh r + d_{2,in}^\dagger \sinh r, \;\; d'_{in} = d_{1,in}^\dagger \sinh r + d_{2,in} \cosh r
$$
$$
\langle d_{in}(\omega) d'_{in}(\omega) \rangle = (\langle d_{1,in}^\dagger d_{1,in} \rangle + 1) \cosh(r) \sinh(r) + \langle d_{2,in}^\dagger d_{2,in} \rangle \cosh(r) \sinh(r)
$$
$$
\langle d'_{in}(\omega) d_{in}(\omega) \rangle = \langle d_{1,in}^\dagger d_{1,in} \rangle \cosh(r) \sinh(r) + (\langle d_{2,in}^\dagger d_{2,in} \rangle + 1) \cosh(r) \sinh(r)
$$
$$
\langle \beta_A(t) \beta_B(t) \rangle = \frac{\kappa(\kappa+\gamma)}{\kappa(\kappa+\gamma) + 2\widetilde{G}^2} \left(\langle d_{1,in}^\dagger d_{1,in} \rangle + 1 + \langle d_{2,in}^\dagger d_{2,in} \rangle \right) \cosh(r) \sinh(r)
$$

$$
\langle \beta_A(t) \beta_B(t) \rangle = \langle \beta_B(t) \beta_A(t) \rangle
$$

$$
\begin{aligned}
d_2 d_1 &= \beta_B \beta_A \cosh^2 r + \beta_A^\dagger \beta_B^\dagger \sinh^2 r - (\beta_B^\dagger \beta_B + \beta_A^\dagger \beta_A + 1) \sinh r \cosh r \\
d_2^\dagger d_1^\dagger &= \beta_B^\dagger \beta_A^\dagger \cosh^2 r + \beta_A \beta_B \sinh^2 r - (\beta_B^\dagger \beta_B + \beta_A^\dagger \beta_A + 1) \sinh r \cosh r
\end{aligned}
$$

let me try to write down β_A, β_B again

$$\beta_A(\omega) = \frac{-i(\omega + i\frac{\gamma}{2})\sqrt{\kappa} d_{in} + i\widetilde{G}\sqrt{\gamma} b_{in}}{(\omega + i\frac{\kappa}{2})(\omega + i\frac{\gamma}{2}) - \widetilde{G}^2}$$

$$\beta_B(\omega) = \frac{i\sqrt{\kappa} d'_{in}(\omega)}{\omega - i\frac{\kappa}{2}}$$

this is consistent with mma calculation below using the most general expression we calculated later.

$$\begin{aligned}
\langle \beta_A(t)\beta_B(t)\rangle &= \frac{1}{2\pi}\int d\omega \frac{\omega + i\frac{\gamma}{2}}{\left((\omega + i\frac{\kappa}{2})(\omega + i\frac{\gamma}{2}) - \widetilde{G}^2\right)(\omega - i\frac{\kappa}{2})}\kappa\langle d_{in}(\omega) d'_{in}(\omega)\rangle \\
&= \frac{1}{2\pi}\int d\omega \left(\frac{i\frac{\kappa}{2} + i\frac{\gamma}{2}}{\left((\omega + i\frac{\kappa}{2})(\omega + i\frac{\gamma}{2}) - \widetilde{G}^2\right)(\omega - i\frac{\kappa}{2})} + \frac{1}{(\omega + i\frac{\kappa}{2})(\omega + i\frac{\gamma}{2}) - \widetilde{G}^2}\right)\kappa\langle d_{in}(\omega) d'_{in}(\omega)\rangle
\end{aligned}$$

$$\begin{aligned}
\langle \beta_A^\dagger(t)\beta_B^\dagger(t)\rangle &= \frac{1}{2\pi}\int d\omega \frac{\omega - i\frac{\gamma}{2}}{\left((\omega - i\frac{\kappa}{2})(\omega - i\frac{\gamma}{2}) - \widetilde{G}^2\right)(\omega + i\frac{\kappa}{2})}\kappa\langle d_{in}^\dagger(\omega) d'^\dagger_{in}(\omega)\rangle \\
&= \frac{1}{2\pi}\int d\omega \left(\frac{-i\frac{\kappa}{2} - i\frac{\gamma}{2}}{\left((\omega - i\frac{\kappa}{2})(\omega - i\frac{\gamma}{2}) - \widetilde{G}^2\right)(\omega + i\frac{\kappa}{2})} + \frac{1}{(\omega - i\frac{\kappa}{2})(\omega - i\frac{\gamma}{2}) - \widetilde{G}^2}\right)\kappa\langle d_{in}^\dagger(\omega) d'^\dagger_{in}(\omega)\rangle \\
&= \frac{-2i(\kappa+\gamma)}{(2\omega_1 + i\kappa)(2i\omega_2 - \kappa)}\kappa\langle d_{in}^\dagger(\omega) d'^\dagger_{in}(\omega)\rangle \\
&= \frac{\kappa+\gamma}{2\widetilde{G}^2 + \kappa(\kappa+\gamma)}\kappa\langle d_{in}^\dagger(\omega) d'^\dagger_{in}(\omega)\rangle
\end{aligned}$$

Appendix E. thermal state

For two-mode squeezed thermal states, we write density matrix for canonical ensemble of this type of gaussian state in terms of average photon number $\overline{n}$:

$$\rho = S(r)\rho_{1,th}\otimes\rho_{2,th}S(r)^\dagger$$

$$\rho_{i,th} = \sum_{n_i=1}^{\infty}\frac{\overline{n}^{n_i}}{(1+\overline{n})^{n_i+1}}|n_i\rangle\langle n_i|$$

covariance matrix V in the $\{x_1, p_1, x_2, p_2\}$ basis, $x_i = \frac{a_i + a_i^\dagger}{\sqrt{2}}$, $p_i = -i\frac{a_i - a_i^\dagger}{\sqrt{2}}$ is [?],

$$V = \begin{pmatrix} B & C \\ C^T & B' \end{pmatrix}, \quad B = \begin{pmatrix} a & \\ & a \end{pmatrix}, \quad B' = \begin{pmatrix} b & \\ & b \end{pmatrix}, \quad C = \begin{pmatrix} c & \\ & -c \end{pmatrix}$$

$$\begin{aligned}
a &= \overline{n}_1\cosh^2 r + \overline{n}_2\sinh^2 r + \frac{1}{2}\cosh 2r \\
b &= \overline{n}_1\sinh^2 r + \overline{n}_2\cosh^2 r + \frac{1}{2}\cosh 2r \\
c &= \frac{1}{2}(\overline{n}_1 + \overline{n}_2 + 1)\sinh 2r
\end{aligned}$$

For fixed average photon number $\overline{n}_1, \overline{n}_2$, when the squeezing parameter r is larger than a critical r_s, the two states are entangled. Above this critical number, entanglement E_N is linearly(?) dependent on squeezing parameter r.

$$r \geq \frac{1}{4}\cosh^{-1}\frac{1 + 2n_1 + n_1^2 + 2n_2 + 10n_1n_2 + 8n_1^2n_2 + n_2^2 + 8n_1n_2^2 + 8n_1^2n_2^2}{(1+n_1+n_2)^2}$$

For two-mode vacuum states $\overline{n}_1 = \overline{n}_2 = 0$, critical $r_s = 0$, we get $E_N = max[0, 2r] = 2r$.

How to relate these two different expressions? (occupancy of bosonic modes, average photon number. a more formal relation might be achieved by expressing number states $|n\rangle$ in terms of a. but thermal states and coherent states are very different states. they might not be related.)

In order to relate between two types of expressions of E_N, we start from two-mode squeezed thermal state, and express c in terms of a and b. We get $c = \frac{1}{2}(a+b)\tanh(2r)$. Suppose this two-mode squeezed thermal state and two bosonic modes share the same covariance matrix, we get $n_{12} = \frac{1}{2}(n_1 + n_2 + 1)\tanh(2r)$. Both are generic gaussian states.

Appendix F. Stability

We have to ensure all eigenvalues of dynamical matrix M have a positive real part, or equivalently, all have negative imaginary part

$$M = \begin{pmatrix} -\mathrm{i}\omega + \mathrm{i}\omega_{\mathrm{M}} + \frac{\gamma}{2} & \mathrm{i}G_1 & \mathrm{i}G_2 \\ \mathrm{i}G_1 & -\mathrm{i}\omega + \mathrm{i}\omega_{\mathrm{M}} + \frac{\kappa_1}{2} & 0 \\ -\mathrm{i}G_2 & 0 & -\mathrm{i}\omega + \mathrm{i}\omega_{\mathrm{M}} + \frac{\kappa_2}{2} \end{pmatrix}$$

set $\det(M) = 0$

$$(\omega + \mathrm{i}\frac{\kappa_1}{2})\left((\omega + \mathrm{i}\frac{\gamma}{2})(\omega + \mathrm{i}\frac{\kappa_2}{2}) + G_2^2\right) - G_1^2(\omega + \mathrm{i}\frac{\kappa_2}{2}) = (\omega - \omega_1)(\omega - \omega_2)(\omega - \omega_3)$$

we have root relations

$$\omega_1 + \omega_2 + \omega_3 = -\mathrm{i}(\frac{\gamma}{2} + \frac{\kappa_1}{2} + \frac{\kappa_2}{2})$$

$$\omega_1\omega_2 + \omega_1\omega_3 + \omega_2\omega_3 = -G_1^2 + G_2^2 - \frac{\gamma}{2}\frac{\kappa_1}{2} - \frac{\gamma}{2}\frac{\kappa_2}{2} - \frac{\kappa_1}{2}\frac{\kappa_2}{2}$$

$$\omega_1\omega_2\omega_3 = \mathrm{i}G_1^2\frac{\kappa_2}{2} - \mathrm{i}G_2^2\frac{\kappa_1}{2} + \mathrm{i}\frac{\gamma}{2}\frac{\kappa_1}{2}\frac{\kappa_2}{2}$$

$\omega_{1,2,3}$ are pure imaginary. We get instability conditions

$$G_2^2\frac{\kappa_1}{2} - G_1^2\frac{\kappa_2}{2} \geq \frac{\gamma}{2}\frac{\kappa_1}{2}\frac{\kappa_2}{2}$$

or

$$\frac{\gamma}{2}\frac{\kappa_1}{2} + \frac{\gamma}{2}\frac{\kappa_2}{2} + \frac{\kappa_1}{2}\frac{\kappa_2}{2} + G_1^2 - G_2^2 \geq 0$$

$$(\frac{\gamma}{2} + \frac{\kappa_2}{2})G_2^2 - (\frac{\gamma}{2} + \frac{\kappa_1}{2})G_1^2 \geq (\frac{\gamma}{2} + \frac{\kappa_1}{2} + \frac{\kappa_2}{2})(\frac{\gamma}{2}\frac{\kappa_1}{2} + \frac{\gamma}{2}\frac{\kappa_2}{2} + \frac{\kappa_1}{2}\frac{\kappa_2}{2}) - \frac{\gamma}{2}\frac{\kappa_1}{2}\frac{\kappa_2}{2}$$

that is

$$G_2 \geq \sqrt{\frac{\kappa_2}{\kappa_1}\left(G_1^2 + \frac{\gamma\kappa_1}{4}\right)}$$

or

$$G_2 \leq \min\left[\sqrt{\frac{\gamma+\kappa_1}{\gamma+\kappa_2}\left(G_1^2 + \frac{(\gamma+\kappa_2)(\kappa_1+\kappa_2)}{4}\right)}, \sqrt{G_1^2 + \frac{\gamma}{2}\frac{\kappa_1}{2} + \frac{\gamma}{2}\frac{\kappa_2}{2} + \frac{\kappa_1}{2}\frac{\kappa_2}{2}}\right]$$

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