Floquet theorem ensures quasienergy due to periodicity in time, just as quasimomentum exists from periodic structure in real space in Bloch theorem. In contrast to possessing good quantum number in energy-conserving systems, we denote driven system with corresponding driving frequency, as a description of emerging dimension in parameter space. This setup, with d dimension of Hilbert space described by internal degrees of freedom of system states, and N synthetic dimension from possible manipulation and modulation of system parameters, forms high dimensional Floquet lattice, and is found to specially allow for non-trivial topological phenomena, possibly due to enhanced dimensionality and complex interactions between them, for example, one parameter might act as effective gauge potential to the other. The formation of topological nontrivial phases in high dimensional spaces reveals itself through real space phenomena, most typically, quantized pump of energy or current, tracing back to Thouless. Interestingly as result, we are gaining extra degrees of freedom or even novel phases by applying seemingly constrained drives, each obeys exact periodicity. By applying more and various types of drives, we are tailoring structure of parameter space, creating layers in complexity and allowing for the unfolding and emergence of novel phenomena.

In traditional Floquet theorem, to ensure unitarity, the diagonalization of evolution operator U_T requires Hermitian and periodic Hamiltonian $H(t)$, results in below spectral decomposition form:

$$U_T = \sum_{n=1}^{d} e^{-i\epsilon_n T/\hbar} \left|\phi_n\right\rangle \left\langle\phi_n\right| , \quad \left|\phi_n(t)\right\rangle = \left|\phi_n(t+T)\right\rangle \tag{1}$$

the eigenstates $|\phi_n\rangle$ of eligible U_T follows periodicity, and Floquet Hamiltonian H_F is constructed with corresponding quasienergy ϵ_n. When the system is driven by N incommensurate frequencies $\omega_1, \ldots, \omega_N$, the periodic part of the wavefunction can be generalized to a multidimensional Fourier expansion following number of absorbed photons $\vec{m} = (m_1, m_2, \ldots, m_N) \in \mathbb{Z}^N$:

$$|\phi_n(t)\rangle = \sum_{\vec{m}} e^{-i\,\vec{m}\cdot\vec{\omega}t} |\phi_{n,\vec{m}}\rangle, \quad |\psi_n(t)\rangle = \sum_{\vec{m}} e^{-i\,(\epsilon_n/\hbar + \vec{m}\cdot\vec{\omega})t} |\phi_{n,\vec{m}}\rangle \tag{2}$$

where $|\phi_{n,\vec{m}}\rangle$ represents a Floquet lattice site, and Floquet Hilbert space is the direct product of $\mathcal{H}_{\text{system}}$ with internal degree of freedom d, and $\mathcal{H}_{\text{drive}}$ with infinite dimension $\mathbb{Z}^N$. plug back to the Schrodinger equation to get

$$\epsilon_n |\phi_{n,\vec{m}}\rangle = \sum_{\vec{m}'} H_{\vec{m}-\vec{m}'} |\phi_{n,\vec{m}'}\rangle + \hbar\,(\vec{m}\cdot\vec{\omega}) |\phi_{n,\vec{m}}\rangle \tag{3}$$

the equation of this eigenstate problem has a tight-binding form, with $H_{\vec{m}-\vec{m}'}$ corresponds to the hopping matrix, and $\vec{m}\cdot\vec{\omega}\hbar$ represents the linear potential. such dynamics is analogous to that of electron motion in superlattice potential, Wannier-Stark ladder, or photon motion under effective magnetic field. While this analogy could have profound implications or potential applications, we focus on its direct consequence on dynamics of Floquet lattice.

The eigen-equation above reveals the effective band structure in the synthetic lattice. While this picture is comparable to metallic balls falling down a tilted grid, the quantum nature of this motion prevents it falling down all the way, but rather turning back periodically, in Wannier-Stark localization/Bloch oscillation, when the drive/hopping is weak. However, only with strong drive does the tight-binding model holds, and this is the regime of interest, where hopping is strong enough crossing through multiple m lattice sites, resulting in interference between wavefunctions and extended Floquet bands.

If this band is topologically trivial, what we have is coupling between Floquet modes periodically, the wave package move back and forth without net transport. However, if the Floquet band is associated with a nontrivial Chern number C, Berry curvature leads to a path-dependent physical result, accumulating net drift from each periodic motion. Specificly,

$$\vec{v} = \nabla_{\vec{k}}\epsilon(\vec{k}) + \vec{E} \times \vec{\Omega}_{\vec{k}} \tag{4}$$

here, the effective force E becomes $\hbar\vec{\omega}$, Berry curvature is $\vec{\Omega}_{\vec{q}}$, and $\vec{v}_{\text{anomalous}} = \hbar\vec{\omega} \times \vec{\Omega}_{\vec{q}}$. this transverse drift velocity is in Floquet space denoted by $(n, \vec{m})$, and reveals transition between different photon-number sectors m, that is, pumping of energy between driving frequency ω.

Assuming the system is initialized in a single topological Floquet band, the long-term average anomalous velocity is given by integrating the Berry curvature over the Floquet Brillouin zone:

$$\langle \vec{v} \rangle = \hbar\,\vec{\omega} \times \frac{1}{(2\pi)^2} \int_{\text{BZ}} \vec{\Omega}_{\vec{q}}\, d^2q = \frac{C}{2\pi}\,\vec{\omega}_1. \tag{5}$$

the resulting energy pumping rate between the drives is then

$$P_{12} = -P_{21} = \sum_i \hbar\omega_i \langle v_i \rangle = \hbar\,\vec{\omega} \cdot \langle \vec{v} \rangle = \frac{C}{2\pi}\hbar\omega_1\omega_2 \tag{6}$$

let's try to verify this conclusion via numeric results.

Consider a spin-1/2 particle driven by two elliptically polarized fields, corresponding to two incommensurate frequencies ω_1 and ω_2. the time-dependent effective magnetic field generated by the drive causes the spin state trajectory a path on the Bloch sphere. For chiral Bernevig-Hughes-Zhang (BHZ) model, proper choose of mass m results in a trajectory fully covering the Bloch sphere, manifesting topological phase. We choose a simplified version, half of the BHZ model, that the tight-binding band structure is topologically nontrivial.

$$\mathcal{H} = v_x \sin(k_1)\sigma_x + v_y \sin(k_2)\sigma_y + \left[m - b_x \cos(k_1) - b_y \cos(k_2)\right]\sigma_z. \tag{7}$$

this Hamiltonian could be mapped to Floquet space by substituting momentum k with time parameter $\omega t + \phi$. Upon further simplification we choose this special form

$$\frac{\mathcal{H}(t)}{\eta} = \sin(\omega_1 t + \phi_1)\sigma_x + \sin(\omega_2 t + \phi_2)\sigma_y + \left[m - \cos(\omega_1 t + \phi_1) - \cos(\omega_2 t + \phi_2)\right]\sigma_z \tag{8}$$

Since the Hamiltonian is explicitly time-dependent and not amenable to direct diagonalization, we employ a Trotterized time evolution approach:

$$U(t + \delta t, t) \approx e^{-iH(t)\,\delta t/\hbar} \approx I - iH(t)\delta t \tag{9}$$

At each time step t, the wavefunction is updated iteratively from an initial spin state. while this state might not be a Floquet eigenstate, the Trotterized scheme can approximate adiabatic evolution along a single Floquet band in the topological regime, thereby enabling robust energy pumping.

To extract energy pumping between the two drives, we compute the instantaneous power from each frequency component and integrate over time to obtain the accumulated work:

$$W_i(t) = \int_0^t P_i(t')\,dt' = \int_0^t \left\langle \psi(t') \middle| \frac{\partial H(t')}{\partial t_i} \middle| \psi(t') \right\rangle dt' \tag{10}$$

We pay specific attention to the long-time transport behavior between the two drives. In the topological regime, this reflects quantized energy pumping associated with a nonzero Berry curvature of the Floquet bands. The net energy transfer can be interpreted as a geometric effect connected to photon-assisted transitions between Floquet modes.

Below figures contain numerical simulation results, showing normalized energy pumping in and outside of topological regimes. These results indicate balanced energy transfer rates for initial state $(|0\rangle + |1\rangle)/2$, where residing in the topological regime ensures linear and robust energy extracting/feeding process.

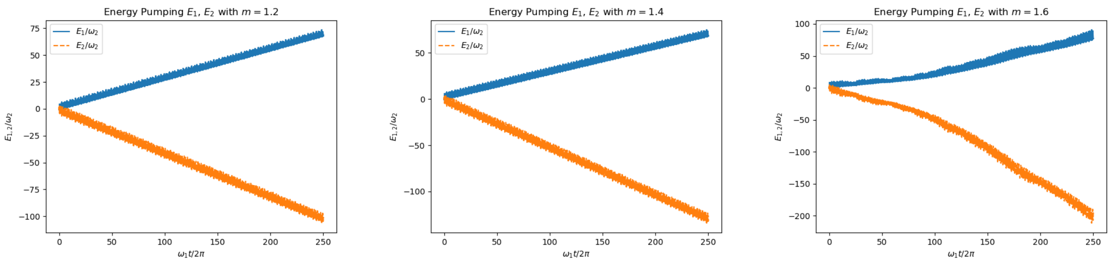

Figure 1: $E_{1,2}/\omega_2$ for $m = 1.2, 1.4, 1.6$

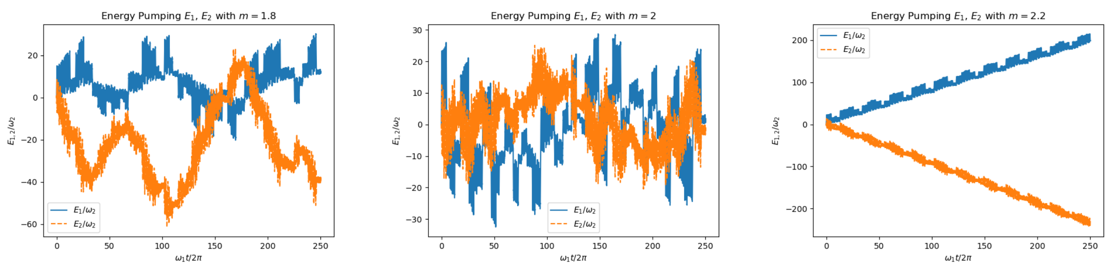

Figure 2: $E_{1,2}/\omega_2$ for $m = 1.8, 2, 2.2$

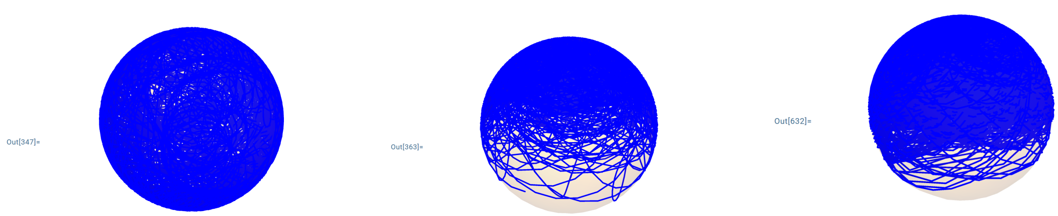

Figure 3: spin trajectory for $m = 0, 1.8, 2.2$