PHYS 364 Pset 1 yuxiang@uchicago.edu

1. (25 pts) A cosmic-ray proton (mass 940 MeV, in natural units where $c = 1$) travels through space at high velocity. If the center-of-mass energy is high enough, it can collide with a cosmic microwave background (CMB) photon (the temperature of the CMB is 2.74K in its overall rest frame) and convert into a proton plus a neutral pion (mass 140 MeV). The pion will then decay into unobserved particles, while the proton will have a lower energy than before the collision. What is the cosmic-ray energy above which we expect this process to occur, and therefore provide a cutoff in the cosmic-ray energy spectrum? (This is known as the Griesen-Zatsepin-Kuzmin, or GZK, cutoff.)

reference: Lecture 5, E. Daw

$$p + \gamma \to n + \pi^+$$

Introduce 4-momentum $\vec{P} = \begin{pmatrix} E/c \\ p \end{pmatrix}$.

$$\vec{P}^2 = -\frac{E^2}{c^2} + p^2 = -m_0^2 c^2.$$

We have $\vec{P}_p + \vec{P}_\gamma = \vec{P}_n + \vec{P}_\pi$.

Threshold: In CoM frame, n and π^+ both at rest.

$$\underbrace{\vec{P}_p^2}_{-m_p^2 c^2} + 2\vec{P}_p \cdot \vec{P}_\gamma + \underbrace{\vec{P}_\gamma^2}_{0} = -(m_n + m_\pi)^2 c^2 \quad (*)$$

$$\vec{P}_p = \begin{pmatrix} E_p/c \\ p_p \end{pmatrix} \sim \begin{pmatrix} E_p/c \\ E_p/c \end{pmatrix} \quad \text{(deep relativistic regime)}$$

$$\vec{P}_\gamma = \begin{pmatrix} E_\gamma/c \\ -E_\gamma/c \end{pmatrix}. \quad \Rightarrow \quad \vec{P}_p \cdot \vec{P}_\gamma = -2\frac{E_p \cdot E_\gamma}{c^2}$$

$$(*) \Rightarrow \quad -m_p^2c^2 - 4\frac{E_p \cdot E_\gamma}{c^2} = -(m_n + m_\pi)^2c^2.$$

$$\Rightarrow \quad E_p = \frac{\left[(m_n+m_\pi)^2 - m_p^2\right]c^4}{4E_\gamma}.$$

In our case, $E_\gamma = k_B T = 1.38 \times 10^{-23} \times 2.74 \text{ J}$

$$= 2.36 \times 10^{-10} \text{ MeV}.$$

$m_\pi = 140$ MeV. $m_p = 940$ MeV. $\approx m_n$.

$$\Rightarrow \quad E_p = \frac{(140+940)^2 - 940^2}{2.36 \times 10^{-10}} \text{ MeV}$$

$$= 1.20 \times 10^{15} \text{ MeV}.$$

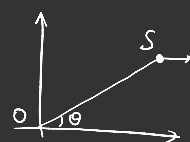
↑ cutoff energy

2. A radar gun measures the speed of passing cars via the Doppler effect, measuring the beat frequency between the emitted microwave radiation of frequency ν and the returning reflected radiation.

(a) (15 pts) Find the beat frequency as a function of the velocity v of the car as it approaches the position of the gun. Take the radar gun to lie on the linear trajectory of the car.

(b) (10 pts) Estimate the accuracy of the radar gun in determining a car's velocity at typical highway speeds (say 30 m/s). Assume that the gun uses microwaves at 10 GHz and takes about a tenth of a second to make its measurement.

(a) Let's derive Doppler effect for light.



$$r^2 = c^2(t^* - t)^2$$

$$\Rightarrow \quad r(v\,dt) = c^2(t^* - t)(dt^* - dt)$$

$$dt^* = \left(1 + \frac{v}{c}\cos\theta\right)dt \qquad\qquad dt_0 = \frac{1}{\sqrt{1-\beta^2}}\,dt$$

$$\Rightarrow \quad \frac{dt^*}{dt_0} = \frac{1 + \frac{v}{c}\cos\theta}{\sqrt{1-\beta^2}} = \frac{\nu_0}{\nu}$$

When $\theta = 0$, $\quad \dfrac{\nu_0}{\nu} = \dfrac{1+\beta}{\sqrt{1-\beta^2}} = \sqrt{\dfrac{1+\beta}{1-\beta}}$

not quite, take into account of reflected wave another transformation

beat frequency $\quad \Delta\nu = \left|\sqrt{\dfrac{1-\beta}{1+\beta}} - 1\right|\nu_0 = \left(1 - \sqrt{\dfrac{c-v}{c+v}}\right)\nu_0$

include reflection: $\quad \Delta\nu = \left|\sqrt{\dfrac{1-\beta}{1+\beta}} - \sqrt{\dfrac{1+\beta}{1-\beta}}\right|\nu_0 = \left(\sqrt{\dfrac{1+\beta}{1-\beta}} - \sqrt{\dfrac{1-\beta}{1+\beta}}\right)\nu_0$

for $\beta = \dfrac{v}{c} \ll 1$, $\quad \Delta\nu \approx (1+\beta - 1+\beta)\,\nu_0 = \dfrac{2v}{c}\nu_0$

ν_{beat}

(b) When $\frac{v}{c} \ll 1$. $\Delta\nu \approx \frac{2v}{c}\nu_0 = \frac{2\times 30}{3\times 10^8}\nu_0 = 2\times 10^{-7}\nu_0 = 2\times 10^3$ Hz.

$\Delta t = 10^{-1}$ s. $\Rightarrow \Delta f = 10$ Hz.

this will affect speed expectation:

$$V_{err} = \frac{c}{2}\frac{\Delta f}{\nu_0} = \frac{1}{2}\times 3\times 10^8 \times \frac{10}{10^9} = 1.5 \text{ m/s}.$$

this is the uncertainty in speed caused

by 0.1 s measurement time.

3. (25 pts) A common phenomenon in particle physics is the scattering of two particles A+B into two new particles C+D. For such events it is convenient to define the *Mandelstam variables*:

$$\begin{aligned} s &= -\eta_{\mu\nu}(p_A^\mu + p_B^\mu)(p_A^\nu + p_B^\nu) \\ t &= -\eta_{\mu\nu}(p_A^\mu - p_C^\mu)(p_A^\nu - p_C^\nu) \\ u &= -\eta_{\mu\nu}(p_A^\mu - p_D^\mu)(p_A^\nu - p_D^\nu) \ , \end{aligned} \tag{1}$$

where p_i^μ are the 4-momenta. The beauty of these variables is that they are all Lorentz-invariant, and so can be evaluated in whatever frame is most convenient.

(a) Show that $s + t + u = m_A^2 + m_B^2 + m_C^2 + m_D^2$.

(b) Express the energy of A in the center-of-mass (CM) frame (in which the spatial components of the total momentum vanish), in terms of the masses and the Mandelstam variables.

(c) Express the energy of A in the lab frame, in which B is at rest.

(d) Express the total energy in the CM frame.

(e) For scattering of identical particles, $A + A \to A + A$, show that in the CM frame we have

$$\begin{aligned} s &= 4(p^2 + m_A^2) \\ t &= -2p^2(1 - \cos\theta) \\ u &= -2p^2(1 + \cos\theta) \ , \end{aligned} \tag{2}$$

where p is the 3-momentum of one of the incident particles, and θ is the scattering angle.

(a) $s = -\eta_{\mu\nu}\left(P_A^\mu P_A^\nu + P_B^\mu P_A^\nu + P_A^\mu P_B^\nu + P_B^\mu P_B^\nu\right)$ $\qquad \eta_{\mu\nu} = \mathrm{diag}(-1,1,1,1)$.

$= -\left(P_A \cdot P_A + 2 P_A \cdot P_B + P_B \cdot P_B\right)$

$= -\left(-E_A^2 + \vec{P}_A^2 - 2E_A \cdot E_B + 2\vec{P}_A \cdot \vec{P}_B - E_B^2 + \vec{P}_B^2\right)$. $\qquad c = 1$.

$t = -\eta_{\mu\nu}\left(P_A^\mu P_A^\nu - P_C^\mu P_A^\nu - P_A^\mu P_C^\nu + P_C^\mu P_C^\nu\right)$

$= -\left(P_A \cdot P_A - 2P_A \cdot P_C + P_C \cdot P_C\right)$

$= -\left(-E_A^2 + \vec{P}_A^2 + 2E_A \cdot E_C - 2\vec{P}_A \cdot \vec{P}_C - E_C^2 + \vec{P}_C^2\right)$.

$u = -\eta_{\mu\nu}\left(P_A^\mu P_A^\nu - P_D^\mu P_A^\nu - P_A^\mu P_D^\nu + P_D^\mu P_D^\nu\right)$

$= -\left(P_A \cdot P_A - 2P_A \cdot P_D + P_D \cdot P_D\right)$

$= -\left(-E_A^2 + \vec{P}_A^2 + 2E_A \cdot E_D - 2\vec{P}_A \cdot \vec{P}_D - E_D^2 + \vec{P}_D^2\right)$

$$\Rightarrow \quad s = (E_A + E_B)^2 - (\vec{p}_A + \vec{p}_B)^2.$$

$$t = (E_A - E_C)^2 - (\vec{p}_A \oplus \vec{p}_C)^2. \qquad \text{minus sign!}$$

$$u = (E_A - E_D)^2 - (\vec{p}_A - \vec{p}_D)^2.$$

Also, $\vec{p}_A + \vec{p}_B = \vec{p}_C + \vec{p}_D$, $\quad E_A + E_B = E_C + E_D$

$$-2E_A(E_A+E_B)$$

$$\Rightarrow s+t+u = E_A^2 + E_B^2 + E_C^2 + E_D^2 + \underbrace{E_A^2 + E_A^2 + 2E_A \cdot E_B}\ \underbrace{-2E_A \cdot E_C - 2E_A \cdot E_D}$$

cancell

$$-\left(\vec{p}_A^{\,2} + \vec{p}_B^{\,2} + 2\vec{p}_A \cdot \vec{p}_B + \vec{p}_A^{\,2} + \vec{p}_C^{\,2} - 2\vec{p}_A \cdot \vec{p}_C + \vec{p}_A^{\,2} + \vec{p}_D^{\,2} - 2\vec{p}_A \cdot \vec{p}_D\right)$$

$$2\vec{p}_A \cdot (\vec{p}_B - \vec{p}_A - \vec{p}_B)$$

$$= E_A^2 + E_B^2 + E_C^2 + E_D^2 - \left(\vec{p}_A^{\,2} + \vec{p}_B^{\,2} + \vec{p}_A^{\,2} + \vec{p}_C^{\,2} + \vec{p}_A^{\,2} + \vec{p}_D^{\,2} - 2\vec{p}_A^{\,2}\right)$$

$$= \sum_i (E_i^2 - \vec{p}_i^{\,2}) = m_A^2 + m_B^2 + m_C^2 + m_D^2.$$

(b) In CM frame, $\sum_i \vec{p}_i = 0$. $\Rightarrow$ $\vec{p}_A + \vec{p}_B = \vec{p}_C + \vec{p}_D = 0$.

$$\Rightarrow \quad s = (E_A + E_B)^2. \qquad E_A + E_B = \sqrt{s}.$$

$$t = (E_D - E_B)^2 - (\vec{p}_A + \vec{p}_D)^2. \qquad s+t+u = \sum_i m_i^2.$$

$$u = (E_A - E_D)^2 - (\vec{p}_A - \vec{p}_D)^2.$$

how to express E_B in m and stu?

$$E_A + E_B = E_C + E_D .$$

$$E_A^2 = \vec{p}_A^{\,2} + m_A^2 .$$

$$\vec{p}_A + \vec{p}_B = 0 .$$

$$E_B^2 = \vec{p}_B^{\,2} + m_B^2 .$$

equal

$$\begin{cases} E_A^2 - m_A^2 = E_B^2 - m_B^2 . \\ E_A + E_B = \sqrt{s} . \end{cases} \Rightarrow E_A^2 - m_A^2 = E_A^2 - 2\sqrt{s}\,E_A + s - m_B^2 .$$

$$\Rightarrow E_A = \frac{m_A^2 - m_B^2 + s}{2\sqrt{s}} .$$

(c) In such lab frame, $\vec{p}_B = 0$. $\vec{p}_A = \vec{p}_C + \vec{p}_D$.

$$s = (E_A + E_B)^2 - \vec{p}_A^{\,2} .$$

$$E_A^2 = p_A^2 + m_A^2 .$$

$$t = (E_A - E_C)^2 - \vec{p}_D^{\,2} .$$

$$E_B^2 = m_B^2 .$$

$$u = (E_A - E_D)^2 - \vec{p}_C^{\,2} .$$

$$t + u = E_A^2 + E_C^2 + E_A^2 + E_D^2 - 2E_A \cdot \underbrace{(E_C + E_D)}_{E_A + E_B} - \vec{p}_C^{\,2} - \vec{p}_D^{\,2} .$$

$$t - u = E_C^2 - E_D^2 - 2E_A \cdot (E_C - E_D) + \vec{p}_C^{\,2} - \vec{p}_D^{\,2} .$$

$$\Rightarrow t + u = -2E_A \cdot E_B + m_C^2 + m_D^2 .$$

$$s = m_A^2 + E_B^2 + 2E_A \cdot E_B . \leftarrow$$ enough to use s!

$$\Rightarrow E_A = \frac{s - m_A^2 - m_B^2}{2m_B} .$$

(d) In CM frame.

$$s = (E_A + E_B)^2.$$

$$t = (E_D - E_B)^2 - (\vec{p}_A + \vec{p}_D)^2.$$

$$u = (E_A - E_D)^2 - (\vec{p}_A - \vec{p}_D)^2.$$

total energy $= E_A + E_B = E_C + E_D.$

$$E_{tot} = \sqrt{s}.$$

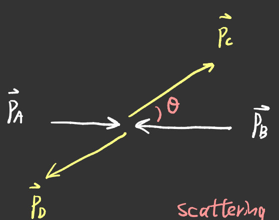
(e) For identical particle. $m_A = m_B = m_C = m_D.$

$|\vec{p}_A| = |\vec{p}_B| = |\vec{p}_C| = |\vec{p}_D| = p.$ $\quad E_A = E_B = E_C = E_D = E.$

$$s = (E_A + E_B)^2 = 4E_A^2 = 4(\vec{p}_A^{\,2} + m_A^2).$$

$$t = (E_A - E_C)^2 - (\vec{p}_A - \vec{p}_C)^2 = (\vec{p}_A - \vec{p}_C)^2.$$

$$u = (E_A - E_D)^2 - (\vec{p}_A - \vec{p}_D)^2 = -(\vec{p}_A - \vec{p}_D)^2.$$



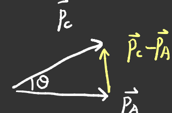
scattering angle θ

$$\Rightarrow |\vec{p}_A - \vec{p}_C| = 2p\sin\frac{\theta}{2}.$$

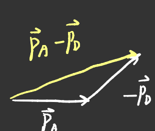
$$\Rightarrow \quad s = 4(p^2 + m_A^2).$$

$$t = -4p^2\sin^2\frac{\theta}{2} = -2p^2(1-\cos\theta).$$

$$u = -4p^2\cos^2\frac{\theta}{2} = -2p^2(1+\cos\theta).$$

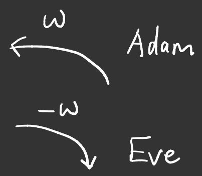


Δ

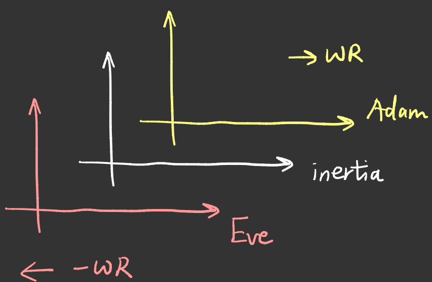


$$|\vec{p}_A - \vec{p}_D| = 2p\cos\frac{\theta}{2}$$

4. (25 pts) Adam and Eve are undergoing uniform circular motion about the $\hat{\mathbf{z}}$ axis at angular velocity ω and $-\omega$ respectively (*i.e.* in opposite directions). At the moment they pass, Eve sees Adam's clock running more slowly, and so she expects that his elapsed time will be smaller the next time they meet. Adam expects the opposite. What really happens? Explain by relating their proper time intervals $d\tau_A$, $d\tau_E$ to the time interval dt of the inertial frame at rest with respect to their axis of rotation.



In each's frame, they see the clock on the other side slower.



$$d\tau_A = \sqrt{1-\left(\frac{\omega R}{c}\right)^2}\, dt.$$

$$d\tau_E = \sqrt{1-\left(\frac{\omega R}{c}\right)^2}\, dt.$$

they should have same time dilation effect.

However, if we move to Adam's frame.

$$dt = \sqrt{1-\left(\frac{\omega R}{c}\right)^2}\, d\tau_A.$$

$$v_{EA} = \frac{\omega R + \omega R}{1+\left(\frac{\omega R}{c}\right)^2}. \qquad d\tau_E = \sqrt{1-\left(\frac{v_{EA}}{c}\right)^2}\, d\tau_A.$$

this relative speed make τ_E seems slower than τ_A.

But seen from inertia frame. they have equal time dilation effect.

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1. (a) (10 pts) Show that the Lorentz (*a.k.a.* Minkowski) metric $\eta_{\alpha\beta}$ is invariant under Lorentz transformations

$$\Lambda^{\alpha'}_{\alpha}\eta_{\alpha'\beta'}\Lambda^{\beta'}_{\beta} = \eta_{\alpha\beta}$$

Consider both boost transformations and rotations. It is sufficient to prove the result for boosts and rotations around the three spatial coordinate axes, since a sequence of such transformations is sufficient to generate the arbitrary Lorentz transsformation

(b) (15 pts) The electromagnetic field strengths $\mathbf{E}, \mathbf{B}$ package into an antisymmetric tensor of type $(0, 2)$

$$F_{\alpha\beta} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{pmatrix} = \partial_\alpha A_\beta - \partial_\beta A_\alpha$$

where $A_\alpha = (-\Phi, \mathbf{A})$ is the four-vector comprising the electrostatic potential Φ and the vector potential $\mathbf{A}$. Use these properties to derive the Lorentz transformation laws of the electric and magnetic fields $\mathbf{E}$ and $\mathbf{B}$.

(a) Lorentz boost along x:

$$\Lambda_{\text{boost}_x} = \begin{pmatrix} \gamma & -\beta\gamma & & \\ -\beta\gamma & \gamma & & \\ & & 1 & \\ & & & 1 \end{pmatrix}.$$

show $\Lambda^{\alpha'}_{\alpha}\eta_{\alpha'\beta'}\Lambda^{\beta'}_{\beta} = \eta_{\alpha\beta}$.

$$\Rightarrow \begin{pmatrix} \gamma & -\beta\gamma & & \\ -\beta\gamma & \gamma & & \\ & & 1 & \\ & & & 1 \end{pmatrix}^T \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \begin{pmatrix} \gamma & -\beta\gamma & & \\ -\beta\gamma & \gamma & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \gamma & -\beta\gamma & & \\ -\beta\gamma & \gamma & & \\ & & 1 & \\ & & & 1 \end{pmatrix}^T \begin{pmatrix} -\gamma & \beta\gamma & & \\ -\beta\gamma & \gamma & & \\ & & 1 & \\ & & & 1 \end{pmatrix} = \begin{pmatrix} \underbrace{-\gamma^2(1-\beta^2)}_{=-1} & & & \\ & \underbrace{\gamma^2(1-\beta^2)}_{=1} & & \\ & & 1 & \\ & & & 1 \end{pmatrix}.$$

equation hold.

then consider Lorentz rotation along z:

$$\Lambda_{\text{rotation\_z}} = \begin{pmatrix} 1 & & & \\ & \cos\alpha & \sin\alpha & \\ & -\sin\alpha & \cos\alpha & \\ & & & 1 \end{pmatrix}. \qquad \text{show } \Lambda^{\alpha'}{}_{\alpha}\, \eta_{\alpha'\beta'}\, \Lambda^{\beta'}{}_{\beta} = \eta_{\alpha\beta}.$$

$$\begin{pmatrix} 1 & & & \\ & \cos\alpha & \sin\alpha & \\ & -\sin\alpha & \cos\alpha & \\ & & & 1 \end{pmatrix}^{T} \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \begin{pmatrix} 1 & & & \\ & \cos\alpha & \sin\alpha & \\ & -\sin\alpha & \cos\alpha & \\ & & & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & & & \\ & \cos\alpha & -\sin\alpha & \\ & \sin\alpha & \cos\alpha & \\ & & & 1 \end{pmatrix} \begin{pmatrix} -1 & & & \\ & \cos\alpha & \sin\alpha & \\ & -\sin\alpha & \cos\alpha & \\ & & & 1 \end{pmatrix} = \begin{pmatrix} -1 & & & \\ & \underbrace{\cos^2\alpha + \sin^2\alpha}_{=1} & & \\ & & \underbrace{\sin^2\alpha + \cos^2\alpha}_{=1} & \\ & & & 1 \end{pmatrix}.$$

equation hold.

(b) Derive Lorentz transformation for $\vec{E}$ and $\vec{B}$.

$F_{\alpha\beta}$ satisfies $\quad F'_{\mu\nu} = \Lambda^{\alpha}{}_{\mu}\, \Lambda^{\beta}{}_{\nu}\, F_{\alpha\beta}.$

Apply Lorentz boost in x direction to F, we have

$$F' = \begin{pmatrix} \gamma & -\beta\gamma & & \\ -\beta\gamma & \gamma & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \begin{pmatrix} & -E_x & -E_y & -E_z \\ E_x & & B_z & -B_y \\ E_y & -B_z & & B_x \\ E_z & B_y & -B_x & \end{pmatrix} \begin{pmatrix} \gamma & -\beta\gamma & & \\ -\beta\gamma & \gamma & & \\ & & 1 & \\ & & & 1 \end{pmatrix}^{T}$$

$$= \begin{pmatrix} \gamma & -\beta\gamma & & \\ -\beta\gamma & \gamma & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \begin{pmatrix} \beta\gamma E_x & -\gamma E_x & -E_y & -E_z \\ \gamma E_x & -\beta\gamma E_x & B_z & -B_y \\ \gamma E_y + \beta\gamma B_z & -\beta\gamma E_y - \gamma B_z & & B_x \\ \gamma E_z - \beta\gamma B_y & -\beta\gamma E_z + \gamma B_y & -B_x & \end{pmatrix}$$

$$= \begin{pmatrix} & -(1-\beta^2)\gamma^2 E_x & -\gamma E_y - \beta\gamma B_z & -\gamma E_z + \beta\gamma B_y \\ (1-\beta^2)\gamma^2 E_x & & \beta\gamma E_y + \gamma B_z & \beta\gamma E_z - \gamma B_y \\ \gamma E_y + \beta\gamma B_z & -\beta\gamma E_y - \gamma B_z & & B_x \\ \gamma E_z - \beta\gamma B_y & -\beta\gamma E_z + \gamma B_y & -B_x & \end{pmatrix}$$

Thus, under x-Lorentz boost, we have

$$E_x' = E_x, \qquad B_x' = B_x$$

$$E_y' = \gamma(E_y + \beta B_z) \qquad B_y' = \gamma(B_y - \beta E_z)$$

$$E_z' = \gamma(E_z - \beta B_y). \qquad B_z' = \gamma(B_z + \beta E_y).$$

We could try to write similar transformation for Lorentz boost under different direction, but let's try to use the general form:

Λ along $\vec{v} = (v_x, v_y, v_z)$ is

(below derivation is based on reference to online materials)

$$\Lambda_{\vec{v}} = \begin{pmatrix} \gamma & -\gamma\beta_x & -\gamma\beta_y & -\gamma\beta_z \\ -\gamma\beta_x & 1+(\gamma-1)\frac{v_x^2}{v^2} & (\gamma-1)\frac{v_x v_y}{v^2} & (\gamma-1)\frac{v_x v_z}{v^2} \\ -\gamma\beta_y & (\gamma-1)\frac{v_x v_y}{v^2} & 1+(\gamma-1)\frac{v_y^2}{v^2} & (\gamma-1)\frac{v_y v_z}{v^2} \\ -\gamma\beta_z & (\gamma-1)\frac{v_x v_z}{v^2} & (\gamma-1)\frac{v_x v_y}{v^2} & 1+(\gamma-1)\frac{v_z^2}{v^2} \end{pmatrix}.$$

E field: $\quad F_{0i}' = \sum_{\alpha,\beta} \Lambda_0^{\alpha} \Lambda_i^{\beta} F_{\alpha\beta}.$

$$\Lambda_0^{\alpha} F_{\alpha i} = \gamma \underbrace{F_{0i}}_{-E_i} - \gamma \frac{v_j}{c} \underbrace{F_{ji}}_{\varepsilon_{ijk} B_k}. \quad \star$$

$$\Rightarrow E_i' = \gamma\left(E_i + (\vec{v}\times\vec{B})_i\right) - \frac{\gamma^2}{\gamma+1}(\vec{E}\cdot\vec{v})\frac{v_i}{v^2}.$$

$$\Rightarrow \quad \vec{E}' = \gamma(\vec{E} + \vec{v}\times\vec{B}) - \frac{\gamma^2}{\gamma+1}\frac{(\vec{E}\cdot\vec{v})\,\vec{v}}{v^2}.$$

B field : $F'_{ij} = \sum_{\alpha,\beta} \Lambda^{\alpha}_{i} \Lambda^{\beta}_{j} F_{\alpha\beta}$

$$\Rightarrow \quad B'_i = \gamma\left(B_i - \frac{1}{c^2}(\vec{v}\times\vec{E})_i\right) - \frac{\gamma^2}{\gamma+1}(\vec{B}\cdot\vec{v})\frac{v_i}{v^2}.$$

Overall.

$$\vec{E}' = \gamma(\vec{E} + \vec{v}\times\vec{B}) - \frac{\gamma^2}{\gamma+1}\frac{(\vec{E}\cdot\vec{v})\vec{v}}{v^2}.$$

$$\vec{B}' = \gamma\left(\vec{B} - \frac{\vec{v}\times\vec{E}}{c^2}\right) - \frac{\gamma^2}{\gamma+1}\frac{(\vec{B}\cdot\vec{v})\vec{v}}{v^2}.$$

* supplement for ☆ : $F'_{0i} = \Lambda^{0}_{0}\Lambda^{0}_{i}\underbrace{F_{00}}_{=0} + \Lambda^{0}_{0}\Lambda^{j}_{i}\underbrace{F_{0j}}_{-E_j} + \Lambda^{k}_{0}\Lambda^{0}_{i}F_{k0} + \Lambda^{k}_{0}\Lambda^{j}_{i}F_{kj}$

$$= \gamma\left(1 + (\gamma-1)\frac{v_i v_j}{v^2}\right)\cdot(-E_j) - \left(-\gamma\frac{v_k}{c}\right)\left(-\gamma\frac{v_i}{c}\right)E_k + \left(-\gamma\frac{v_k}{c}\right)\left(\delta^{j}_{i} + \frac{\gamma-1}{v^2}v_j v_i\right)\varepsilon_{kjm}B_m$$

$$= -\gamma E_i - \frac{\gamma}{c}(\vec{v}\times\vec{B})_i + \frac{\gamma^2}{c^2}(\vec{E}\cdot\vec{v})v_i.$$

2. (25 pts) Consider the action

$$\mathcal{S} = -m\int d\lambda\sqrt{-\frac{dx^\alpha}{d\lambda}\eta_{\alpha\beta}\frac{dx^\beta}{d\lambda}} + q\int d\lambda\, A_\alpha(x(\lambda))\frac{dx^\alpha}{d\lambda}$$

for a particle moving along a worldline $x^\alpha(\lambda)$ in inertial coordinates x^α. Show that this action is invariant under (i) worldline reparametrizations $\lambda \to \xi(\lambda)$, and (ii) "gauge transformations" $A_\alpha \to A_\alpha + \partial_\alpha\Lambda$ for any smooth function $\Lambda(x)$ which vanishes at infinity. Show that the variational principle for the above action leads to the equation of motion for a relativistic charged particle of charge q and mass m moving under the influence of the electromagnetic Lorentz force *(Hint: Choose proper time for the parametrization of the worldline.)*

(i) $\lambda \to \xi(\lambda)$. $\quad \frac{dM}{d\lambda} = \frac{dM}{d\xi}\frac{d\xi}{d\lambda} = \xi'(\lambda)\frac{dM}{d\xi}$.

$$S \to -m\int \underbrace{d\lambda\, \xi'(\lambda)}_{=d\xi}\sqrt{\frac{-dx^\alpha}{d\xi}\eta_{\alpha\beta}\frac{dx^\beta}{d\xi}} + q\int d\lambda\, A_\alpha(x(\xi))\,\xi'(\lambda)\frac{dx^\alpha}{d\xi}$$

$$= -m\int d\xi\sqrt{\frac{-dx^\alpha}{d\xi}\eta_{\alpha\beta}\frac{dx^\beta}{d\xi}} + q\int d\xi\, A_\alpha(x(\xi))\frac{dx^\alpha}{d\xi}.$$

Compare to original action, we know it is invariant.

(ii) $A_\alpha \longrightarrow A_\alpha + \partial_\alpha\Lambda$. $\quad \Lambda$ smooth, $\to 0$ at infinity.

focus on $q\int d\lambda\, A_\alpha(x(\lambda))\frac{dx^\alpha}{d\lambda}$ term.

$$\longrightarrow \quad q\int d\lambda\left(A_\alpha(x(\lambda))\frac{dx^\alpha}{d\lambda} + \partial_\alpha\Lambda\frac{dx^\alpha}{d\lambda}\right)$$

$$= q\int d\lambda\, A_\alpha(x(\lambda))\frac{dx^\alpha}{d\lambda} + \underline{q\int dx^\alpha\,\partial_\alpha\Lambda}. \qquad = q\Lambda\Big|_{-\infty}^{+\infty} = 0.$$

We know this action is invariant under this "gauge transformation"

Let's apply $\delta S = 0$ and find eqn of motion.

$$x^\alpha \longrightarrow x^\alpha + \delta x^\alpha.$$

$$\delta\sqrt{-\eta_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda}} = \frac{\delta\left(\underline{-\eta_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda}}\right)}{2\sqrt{-\eta_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda}}} \quad \longrightarrow \quad -2\eta_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{d(\delta x^\beta)}{d\lambda}$$

$$\Rightarrow \quad \delta S = m\int d\lambda \frac{\eta_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{d(\delta x^\beta)}{d\lambda}}{\underbrace{\sqrt{-\eta_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{d(\delta x^\nu)}{d\lambda}}}_{\text{metric of path}}} + q\int d\lambda\left(\frac{\partial A_\alpha}{\partial x^\beta}\delta x^\beta + A_\alpha\frac{d}{d\lambda}(\delta x^\alpha)\right)$$

$$= m\int\frac{1}{c}d\lambda\,\cancel{\eta_{\alpha\beta}}\frac{dx^\alpha}{d\lambda}\delta\left(\frac{dx^\beta}{d\lambda}\right) + q\int d\lambda\frac{\partial A_\alpha}{\partial x^\beta}\frac{dx^\alpha}{d\lambda}\delta x^\beta + \underbrace{q\left[A_\alpha\delta x^\alpha\right]}_{=0} - q\int dA_\alpha\,\delta x^\alpha$$

$$= m\int d\lambda\frac{d}{d\lambda}\left(\frac{1}{c}\eta_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\right)\delta x^\beta + q\int d\lambda\left(\frac{\partial A_\alpha}{\partial x^\beta}\frac{dx^\alpha}{d\lambda}\delta x^\beta - \frac{dA_\alpha}{d\lambda}\delta x^\beta\right)$$

$$= \int d\lambda\left[\underbrace{\frac{d}{d\lambda}\left(\frac{m}{c}\eta_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\right)}_{m\frac{d^2x^\alpha}{d\lambda^2}} + q\underbrace{\left(\frac{\partial A_\alpha}{\partial x^\beta} - \frac{\partial A_\beta}{\partial x^\alpha}\right)}_{F_{\beta\alpha}}\frac{dx^\alpha}{d\lambda}\right]\delta x^\beta.$$

$\longrightarrow$ this step is tricky — choose proper time? might help?

We have

$$\delta S = \int d\lambda \left(\underline{m \frac{d^2 x^\alpha}{d\lambda^2} + q F_{\beta\alpha} \frac{dx^\alpha}{d\lambda}} \right) \delta x^\beta$$

$$\frac{d^2 x^\alpha}{d\lambda^2} + \frac{q}{m} F_{\beta\alpha} \frac{dx^\alpha}{d\lambda} = 0 .$$ ☆

this describes motion of relativistic charged particle moving under EM Lorentz force.

should read again on covariance derivative to make above derivation clearer.

3. (25 pts)

(a) Explain why a the density ρ of a conserved quantity and its flux $\mathbf{j}$ transform as the time and space components of a 4-vector $\underline{\mathrm{J}}$. Show that conservation of this quantity amounts to the continuity equation $\partial_\alpha J^\alpha = 0$, by considering the integral of this equation over a spatial volume $\mathcal{V}$ and interpreting $\int_{\mathcal{V}} \rho$ as the amount of the conserved quantity contained in $\mathcal{V}$ and the remaining terms as constituting the flux of that quantity into or out of $\mathcal{V}$.

(b) The energy and momentum of a free particle transform as a four-vector, and each is conserved separately in any given Lorentz frame, $dp_\alpha/d\tau = 0$. Therefore there must be a four-vectors' worth of four-vectors – the energy-momentum currents – consisting of the energy, the energy flux, the momentum, and the momentum flux. Find these quantities for a free relativistic particle of mass m, and show that they form a symmetric two-tensor – the *energy-momentum tensor* – of type (0,2) with components $T_{\alpha\beta}$ in an inertial frame.

(c) Show that the energy momentum tensor you found in part (b) satisfies the continuity equation, as a consequence of the particle's equation of motion.

(a) $J^\mu = \rho U^\mu = (c\rho, \vec{j})$, is a 4-vector.

As verified by $\eta^{\mu\nu} J^\mu J^\nu = c^2\rho^2 - \vec{j}^{\,2} = c^2 \rho_0^2$.

Under Lorentz transformation along x.

$J' \to \left(\gamma(c\rho - \beta j_x), \gamma(j_x - \beta c\rho), j_y, j_z\right)$.

this reveals how ρ and $\vec{j}$ transforms correspondingly.

not quite, should be based on the "conserved quantity". ?

the flux $\vec{j}$ should be proportional to $\rho\vec{v}$. it is, $\rho \frac{dx^i}{d\tau}$.

$Q = \int_V \rho_0 \, dV$ conserved. $\quad \int_V \left(\frac{\partial}{\partial t}\rho + \nabla\cdot\vec{j}\right) dV = 0$.

✗

?

$\therefore \quad \frac{\partial}{\partial t}\int_V \rho \, dV + \int_{\partial V} \vec{j}\cdot d\vec{S} = 0 . \qquad \int_{\partial V} \vec{p}\cdot d\vec{S} = \int_V \nabla\cdot\vec{p}\, dV$

$\int_V \underbrace{\left(\frac{\partial}{\partial t}\rho + \nabla\cdot\vec{j}\right)} dV = 0 .$ As we showed $J^\mu = \rho U^\mu$.

$$U^\mu = \left(\frac{dt}{d\tau}, \frac{dx^i}{d\tau}\right) = \gamma(c, v^i)$$

$$\partial_\alpha J^\alpha = \partial_\alpha\left(\gamma\rho(c, v^i)\right) = \gamma\frac{\partial}{\partial t}(\rho c) + \gamma\frac{\partial}{\partial x_i}(\rho v^i)$$

☆

$$= \gamma\left(\frac{\partial}{\partial t}(\rho c) + \nabla\cdot(\rho\vec{v})\right).$$

Let's integrate this quantity.

$$\Rightarrow \quad \gamma\int_V c\frac{\partial}{\partial t}\rho \, dV + \gamma\int_V \nabla\cdot(\rho\vec{v})\, dV$$

$$= \gamma\left(\frac{\partial}{\partial t}\int_V \rho\, dV + \int_V \nabla\cdot(\rho\vec{v})\, dV\right) = 0. \quad \text{conserved.}$$

Q in V $\qquad$ Q into/out of V.

Thus $\partial_\alpha J^\alpha = 0$.

(b) $P_\mu = \left(-\frac{E}{c}, P_i\right). \qquad \frac{dP_\alpha}{d\tau} = 0 .$

Let's find energy-momentum current. $\qquad$ (reference : Landau, Field Theory).

As we showed, $\partial_\alpha A^\alpha = 0 \iff \int A^\alpha dS_\alpha = \text{const}$.

(A^α is 4-vector)

Thus $\partial_\alpha T_i^{\ \alpha} = 0 \iff \int T_i^{\ \alpha} dS_\alpha = \text{const}. \sim p^i$.

we set $p^i = \frac{1}{c}\int T^{i\alpha} dS_\alpha$.

for integration on $x^0 = \text{const}$, $p^i = \frac{1}{c}\int T^{i0} dV$.

T^{00} : energy density. $\frac{1}{c}T^{i0}$, momentum density.

for other components, write

$$\frac{1}{c}\frac{\partial T^{00}}{\partial t} + \frac{\partial T^{0\alpha}}{\partial x^\alpha} = 0$$

$$\frac{1}{c}\frac{\partial T^{\alpha 0}}{\partial t} + \frac{\partial T^{\alpha\beta}}{\partial x^\beta} = 0.$$

compare to $\frac{\partial \rho}{\partial t} + \nabla\cdot\vec{j} = 0$ we know

$T^{0\alpha}$: energy current.

$T^{\alpha\beta}$: momentum current.

energy density W

$$\text{Overall,}\quad T^{i\alpha} = \begin{pmatrix} \boxed{T_{00}} & \boxed{T_{01} \quad T_{02} \quad T_{03}} \\ \boxed{\begin{matrix} T_{10} \\ T_{20} \\ T_{30} \end{matrix}} & T_{\alpha\beta} \end{pmatrix}.$$

energy current S_i

momentum current σ_{ij}

(c) Verify $\partial_\alpha T^{i\alpha} = 0$.

$\partial_\alpha T^{0\alpha} = 0$. energy conservation.

$\partial_\alpha T^{i\alpha} = 0$. momentum conservation.

$$\partial_\alpha T^{0\alpha} = \partial_t T^{00} + \nabla\cdot(T^{01}, T^{02}, T^{03})$$

$$= \partial_t W + \nabla \cdot \vec{S} = 0.$$

$$\partial_\alpha T^{i\alpha} = \partial_t T^{i0} + \nabla \cdot \left(T^{i1}, T^{i2}, T^{i3}\right)$$

$$= \frac{1}{c} \partial_t S_i + \nabla \cdot \sigma_{ij} = 0.$$

Overall, $\partial_\alpha T^{i\alpha} = 0$. satisfies continuity eqn.

4. (25 pts) While the description of physical processes might be most convenient in Cartesian inertial frames, one can in principle use other frames of reference. For instance, you are currently in an accelerated frame that is rotating with the angular velocity of the earth's rotation. Consider a free particle described in the rotating frame

$$(t, x, y, z) = (t', x' \cos \omega t' + y' \sin \omega t', y' \cos \omega t' - x' \sin \omega t', z') .$$

Expand the Minkowski interval ds^2 in the new coordinate system and evaluate the action for a relativistic (uncharged) free particle.

Vary this action to obtain equations of motion of the form

$$\ddot{x}^{\alpha'} + \Gamma^{\alpha'}_{\beta'\gamma'} \dot{x}^{\beta'} \dot{x}^{\gamma'} = 0$$

Here the overdots denote derivatives with respect to the parameter λ of the particle trajectory. The second term is often moved to the RHS of the equation to look more like Newton's equations of motion, where it is interpreted as a "fictitious force" due to the acceleration. According to Einstein's equivalence principle, this force is no more fictitious than gravity! Evaluate $\Gamma^{\alpha'}_{\beta'\gamma'}$ and dentify the terms in your expression that are interpreted as the centrifugal force and the Coriolis force.

$t = t'$. $\qquad$ $t' = t$.

$x = x' \cos wt' + y' \sin wt'$. $\Rightarrow$ $x' = x \cos wt - y \sin wt$.

$y = y' \cos wt' - x' \sin wt'$. $\qquad$ $y' = x \sin wt + y \cos wt$.

$z = z'$. $\qquad$ $z' = z$.

actually no need ?

yes.

$$dx = -wx' \sin wt' \, dt' + wy' \cos wt' \, dt' + dx' \cos wt' + dy' \sin wt'.$$

$$dy = -wy' \sin wt' \, dt' - wx' \cos wt' \, dt' + dy' \cos wt' - dx' \sin wt'.$$

$$dx^2 + dy^2 = w^2 (x'^2 + y'^2) \, dt'^2 + dx'^2 + dy'^2$$

$$+ 2w(-x' \sin wt' + y' \cos wt') \, dt' \cdot (dx' \cos wt + dy' \sin wt')$$

$$+ 2w(-y' \sin wt' - x' \cos wt') \, dt' \cdot (dy' \cos wt' - dx' \sin wt)$$

$$= w^2 (x'^2 + y'^2) \, dt'^2 + dx'^2 + dy'^2$$

$$+ 2wy' \, dt' dx' - 2wx' \, dt' dy'.$$

$$\Rightarrow \quad ds^2 = \eta_{\alpha\beta}\, dx^\alpha dx^\beta = -dt'^2 + \omega^2(x'^2+y'^2)\,dt'^2 + dx'^2 + dy'^2 + dz'^2$$

$$+ 2\omega y' dt' dx' - 2\omega x' dt' dy' .$$

$$S = -m\int d\lambda \sqrt{-\frac{dx^\alpha}{d\lambda}\eta_{\alpha\beta}\frac{dx^\beta}{d\lambda}}$$ we do not have to parameterize

Use $\quad S = -m\int ds .$

$$= -m\int \left[\left(\omega^2(x'^2+y'^2)-1\right) + \left(\frac{dx'}{dt}\right)^2 + \left(\frac{dy'}{dt}\right)^2 + \left(\frac{dz'}{dt}\right)^2 + 2\omega\left(y'\frac{dx'}{dt} - x'\frac{dy'}{dt}\right)\right]^{\frac{1}{2}} dt .$$

Variation : consider $\mathcal{L}$ inside the square root.

$S = \int \sqrt{f}\, dt .$

$$\delta S = \int \left(\frac{\partial\sqrt{f}}{\partial x^\mu} - \frac{d}{dt}\frac{\partial}{\partial \dot{x}^\mu}\sqrt{f}\right)\delta x^\mu +$$

Cancell $\quad \frac{1}{2\sqrt{f}}\frac{\partial f}{\partial x^\mu}$

$$\mathcal{L} = \omega^2(x'^2+y'^2) - 1 + \dot{x}'^2 + \dot{y}'^2 + \dot{z}'^2 + 2\omega(y'\dot{x}' - x'\dot{y}') .$$

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}} - \frac{\partial \mathcal{L}}{\partial q} = 0 . \quad \Rightarrow \quad \frac{d}{dt}\left(2\dot{x}' + 2\omega y'\right) - 2\omega^2 x' - 2\omega\dot{y} = 0 .$$

$$\Rightarrow \quad \ddot{x}' - \omega^2 x' = 0 .$$

similarly. $\quad \ddot{y} - \omega^2 y = 0 . \quad$ and $\quad \ddot{z} = 0 .$

the Coriolis force cancell out ?

The form from online reference :

$$\ddot{t} + \Gamma^{t}_{tt}\,\dot{t}^2 + \Gamma^{t}_{tx}\,\dot{t}\dot{x} + \Gamma^{t}_{ty}\,\dot{t}\dot{y} = 0 .$$

$$\ddot{x} + \Gamma^{x}_{tt}\,\dot{t}^2 + \Gamma^{x}_{tx}\,\dot{t}\dot{x} + \Gamma^{x}_{ty}\,\dot{t}\dot{y} = 0 .$$

$$\ddot{y} + \Gamma^{y}_{tt}\,\dot{t}^2 + \Gamma^{y}_{tx}\,\dot{t}\dot{y} + \Gamma^{y}_{ty}\,\dot{t}\dot{y} = 0 .$$

$$\ddot{z} = 0 .$$

$$\Rightarrow \quad \ddot{x}^{\alpha'} + \Gamma^{\alpha'}_{\beta'\gamma'}\,\dot{x}^{\beta'}\dot{x}^{\gamma'} = 0 .$$

PHYS 364 Pset 3 yuxiang@uchicago.edu

1. (25 pts) (a) Consider the unit sphere in Cartesian coordinates in $\mathbb{R}^3$:

$$S^2 \equiv \left\{(x^1, x^2, x^3) : (x^1)^2 + (x^2)^2 + (x^3)^2 = 1\right\}. \tag{1}$$

One can introduce "stereographic" coordinate systems defined on $\mathbb{S}^2 \backslash N$ and $\mathbb{S}^2 \backslash S$ (*i.e.* the sphere minus one or the other pole) where N and S are the North and South poles (at $x^3 = \pm 1$):

$$(y^1, y^2) \equiv \left(\frac{2\,x^1}{(1-x^3)}, \frac{2\,x^2}{(1-x^3)}\right), \qquad (z^1, z^2) \equiv \left(\frac{2\,x^1}{(1+x^3)}, \frac{2\,x^2}{(1+x^3)}\right). \tag{2}$$

There is also the natural chart of polar coordinates in which

$$(x^1, x^2, x^3) = (\sin\theta\,\cos\phi\,, \sin\theta\,\sin\phi\,, \cos\theta)\,, \tag{3}$$

for $0 < \theta < \pi$, $-\pi < \phi < \pi$.

(i) Compute, and simplify $y^\mu(\theta,\phi)$ and $z^\mu(\theta,\phi)$. (**Comment:** *You might find it easier to parametrize everything using* $\frac{\theta}{2}$ *rather than* θ.)

(ii) Suppose $W^\mu = (0,1)$ are the components of a vector in the (y^1, y^2) system in the coordinate basis. What are the components in the (θ,ϕ) coordinate system? Suppose $V_\mu = (0,1)$ are the components of a dual vector (a.k.a. covector) in the (y^1, y^2) system in the coordinate basis. What are the components in the (θ,ϕ) system?

(iii) Now define a Maxwell field strength on the sphere via

$$F_{\theta\phi} = m\,\sin\theta\,. \tag{4}$$

Find the integral of F over the two-sphere in these standard spherical coordinates by performing the integral

$$\int F_{\theta\phi}\,d\theta\wedge d\phi\,. \tag{5}$$

Comment: *This integral arises as the integral over a Gaussian sphere that determines the charge on a magnetic monopole.*

(i) $$y^\mu(\theta,\phi) = (y^1, y^2) = \left(\frac{2x^1}{1-x^3}, \frac{2x^2}{1-x^3}\right) = \left(\frac{2\sin\theta\cos\phi}{1-\cos\theta}, \frac{2\sin\theta\sin\phi}{1-\cos\theta}\right)$$

$$= \left(\frac{2\cdot 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\cos\phi}{2\sin^2\frac{\theta}{2}}, \frac{2\cdot 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\sin\phi}{2\sin^2\frac{\theta}{2}}\right) = \left(\frac{\cos\phi}{\tan\frac{\theta}{2}}, \frac{\sin\phi}{\tan\frac{\theta}{2}}\right).$$

$$z^\mu(\theta,\phi) = (z^1, z^2) = \left(\frac{2\sin\theta\cos\phi}{1+\cos\theta}, \frac{2\sin\theta\sin\phi}{1+\cos\theta}\right) = \left(\tan\frac{\theta}{2}\cos\phi, \tan\frac{\theta}{2}\sin\phi\right).$$

(ii) General coordinate transformation:

$$d\tilde{x}^\mu = \frac{\partial \tilde{x}^\mu}{\partial x^\nu} dx^\nu .$$

vector transformation:

$$\tilde{T}^\mu(\tilde{x}) = \frac{\partial \tilde{x}^\mu}{\partial x^\nu} T^\nu(x), \qquad \tilde{T}_\mu(\tilde{x}) = \frac{\partial x^\nu}{\partial \tilde{x}^\mu} T_\nu(x).$$

here $W^\nu = \frac{\partial x^\nu}{\partial y^\mu} W^\mu$. $\qquad W^{y^1} = 0, \quad W^{y^2} = 1. \qquad W^\theta, W^\phi = ?$

$$W^\theta = \frac{\partial \theta}{\partial y^1} W^1 + \frac{\partial \theta}{\partial y^2} W^2 = \frac{\partial \theta}{\partial \left(\frac{\sin\phi}{\tan\frac{\theta}{2}}\right)} = -\frac{2\sin^2\frac{\theta}{2}}{\sin\phi}.$$

$$\sin\phi \left(\cot\frac{\theta}{2}\right)' = \sin\phi \cdot \frac{-1}{2} \frac{1}{\sin^2\frac{\theta}{2}}$$

$$W^\phi = \frac{\partial \phi}{\partial y^1} W^1 + \frac{\partial \phi}{\partial y^2} W^2 = \left(\cos\phi \cdot \frac{1}{\tan\frac{\theta}{2}}\right)'^{-1} = \frac{\tan\frac{\theta}{2}}{\cos\phi}.$$

$$V_\nu = \frac{\partial y^\mu}{\partial x^\nu} V_\mu .$$

$$V_\theta = \frac{\partial y^1}{\partial \theta} V_1 + \frac{\partial y^2}{\partial \theta} V_2 = \frac{\partial y^2}{\partial \theta} = -\frac{\sin\phi}{2\sin^2\frac{\theta}{2}}.$$

$$V_\phi = \frac{\partial y^1}{\partial \phi} V_1 + \frac{\partial y^2}{\partial \phi} V_2 = \frac{\partial y^2}{\partial \phi} = \frac{\cos\phi}{\tan\frac{\theta}{2}}.$$

(iii) $F_{\theta\phi}\, d\theta \wedge d\phi = \iint m \sin\theta \, d\theta \, d\phi$

$$= 2\pi \int_0^\pi m \sin\theta \, d\theta = 4\pi m.$$

(iv) Determine F in both sets of stereographic coordinates $y^\mu(\theta,\phi)$ and $z^\mu(\theta,\phi)$ defined above. Locally, F can be written in any given coordinate patch as the skew-derivative of a one-form potential, $F = dA$ (*i.e.* $F_{\theta\phi} = \partial_\theta A_\phi - \partial_\phi A_\theta$). Find such one-form or vector potentials, A and $\tilde{A}$, with $F = dA$ and $F = d\tilde{A}$, where A is smooth on the patch $\mathbb{S}^2\backslash N$ and $\tilde{A}$ is smooth on the patch $\mathbb{S}^2\backslash S$. Thus locally in each patch, the field strength is a total derivative. So you might wonder, how is it that you got a nonzero answer for part (iii)? The answer is that while it is true that locally F is a total derivative of some vector potential, you cannot find such an expression *globally*. You should find that A is ill-defined at N and $\tilde{A}$ is ill-defined at S, so neither one can be used over the whole sphere to define F. Instead, what one has is

$$\int_{\mathbb{S}^2} F = \int_{\mathbb{D}_N} d\tilde{A} + \int_{\mathbb{D}_S} dA = \oint (A - \tilde{A}) \,, \tag{6}$$

where the intermediate integrals are over the northern and southern hemispheres, and in the last expression we have used Stokes' theorem to write this as an integral over the boundary of these hemispherical disks (the equator), being careful to keep track of the orientation.

(**Hint:** *You will probably find simpler expressions in this and other parts of this problem if you work out the corresponding expressions in the (θ,ϕ) system first and translate back to $y^\mu(\theta,\phi)$ and $z^\mu(\theta,\phi)$. Alternatively, where possible, skew-symmetrize expressions in (y^1,y^2), (z^1,z^2).*)

(v) Compute the difference $A - \tilde{A}$ in the overlap of the the two charts: $S^2\backslash\{N,S\}$ and verify that $d(A-\tilde{A}) = 0$ (*i.e.* that on the overlap, both A and $\tilde{A}$ represent the same magnetic field strength F). Can we write $(A - \tilde{A}) = d\Lambda$ in the overlap of the two coordinate patches, so that they are related by a gauge transformation? After all, ordinarily two vector potentials that represent the same magnetic field are related by a gauge transformation in this way. (In the language of differential forms – see pp. 48-50 of the lecture notes – one says that $d(A-\tilde{A}) = 0$ usually implies that $A - \tilde{A}$ is the differential of a scalar Λ.) The problem is that the overlap is a band along the equator that has a non-trivial (circle) topology, and the Λ one is looking for is not a single-valued function around this circle; thus there is no well-defined function on the overlap having the required property.

Perform the line integral:

$$\frac{1}{2\pi}\oint (A - \tilde{A}) \tag{7}$$

around the equator of the sphere to recover the magnetic charge.

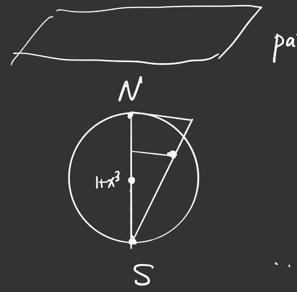
Comment: *The above construction is one of the simplest examples of a "topological charge" – the magnetic charge is determined by the integral of a field strength that is locally trivial i.e. a total derivative, but manages not to vanish when integrated over a closed manifold (i.e. without boundary) because the trivializations (i.e. the potentials $A_{(\alpha)}$ in terms of which one writes $F = dA_{(\alpha)}$ in the patch $\mathbb{O}_\alpha$) differ from patch to patch in such a way that the aggregate integral is nontrivial. This is a key reason why it is important to define quantities locally in patches on a manifold, and to be careful about the relations between definitions on patch overlaps.*

(iv) $F_{\theta\phi} = \partial_\theta A_\phi - \underbrace{\partial_\phi A_\theta}_{\text{set to } 0}$ $\qquad F_{\theta\phi} = m\sin\theta$.

$\partial_\theta A_\phi = m\sin\theta$. $\qquad A_\phi = -m\cos\theta + C_1$.

If set $C_1 = 0$.

$$A(\theta,\phi) = (0, -m\cos\theta).$$



patch $S^2\setminus S$.

is A smooth on both patch?

$$y^\mu(\theta,\phi) = \left(\frac{\cos\phi}{\tan\frac{\theta}{2}}, \frac{\sin\phi}{\tan\frac{\theta}{2}}\right).$$

these forms ensure smoothness.

$$z^\mu(\theta,\phi) = \left(\tan\tfrac{\theta}{2}\cos\phi, \tan\tfrac{\theta}{2}\sin\phi\right).$$

try to find A that satisfies.

$$-m\cos\theta = \frac{1}{\tan\frac{\theta}{2}} \; ? \qquad -m\cos\theta = \tan\frac{\theta}{2} \; ?$$

can not satisfy both.

$A = (0, -m\cos\theta)$ is for $S^2\setminus N$.

We could set

$\tilde{A} = (0, +m(1-\cos\theta))$ for $S^2\setminus S$

from symmetry.

$A - \tilde{A} = (0, m(1-2\cos\theta))$.

$d(A-\tilde{A}) = 2m\sin\theta \neq 0$ at $\theta = \frac{\pi}{2}$.

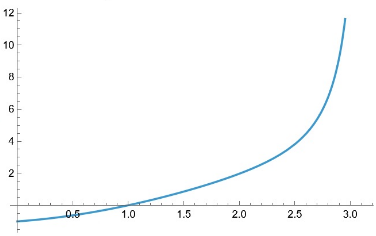
(v) $A - \tilde{A} = (0, -m)$.

$d(A-\tilde{A}) = 0$ at $\theta = \frac{\pi}{2}$.

$\frac{1}{2\pi}\oint (A-\tilde{A}) = -m$. is the charge.

```
ClearAll["Global`*"]
Plot[Tan[θ/2] - Cos[θ], {θ, 0, π}]
FindRoot[Tan[θ/2] - Cos[θ] == 0, {θ, 1}]
```

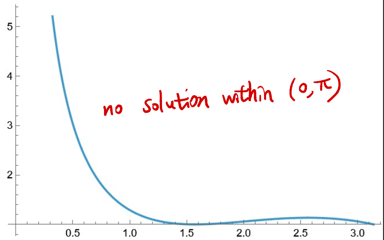
Out[20]=



Out[21]= {θ → 0.99597}

```
In[22]:= Plot[1/Tan[θ/2] - Cos[θ], {θ, 0, π}]
FindRoot[1/Tan[θ/2] - Cos[θ] == 0, {θ, 1.5}]
```

Out[22]=



Out[23]= {θ → 4.13756}

2. (25 pts) Consider a coordinate basis for tangent vectors $\underset{\sim}{e}_\alpha \in T_p\mathcal{M}$ in some coordinate chart $\mathcal{O}_{(A)}$. This basis maps to operators on functions $f : \mathcal{M} \to \mathbb{R}$ via $\underset{\sim}{e}_\alpha \leftrightarrow \partial/\partial x^\alpha$, so that a generic vector can be associated to the operator

$$\underset{\sim}{v} = v^\alpha \frac{\partial}{\partial x^\alpha} \quad .$$

One can think of the vector as implementing an infinitesimal translation via

$$(1 + \epsilon \underset{\sim}{v}) f(x^\alpha) = f(x^\alpha + \epsilon v^\alpha)$$

For two vectors $\underset{\sim}{v}$, $\underset{\sim}{w}$, the commutator or *Lie bracket* $[\underset{\sim}{v}, \underset{\sim}{w}]$ is defined by its action on functions f as

$$[\underset{\sim}{v}, \underset{\sim}{w}](f) = \underset{\sim}{v}(\underset{\sim}{w}(f)) - \underset{\sim}{w}(\underset{\sim}{v}(f)) \quad .$$

It is written in the coordinate basis as

$$\left[v^\alpha \frac{\partial}{\partial x^\alpha}, w^\beta \frac{\partial}{\partial x^\beta}\right] = (v^\alpha \partial_\alpha w^\beta - w^\alpha \partial_\alpha v^\beta) \frac{\partial}{\partial x^\beta} \equiv [\underset{\sim}{v}, \underset{\sim}{w}]^\beta \frac{\partial}{\partial x^\beta}$$

(a) Show that $[\underset{\sim}{v}, \underset{\sim}{w}]^\beta$ transform as the components of a vector under coordinate transformations.

(b) Show that the vectors $J_x = (y\partial_z - z\partial_y)$, $J_y = (z\partial_x - x\partial_z)$, and $J_z = (x\partial_y - y\partial_x)$ act on functions to implement or *generate* infinitesimal rotations around the $\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}$ axes. Compute the commutators of these vector fields. Construct a similar set of vectors for the generators of Lorentz transformations, and evaluate their commutators with respect to one another, and with respect to the rotation generators.

(a) $$[\underset{\sim}{v}, \underset{\sim}{w}]^\beta = v^\alpha \partial_\alpha w^\beta - w^\alpha \partial_\alpha v^\beta$$

General coordinate transformation:

$$d\tilde{x}^\mu = \frac{\partial \tilde{x}^\mu}{\partial x^\nu} dx^\nu .$$

vector transformation:

$$\tilde{T}^\mu(\tilde{x}) = \frac{\partial \tilde{x}^\mu}{\partial x^\nu} T^\nu(x), \qquad \tilde{T}_\mu(\tilde{x}) = \frac{\partial x^\nu}{\partial \tilde{x}^\mu} T_\nu(x) .$$

$$\underset{\sim}{v} = v^\alpha \frac{\partial}{\partial x^\alpha} . \qquad \underset{\sim}{w} = w^\alpha \frac{\partial}{\partial x^\alpha} .$$

Under $$d\tilde{x}^\gamma = \frac{\partial \tilde{x}^\gamma}{\partial x^\alpha} dx^\alpha .$$

$$\underset{\sim}{v} \to v^\gamma \frac{\partial \tilde{x}^\gamma}{\partial x^\alpha} \frac{\partial}{\partial x^\gamma} . \qquad \underset{\sim}{w} \to w^\gamma \frac{\partial \tilde{x}^\gamma}{\partial x^\alpha} \frac{\partial}{\partial x^\gamma} .$$

$$[\underset{\sim}{v}, \underset{\sim}{w}]^\beta \to v^\gamma \frac{\partial \tilde{x}^\gamma}{\partial x^\alpha} \frac{\partial}{\partial x^\gamma} \left(w^\delta \frac{\partial \tilde{x}^\delta}{\partial x^\alpha}\right) - w^\gamma \frac{\partial \tilde{x}^\gamma}{\partial x^\alpha} \frac{\partial}{\partial x^\gamma} \left(v^\delta \frac{\partial \tilde{x}^\delta}{\partial x^\alpha}\right)$$

$$= v^\gamma \partial_\gamma \left(w^\delta \frac{\partial \tilde{x}^\delta}{\partial x^\alpha}\right) - w^\gamma \partial_\gamma \left(v^\delta \frac{\partial \tilde{x}^\delta}{\partial x^\alpha}\right)$$

$$= \frac{\partial \tilde{x}^\gamma}{\partial x^\alpha} \frac{\partial \tilde{x}^\delta}{\partial x^\alpha} \left(v^\gamma \partial_\gamma w^\delta - w^\gamma \partial_\gamma v^\delta\right)$$

$$= \frac{\partial \tilde{x}^\delta}{\partial x^\alpha} [v, w]^\gamma .$$ as vector.

(b) $J_x = y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}$. $J_y = z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}$. $J_z = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$.

$$(1+\varepsilon v_\alpha \partial_\alpha) f(x^\alpha) = f(x^\alpha + \varepsilon v^\alpha)$$

$$(1+\varepsilon J_x) f(x,y,z) = f(x, y+\varepsilon J_x y, z+\varepsilon J_x z) = f(x, y-\varepsilon z, z+\varepsilon y).$$

$$(1+\varepsilon J_y) f(x,y,z) = f(x+\varepsilon z, y, z-\varepsilon x).$$

are rotations.

$$(1+\varepsilon J_z) f(x,y,z) = f(x-\varepsilon y, y+\varepsilon x, z).$$

$$[J_x, J_y] = (y\partial_z - z\partial_y)(z\partial_x - x\partial_z) - (z\partial_x - x\partial_z)(y\partial_z - z\partial_y)$$

$$= y\partial_x + yz\,\partial_z\partial_x - z^2\partial_y\partial_x - yx\,\partial_z^2 + x\partial_y + xz\,\partial_y\partial_z$$

$$-y\partial_x - zy\,\partial_x\partial_z + xy\,\partial_z^2 + z^2\partial_x\partial_y - xz\,\partial_z\partial_y - x\partial_y$$

$$= x\partial_y - y\partial_x = J_z.$$ Cannot get it right?

$$\therefore [J_i, J_j] = \varepsilon_{ijk} J_k.$$

For Lorentz groups,

$K_x = t\partial_x + x\partial_t$. $K_y = t\partial_y + y\partial_t$. $K_z = t\partial_z + z\partial_t$.

$[K_x, K_y] = -J_z$. $[K_y, K_z] = -J_x$. $[K_z, K_x] = -J_y$.

$[J_x, K_y] = K_z$, $[J_y, K_z] = K_x$, $[J_z, K_x] = K_y$.

do not have time to verify each...

3. (25 pts)

(a) Find an example of two linearly independent, nowhere-vanishing vector fields in $\mathcal{M} = \mathbb{R}^2$ whose commutator is non-vanishing. Note that these fields provide a basis for the tangent space at each point, but this basis is not a coordinate basis because the basis vectors do not commute.

(b) Show that one can always find a *non-coordinate* basis $\{\underset{\sim}{e}_a\}$ for the tangent space such that

$$\underset{\sim}{e}_a \cdot \underset{\sim}{e}_b = \eta_{ab}$$

where $\boldsymbol{\eta}$ is the Minkowski metric. This basis is not a coordinate basis because the tangent vectors are $e_a^\mu(x)\partial_\mu$ in this basis are not derivatives with respect to the coordinates, but rather linear combinations of them that change from point to point, hence going away from a given point along $\underset{\sim}{e}_a$ deviates from any given coordinate direction. The $\underset{\sim}{e}_a$ are then known as a *tetrad* basis in GR; more generally in n-dimensional geometry they are called *frame fields*. Show that the components e_a^μ of the tetrads in the coordinate basis associated to coordinates x^μ have the properties

$$e_a^\mu e_b^\nu \eta^{ab} = g^{\mu\nu} \quad , \quad e_a^\mu e_b^\nu g_{\mu\nu} = \eta_{ab} \quad .$$

Show that the matrix whose components are the components of the dual ("one-form") basis e_μ^a, is the inverse of the matrix whose components are the components of the tetrad e_a^μ.

(c) Construct the components e_a^μ of the orthonormal frame fields for spherical polar coordinates on $\mathbb{R}^3$, and their dual one-form components e_μ^a.

(a) In $\mathcal{M} = \mathbb{R}^2$, we could conveniently choose

$$v = \partial_x , \quad w = x\,\partial_y .$$

$$[v, w] = \left(v^\alpha \partial_\alpha w^\beta - w^\alpha \partial_\alpha v^\beta\right)\frac{\partial}{\partial x^\beta}$$

$$= \partial_x x \frac{\partial}{\partial y} + (0 - 0)\frac{\partial}{\partial x} = \frac{\partial}{\partial y} \neq 0 .$$

this vector field satisfies requirement.

(b) non-coordinate basis.

from $\frac{\partial}{\partial x^\mu} \longrightarrow e_a^\mu(x)\frac{\partial}{\partial x^\mu}$ terad basis / frame fields

$$e_a \cdot e_b \overset{?}{=} \eta_{ab} , \qquad e_\mu \cdot e_\nu = g_{\mu\nu} .$$

$$e_a \cdot e_b = e_a^{\mu}(x)\, e_b^{\nu}(x)\, g_{\mu\nu} = \eta_{ab}.$$

$$\Rightarrow \quad e_a^{\mu} e_b^{\nu} \eta^{ab} = g^{\mu\nu}.$$

this transformation could always act on coordinates thus exist.

Since $e_\mu^a e_b^\mu = \delta_b^a$. $e_\mu^a e_a^\nu = \delta_\mu^\nu$.

$$\Rightarrow \quad e_\mu^a e_a^\mu = 1.$$

one-form basis matrix terad

(c) spherical polar coordinates on $\mathbb{R}^3$:

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta\, d\phi^2. \qquad g_{\mu\nu} = \begin{pmatrix} 1 & & \\ & r^2 & \\ & & r^2\sin^2\theta \end{pmatrix}.$$

find $e_a^\mu e_b^\nu g_{\mu\nu} = \eta_{ab} = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}.$

could conveniently find $e_a^\mu = \begin{pmatrix} 1 & & \\ & \frac{1}{r} & \\ & & \frac{1}{r\sin\theta} \end{pmatrix}.$

then we have $e_\mu^a = (e_a^\mu)^{-1} = \begin{pmatrix} 1 & & \\ & r & \\ & & r\sin\theta \end{pmatrix}.$

4. (25 pts)

Consider a rocket moving along the positive x-axis on some trajectory in the $(x^0, x^1) \equiv (t, x)$ coordinate system. Let $v^\mu(\tau)$ be the proper 4-velocity of this rocket. Since $v_\mu v^\mu = -1$, this 4-velocity can be parametrized via:

$$v^\mu = \big(\cosh\nu(\tau)\,,\, \sinh\nu(\tau)\big)\,,$$

for some function, $\nu(\tau)$.

(i) Find the function $\nu(\tau)$ that describes the rocket under constant proper acceleration, α. Find $x^\mu(\tau)$ for an observer with constant proper acceleration who starts from rest at $x = x_0$ on the x-axis at proper time $\tau = \tau_0$.

(ii) Show that for small values of t this gives the standard kinematic result for constant acceleration. Show that for large values of t, $u \equiv \frac{dx}{dt}$ limits to c and that the proper time, τ, measured on the rocket behaves as $\tau \sim \frac{1}{\alpha}\log(\alpha\, t)$.

(iii) The goal now is to set up an orthogonal system of coordinates in which the tangents to the time axis are the 4-velocities of a family of uniformly accelerated observers. Find the space-like vector field, $S^\mu(\tau)$, of unit proper length that is orthogonal to the 4-velocity, $v^\mu(\tau)$, obtained in part (i). (Take $\tau_0 = 0$.) Define a transformation to new coordinates, $(\bar{t}, \bar{x})$, by $(t, x) = (S^0(\bar{t})\,\bar{x}, S^1(\bar{t})\,\bar{x})$. Show that the metric can be written in the form

$$-dt^2 + dx^2 = -F(\bar{x})\,d\bar{t}^2 + d\bar{x}^2\,,$$

for some function, $F(\bar{x})$. Determine this function. Now expand about some point $\bar{x}_0$ by writing $\bar{x} = \bar{x}_0 + \hat{x}$ and, at the point $\bar{x}_0$, rescale $\bar{t}$ to $\hat{t}$ so that, around $\bar{x}_0$, one has

$$-dt^2 + dx^2 = -(1 + \phi(\hat{x}))^2\, d\hat{t}^2 + d\hat{x}^2\,.$$

What is the proper acceleration of an observer at fixed $\bar{x} = \bar{x}_0$, following a trajectory parametrized by $\bar{t}$?

Comment: Constant proper acceleration has the nice feature that the inhabitants of the rocket feel the equivalent of a constant gravitation field inside the rocket. You should observe that $\phi(\hat{x})$ has a nice relation to the Newtonian potential for the gravitational field experienced by the inhabitants of the rocket, as we saw in class.

(i) $a^\mu = \dfrac{dv^\mu}{d\tau} = \dfrac{d}{d\tau}(\cosh\nu, \sinh\nu) = (\sinh\nu\cdot\nu', \cosh\nu\cdot\nu')$.

For constant acceleration $\quad a^\mu a_\mu = -\nu'^2 = -\alpha^2$.

$$\frac{d\nu}{d\tau} = \pm\alpha. \quad\Rightarrow\quad \nu(\tau) = \alpha(\tau - \tau_0).$$

$$\frac{dx^\mu}{d\tau} = v^\mu = \big(\cosh\alpha(\tau-\tau_0),\ \sinh\alpha(\tau-\tau_0)\big).$$

$$\Rightarrow\quad x^\mu = \left(\frac{1}{\alpha}\sinh\alpha(\tau-\tau_0) + C_1,\ \frac{1}{\alpha}\cosh\alpha(\tau-\tau_0) + C_2\right).$$

since $x = x_0$ for $\tau = \tau_0$. (τ_0, x_0)

$$x^\mu(\tau) = \left(\frac{1}{\alpha}\sinh\alpha(\tau-\tau_0) + \tau_0,\ \frac{1}{\alpha}\cosh\alpha(\tau-\tau_0) + x_0 - \frac{1}{\alpha}\right).$$

(ii) take usage of $\sinh x \sim x + \frac{x^3}{6}$. $\cosh x \sim 1 + \frac{x^2}{2}$.

small t gives small $\tau - \tau_0$

$$\frac{1}{\alpha}\sinh\alpha(\tau-\tau_0) + \tau_0 \sim \tau. \quad (\text{thus } t \sim \tau)$$

$$\frac{1}{\alpha}\cosh\alpha(\tau-\tau_0) + x_0 - \frac{1}{\alpha} \sim \frac{1}{\alpha}\left(1 + \frac{1}{2}\alpha^2(\tau-\tau_0)^2\right) + x_0 - \frac{1}{\alpha}$$

that is, $x^\mu(\tau) = \left(\tau,\ \frac{1}{2}\alpha(\tau-\tau_0)^2 + x_0\right)$.

constant acceleration.

for large t, $u = \frac{dx}{dt}$.

$$u = \frac{dx}{d\tau}\cdot\frac{d\tau}{dt} = \sinh\alpha(\tau-\tau_0)\cdot\left(\cosh\alpha(\tau-\tau_0)\right)^{-1}$$

$$= \tanh\alpha(\tau-\tau_0). \longrightarrow 1 \text{ at large } \tau.$$

$t = \frac{1}{\alpha}\sinh\alpha(\tau-\tau_0) + \tau_0$, at large τ

$$\sinh x = \frac{e^x - e^{-x}}{2} \approx \frac{1}{2}e^x \quad (x \to \infty)$$

thus $t \sim \frac{e^{\alpha(\tau-\tau_0)}}{2\alpha}$. that is, $\tau \sim \frac{1}{\alpha}\log(\alpha t)$.

(iii) $v^\mu = (\cosh\alpha(\tau-\tau_0), \sinh\alpha(\tau-\tau_0))$.

find $\eta_{\mu\nu} v^\mu S^\mu = 0$. could set

$S^\mu = (\sinh\alpha(\tau-\tau_0), \cosh\alpha(\tau-\tau_0))$. is orthogonal and has unit length.

then $(t, x) = (S^0(\bar{t})\bar{x}, S^1(\bar{t})\bar{x})$

$= (\bar{x}\sinh\alpha(\bar{t}-\tau_0), \bar{x}\cosh\alpha(\bar{t}-\tau_0))$.

find $-dt^2 + dx^2 = -F(\bar{x})\, d\bar{t}^2 + d\bar{x}^2$.

$$dt = \alpha\bar{x}\cosh\alpha(\bar{t}-\tau_0)\, d\bar{t} + d\bar{x}\sinh\alpha(\bar{t}-\tau_0).$$

$$dx = \alpha\bar{x}\sinh\alpha(\bar{t}-\tau_0)\, d\bar{t} + d\bar{x}\cosh\alpha(\bar{t}-\tau_0).$$

$$-dt^2 + dx^2 = -\alpha^2\bar{x}^2 d\bar{t}^2 + d\bar{x}^2. \qquad F(\bar{x}) = \alpha^2\bar{x}^2.$$

expand around $\bar{x}_0$: $\bar{x} = \bar{x}_0 + \hat{x}$.

$$F(\bar{x}) = \alpha^2(\bar{x}_0^2 + 2\bar{x}_0\hat{x} + \hat{x}^2). \qquad \bar{t} \to \frac{\hat{t}}{\beta}.$$

$$-dt^2 + dx^2 = -\alpha^2(\bar{x}_0^2 + 2\bar{x}_0\hat{x} + \hat{x}^2)\frac{d\hat{t}^2}{\beta^2} + d\hat{x}^2$$

$$= -\frac{\alpha^2\bar{x}_0^2}{\beta^2}\left(1+\frac{\hat{x}}{\bar{x}_0}\right)^2 d\hat{t}^2 + d\hat{x}^2. \qquad \text{then } \beta = \alpha\bar{x}_0.$$

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1. (25 pts)
Show that the commutator of covariant derivatives acting on a tensor can be expressed in terms of the curvature as follows:

$$\nabla_\alpha \nabla_\beta T^{\mu_1\cdots\mu_m}_{\nu_1\cdots\nu_n} - \nabla_\beta \nabla_\alpha T^{\mu_1\cdots\mu_m}_{\nu_1\cdots\nu_n} = \sum_i R^{\mu_i}_{\sigma\alpha\beta} T^{\mu_1\cdots\sigma\cdots\mu_m}_{\nu_1\cdots\nu_n} - \sum_j R^{\sigma}_{\nu_j\alpha\beta} T^{\mu_1\cdots\mu_m}_{\nu_1\cdots\sigma\cdots\nu_n}$$

start from contravariant field A^ν reference : book

(1,1) $\nabla_\lambda A^\nu(x) = \dfrac{\partial A^\nu(x)}{\partial \lambda} + \Gamma^\nu_{\mu\lambda} A^\mu(x)$

$$\Rightarrow \nabla_k \nabla_\lambda A^\nu = \nabla_k \left(\partial_\lambda A^\nu + \Gamma^\nu_{\mu\lambda} A^\mu\right)$$

covariant

$$= \partial_k(\partial_\lambda A^\nu) + \Gamma^\nu_{\mu k}\partial_\lambda A^\mu + (\nabla_k \Gamma^\nu_{\mu\lambda})A^\mu + \Gamma^\nu_{\mu\lambda}\nabla_k A^\mu + \nabla_k(\nabla_\lambda A^\nu)$$

use $\nabla_k B_\lambda = \partial_k B_\lambda - \Gamma^\mu_{\lambda k} B_\mu$,

not quite ?

$$\Rightarrow \nabla_k(\nabla_\lambda A^\nu) = \partial_k(\nabla_\lambda A^\nu) + \Gamma_{\mu k}\nabla_\lambda A^\mu - \Gamma^\mu_{\lambda k}\nabla_\mu A^\nu$$

try again $\nabla_k \nabla_\lambda A^\nu = \nabla_k\left(\partial_\lambda A^\nu + \Gamma^\nu_{\mu\lambda}A^\mu\right)$

$$\partial_k A^\mu + \Gamma^\mu_{\rho k} A^\rho$$

$$= \partial_k(\partial_\lambda A^\nu) + \Gamma_{\mu k}\partial_\lambda A^\mu + (\nabla_k \Gamma^\nu_{\lambda\mu})A^\mu + \Gamma^\nu_{\mu\lambda}\nabla_k A^\mu + (*)$$

$$\partial_k \Gamma^\nu_{\lambda\mu} + \Gamma^\nu_{k\rho}\Gamma^\rho_{\lambda\mu} - \Gamma^\rho_{k\lambda}\Gamma^\nu_{\rho\mu}$$

$$= \partial_k(\partial_\lambda A^\nu) + \Gamma^\nu_{\mu k}\partial_\lambda A^\mu + \partial_k\Gamma^\nu_{\lambda\mu}A^\mu + \Gamma^\nu_{k\rho}\Gamma^\rho_{\lambda\mu}A^\mu$$

$$- \Gamma^\rho_{k\lambda}\Gamma^\nu_{\rho\mu}A^\mu + \Gamma^\nu_{\mu\lambda}\partial_k A^\mu + \Gamma^\nu_{\mu\lambda}\Gamma^\mu_{\rho k}A^\rho$$

directly use expansion

$$\nabla_\kappa \nabla_\lambda A^\nu = \partial_\kappa(\nabla_\lambda A^\nu) + \Gamma^\nu_{\mu\kappa}\nabla_\lambda A^\mu - \Gamma^\mu_{\lambda\kappa}\nabla_\mu A^\nu$$

$$= \partial_\kappa\left(\partial_\lambda A^\nu + \Gamma^\nu_{\rho\lambda}A^\rho\right) + \Gamma^\nu_{\mu\kappa}\left(\partial_\lambda A^\mu + \Gamma^\mu_{\rho\lambda}A^\rho\right) - \Gamma^\mu_{\lambda\kappa}\left(\partial_\mu A^\nu + \Gamma^\nu_{\rho\mu}A^\rho\right)$$

$$= \partial_\kappa(\partial_\lambda A^\nu) + (\partial_\kappa \Gamma^\mu_{\rho\lambda})A^\mu + \Gamma^\nu_{\rho\lambda}\partial_\kappa A^\rho + \Gamma^\nu_{\mu\kappa}\partial_\lambda A^\mu$$
$$+ \Gamma^\nu_{\mu\kappa}\Gamma^\mu_{\rho\lambda}A^\rho - \Gamma^\mu_{\lambda\kappa}\partial_\mu A^\nu - \Gamma^\mu_{\lambda\kappa}\Gamma^\nu_{\rho\mu}A^\mu .$$

then $$\nabla_\lambda \nabla_\kappa A^\nu = \partial_\lambda(\partial_\kappa A^\nu) + (\partial_\lambda \Gamma^\mu_{\rho\kappa})A^\mu + \Gamma^\nu_{\rho\kappa}\partial_\lambda A^\rho + \Gamma^\nu_{\mu\lambda}\partial_\kappa A^\mu$$
$$+ \Gamma^\nu_{\mu\lambda}\Gamma^\mu_{\rho\kappa}A^\rho - \Gamma^\mu_{\kappa\lambda}\partial_\mu A^\nu - \Gamma^\mu_{\kappa\lambda}\Gamma^\nu_{\rho\mu}A^\mu .$$

$$\Rightarrow \nabla_\kappa\nabla_\lambda A^\nu - \nabla_\lambda\nabla_\kappa A^\nu = \underline{\partial_\kappa(\partial_\lambda A^\nu) - \partial_\lambda(\partial_\kappa A^\nu)} + (\partial_\kappa \Gamma^\mu_{\rho\lambda})A^\mu - (\partial_\lambda \Gamma^\mu_{\rho\kappa})A^\mu$$
$$\underline{+\Gamma^\nu_{\rho\lambda}\partial_\kappa A^\rho} \underline{+\Gamma^\nu_{\mu\kappa}\partial_\lambda A^\mu} + \Gamma^\mu_{\kappa\lambda}\partial_\mu A^\nu + \Gamma^\mu_{\kappa\lambda}\Gamma^\nu_{\rho\mu}A^\mu + \Gamma^\nu_{\mu\kappa}\Gamma^\mu_{\rho\lambda}A^\rho$$
$$\underline{-\Gamma^\nu_{\rho\kappa}\partial_\lambda A^\rho} \underline{-\Gamma^\nu_{\mu\lambda}\partial_\kappa A^\mu} - \Gamma^\mu_{\lambda\kappa}\partial_\mu A^\nu - \Gamma^\nu_{\mu\lambda}\Gamma^\mu_{\rho\kappa}A^\rho - \Gamma^\mu_{\lambda\kappa}\Gamma^\nu_{\rho\mu}A^\mu$$

$$= \left(\partial_\kappa \Gamma^\mu_{\rho\lambda} - \partial_\lambda \Gamma^\mu_{\rho\kappa} + \Gamma^\mu_{\kappa\lambda}\Gamma^\nu_{\rho\mu} - \Gamma^\mu_{\lambda\kappa}\Gamma^\nu_{\rho\mu}\right)A^\mu + \left(\Gamma^\mu_{\kappa\lambda} - \Gamma^\mu_{\lambda\kappa}\right)\partial_\mu A^\nu$$

$R^\nu_{\rho\kappa\lambda}$ $\qquad$ $T^\mu_{\kappa\lambda}$

below derivation is from reference

$$(D_\kappa D_\lambda - D_\lambda D_\kappa)(A^\nu B_\nu) = T^\mu_{\kappa\lambda}\,\partial_\mu(A^\nu B_\nu) \qquad (*)$$

$$= (D_\kappa D_\lambda A^\nu - D_\lambda D_\kappa A^\nu)\, B_\nu + A^\nu (D_\kappa D_\lambda - D_\lambda D_\kappa) B_\nu$$

$$R^\nu_{\rho\kappa\lambda} A^\mu + T^\mu_{\kappa\lambda}\,\partial_\mu A^\mu$$

$$= R^\nu_{\rho\kappa\lambda} A^\mu B_\nu + T^\mu_{\kappa\lambda}\,\partial_\mu A^\mu B_\nu + A^\nu (D_\kappa D_\lambda - D_\lambda D_\kappa) B_\nu$$

$$= R^\nu_{\rho\kappa\lambda} A^\mu B_\nu + T^\mu_{\kappa\lambda}(\partial_\mu A^\mu B_\nu) - T^\mu_{\kappa\lambda} A^\rho \partial_\mu B_\rho + A^\nu (D_\kappa D_\lambda - D_\lambda D_\kappa) B_\nu \qquad (**)$$

compare to get $D_\kappa D_\lambda B_\mu - D_\lambda D_\kappa B_\mu = -R^\nu_{\rho\kappa\lambda} B_\nu + T^\mu_{\kappa\lambda}\,\partial_\mu B_\rho$.

with both covariant and contravariant fields, we could prove result.

$$D_\kappa D_\lambda (S^{\mu\nu} B_\nu) - D_\lambda D_\kappa (S^{\mu\nu} B_\nu) = R^\rho_{\sigma\kappa\lambda} S^{\sigma\nu} B_\nu + T^\mu_{\kappa\lambda}\,\partial_\mu (S^{\rho\nu} B_\nu)$$

$$(D_\kappa D_\lambda S^{\mu\nu} - D_\lambda D_\kappa S^{\mu\nu})\, B_\nu = R^\nu_{\rho\kappa\lambda} S^{\rho\nu} B_\nu + R^\nu_{\rho\kappa\lambda} S^{\mu\rho} B_\nu + T^\sigma_{\kappa\lambda}\,\partial_\sigma S^{\mu\nu} B_\nu.$$

$$D_\kappa D_\lambda S^{\mu\nu} - D_\lambda D_\kappa S^{\mu\nu} = R^\mu_{\rho\kappa\lambda} S^{\rho\nu} + R^\nu_{\rho\kappa\lambda} S^{\mu\rho} + T^\sigma_{\kappa\lambda}\,\partial_\sigma S^{\mu\nu}.$$

2. (25 pts) A good approximation to the metric outside the surface of the earth is provided by

$$ds^2 = -(1+2\Phi)dt^2 + (1-2\Phi)dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$$

where

$$\Phi = -\frac{GM}{r}$$

is the Newtonian gravitational potential.

A GPS satellite carries an atomic clock in geosynchronous orbit around the earth (this means that the orbital period is 24 hours, so that the satellite sits along a fixed radial vector through the equator in the frame that co-rotates with the earth). Global positioning depends on accurately measuring the time differences between signals broadcast from several such satellites. Attempts to synchronize the clocks on the satellites with one another and with clocks on the ground run into potential issues with Doppler shifts and time dilations due to the motion of the satellites and observers on the earth, and their presence in a gravitational field.

The Doppler shift aspects were covered in HW1 in a similar problem in Minkowski space. Now consider an observer who sits in their lab on the Earth's equator and sees 365 days pass on their atomic clock. How much time has elapsed on the clocks on the satellites? As a first step, calculate the radius of geosynchronous orbits, and look up the radius of the Earth.

$$\omega = \frac{2\pi}{T} = \sqrt{\frac{GM}{r^3}} . \qquad r = \left(\frac{GMT^2}{4\pi^2}\right)^{\frac{1}{3}} .$$

Let's derive again from rotating frame

$$t = \tilde{t},$$
$$x = r\cos(\theta + \omega\tilde{t})$$
$$y = r\sin(\theta + \omega\tilde{t})$$
$$z = \tilde{z}$$

$$\frac{\partial x^\mu}{\partial \tilde{x}^\nu} = \begin{pmatrix} 1 & & & \\ -\frac{\omega r}{c}\sin(\theta+\omega\tilde{t}) & \cos(\theta+\omega t) & -r\sin(\theta+\omega\tilde{t}) & \\ \frac{\omega r}{c}\cos(\theta+\omega t) & \sin(\theta+\omega t) & r\cos(\theta+\omega\tilde{t}) & \\ & & & 1 \end{pmatrix}$$

$$\tilde{g}_{\mu\nu} = \frac{\partial x^k}{\partial \tilde{x}^\mu}\frac{\partial x^\lambda}{\partial \tilde{x}^\nu}\eta_{k\lambda} = \begin{array}{c} g_{00} \\ \begin{pmatrix} \boxed{-1+\frac{\omega^2 r^2}{c^2}} & & \frac{r^2\omega}{c} & \\ & 1 & & \\ \frac{r^2\omega}{c} & & r^2 & \\ & & & 1 \end{pmatrix} \end{array}$$

$$d\tau = \frac{1}{c}\sqrt{g_{00}}\, dx^0 .$$

it is already known as

$\sqrt{1+2\phi}\, dt$...

there's another factor $\sqrt{1-\frac{\omega^2 r^2}{c^2}}$ due to SR.

total factor : $\sqrt{1+2\phi}\,\sqrt{1-\frac{\omega^2 r^2}{c^2}}^{\,-1} = \sqrt{1-\frac{2GM}{rc^2}} \cdot \sqrt{1-\frac{GM}{rc^2}}^{\,-1} .$

Plug in $r = \left(\frac{GMT^2}{4\pi^2}\right)^{\frac{1}{3}}$. $T = 3600 \times 24$ s. $M = 5.972 \times 10^{24}$ kg.

to get $r \approx 4.224 \times 10^7$ m.

$$\text{total factor} = \frac{\sqrt{1-\frac{2GM}{rc^2}}\sqrt{1-\frac{GM}{rc^2}}^{-1}}{\sqrt{1-\frac{2GM}{Rc^2}}\sqrt{1-\frac{GM}{Rc^2}}^{-1}}$$

$(1-x)\cdot(1+\frac{1}{2}x) \approx$

$\frac{1-\alpha}{1-\beta} \approx 1-\alpha+\beta$

$$\approx \left(1-\frac{1}{2}\frac{GM}{rc^2}\right) \Big/ \left(1-\frac{1}{2}\frac{GM}{Rc^2}\right)$$

$$\approx 1 - 5.25\times10^{-11} + 3.46\times10^{-10}$$

$$= 1 + 2.94\times10^{-10}.$$

$\times 365 \times 24 \times 3600 = 9.27 \times 10^{-3}$ s.

3. (25 pts) Consider the Friedman-Robertson-Walker (FRW) metric for a 2+1 dimensional expanding universe of the form

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -dt^2 + a^2(t)\, h_{ij}dx^i dx^j$$

where $\mu, \nu = 0, 1, 2$ label the spacetime coordinates; $i, j = 1, 2$ label the spatial coordinates; and the spatial metric h_{ij} depends only on the spatial coordinates. The *scale factor* $a(t)$ describes the stretching of proper (i.e. physical) distance between freely falling observers who sit at fixed coordinates x^i. For a homogeneous, isotropic geometry this spatial metric can be written

$$h_{ij}dx^i dx^j = \frac{dr^2}{1-kr^2} + r^2 d\theta^2$$

where $k \in \{-1, 0, 1\}$; the above metric then describes a homogeneous *Friedman-Robertson-Walker* (FRW) cosmology.

(a) Calculate the curvature $R^i{}_{jkl}$ of the 2d spatial metric h_{ij}, and the (*Ricci*) scalar curvature $R^{ij}{}_{ij}$, and interpret the meaning of the choice of k.

(b) Calculate the curvature $\mathcal{R}^\mu{}_{\nu\alpha\beta}$ of the spacetime metric $g_{\mu\nu}$, and the corresponding scalar curvature.

(a) From definition $\quad R^i_{klm} = \dfrac{\partial \Gamma^i_{km}}{\partial x^l} - \dfrac{\partial \Gamma^i_{kl}}{\partial x^m} + \Gamma^i_{nl}\Gamma^n_{km} - \Gamma^i_{nm}\Gamma^n_{kl}$.

$$R_{ik} = g^{lm} R_{limk} = R^l_{ilk}.$$

$$R_{ik} = \frac{\partial \Gamma^l_{ik}}{\partial x^l} - \frac{\partial \Gamma^l_{il}}{\partial x^k} + \Gamma^l_{ik}\Gamma^m_{lm} - \Gamma^m_{il}\Gamma^l_{km}.$$

Let's first calculate Γ.

$$\Gamma^i_{kl} = \frac{1}{2} h^{im}\left(\frac{\partial h_{mk}}{\partial x^l} + \frac{\partial h_{ml}}{\partial x^k} - \frac{\partial h_{kl}}{\partial x^m}\right)$$

We have $\quad h_{ij} = \begin{pmatrix} \frac{1}{1-kr^2} & \\ & r^2 \end{pmatrix} \Rightarrow h^{ij} = \begin{pmatrix} 1-kr^2 & \\ & \frac{1}{r^2} \end{pmatrix}$.

$$\Gamma^r_{rr} = \frac{1}{2} h^{rr}\frac{\partial h_{rr}}{\partial r} = \frac{1}{2}(1-kr^2)\cdot \frac{2kr}{(1-kr^2)^2} = \frac{kr}{1-kr^2}.$$

$$\Gamma^r_{\theta\theta} = \frac{1}{2} h^{rr}\left(\frac{\partial h_{r\theta}}{\partial r} + \frac{\partial h_{r\theta}}{\partial \theta} - \frac{\partial h_{\theta\theta}}{\partial r}\right) = \frac{1}{2}(1-kr^2)\cdot(-2r)$$

$$= -r(1-kr^2).$$

$$\Gamma^{\theta}_{rr} = \frac{1}{2} h^{\theta\theta}\left(\frac{\partial h_{\theta r}}{\partial r} + \frac{\partial h_{\theta r}}{\partial r} - \frac{\partial h_{rr}}{\partial \theta}\right) = 0.$$

$$\Gamma^{\theta}_{r\theta} = \frac{1}{2} h^{\theta\theta}\left(\frac{\partial h_{rr}}{\partial \theta} + \frac{\partial h_{\theta\theta}}{\partial r}\right) = \frac{1}{2}\frac{1}{r^2}\cdot 2r = \frac{1}{r}.$$

$$R^{i}_{klm} = \frac{\partial \Gamma^{i}_{km}}{\partial x^{l}} - \frac{\partial \Gamma^{i}_{kl}}{\partial x^{m}} + \Gamma^{i}_{nl}\Gamma^{n}_{km} - \Gamma^{i}_{nm}\Gamma^{n}_{kl}.$$

$$R^{r}_{\theta r\theta} = \frac{\partial \Gamma^{r}_{\theta\theta}}{\partial r} - \frac{\partial \Gamma^{r}_{\theta r}}{\partial \theta} + \Gamma^{r}_{rr}\Gamma^{r}_{\theta\theta} - \Gamma^{r}_{\theta\theta}\Gamma^{\theta}_{\theta r}$$

$$= -(1-kr^2) + 2kr^2 + \frac{kr}{1-kr^2}\cdot -r(1-kr^2) + r(1-kr^2)\cdot\frac{1}{r}$$

$$= 3kr^2 - 1 - kr^2 + 1 - kr^2$$

$$= kr^2.$$

$$R^{\theta}_{r\theta r} = \frac{\partial \Gamma^{\theta}_{rr}}{\partial \theta} - \frac{\partial \Gamma^{\theta}_{r\theta}}{\partial r} + \Gamma^{\theta}_{n\theta}\Gamma^{n}_{rr} - \Gamma^{\theta}_{nr}\Gamma^{n}_{r\theta}$$

$$= \frac{1}{r^2} + \frac{1}{r}\cdot\frac{kr}{1-kr^2} + \Gamma^{\theta}_{r\theta}\Gamma^{r}_{rr} + \Gamma^{\theta}_{\theta\theta}\Gamma^{\theta}_{rr} - \Gamma^{\theta}_{rr}\Gamma^{r}_{r\theta} - \Gamma^{\theta}_{\theta r}\Gamma^{\theta}_{r\theta}$$

$$= \frac{1}{r^2} + \frac{k}{1-kr^2} + \frac{k}{1-kr^2} - \frac{1}{r^2} = \frac{2k}{1-kr^2}.$$

$$R_{ik} = g^{lm}R_{limk} = R^{l}_{ilk}.$$

$$= g^{rr}R_{rr} + g^{\theta\theta}R_{\theta\theta}$$

$$= (1-kr^2)\cdot\frac{2k}{1-kr^2} + \frac{1}{r^2}\cdot kr^2$$

$$= 3k.$$

$k=1$, positive curvature. 2D sphere.

$k=0$. plane. curvature $=0$.

$k=-1$. hyperbolic. negative curvature.

(b) Let's calculate time dependence of Γ.

$$\Gamma^i_{kl} = \frac{1}{2} h^{im}\left(\frac{\partial h_{mk}}{\partial x^l} + \frac{\partial h_{ml}}{\partial x^k} - \frac{\partial h_{kl}}{\partial x^m}\right)$$

$$h_{ij} = a^2 \begin{pmatrix} \frac{1}{1-kr^2} & \\ & r^2 \end{pmatrix}$$

$$h^{ij} = \frac{1}{a^2} \begin{pmatrix} 1-kr^2 & \\ & \frac{1}{r^2} \end{pmatrix}$$

$$\Gamma^t_{rr} = \frac{1}{2} h^{tm}\left(\frac{\partial h_{mr}}{\partial r} + \frac{\partial h_{mr}}{\partial r} - \frac{\partial h_{rr}}{\partial x^m}\right)$$

$$= \frac{1}{2} h^{tt} \cdot -\frac{\partial h_{rr}}{\partial t} = \frac{1}{2} \frac{2a\dot{a}}{a^2(1-kr^2)} = \frac{\dot{a}}{a(1-kr^2)}.$$

$$\Gamma^t_{\theta\theta} = \frac{1}{2} h^{tt}\left(\frac{\partial h_{t\theta}}{\partial \theta} + \frac{\partial h_{t\theta}}{\partial \theta} - \frac{\partial h_{\theta\theta}}{\partial t}\right) = \frac{1}{2} \cdot 2a\dot{a}r^2 = a\dot{a}r^2.$$

$$\Gamma^r_{tr} = \frac{1}{2} h^{rm}\left(\frac{\partial h_{mt}}{\partial r} + \frac{\partial h_{mr}}{\partial t} - \frac{\partial h_{tr}}{\partial m}\right)$$

$$= \frac{1}{2} h^{rr} \cdot \frac{\partial h_{rr}}{\partial t} = \frac{1}{2} \frac{1}{a^2}(1-kr^2) \cdot \frac{2a\dot{a}}{1-kr^2} = \frac{\dot{a}}{a}.$$

$$\Gamma^\theta_{t\theta} = \frac{1}{2} h^{\theta\theta}\left(\frac{\partial h_{\theta t}}{\partial \theta} + \frac{\partial h_{\theta\theta}}{\partial t} - \frac{\partial h_{t\theta}}{\partial \theta}\right) = \frac{1}{2} \frac{1}{a^2 r^2} \cdot 2a\dot{a}r^2 = \frac{\dot{a}}{a}.$$

$$R^i_{klm} = \frac{\partial \Gamma^i_{km}}{\partial x^l} - \frac{\partial \Gamma^i_{kl}}{\partial x^m} + \Gamma^i_{nl}\Gamma^n_{km} - \Gamma^i_{nm}\Gamma^n_{kl}.$$

$$R^t_{rtr} = \frac{\partial \Gamma^t_{rr}}{\partial t} - \frac{\partial \Gamma^t_{rt}}{\partial r} + \Gamma^t_{nt}\Gamma^n_{rr} - \Gamma^t_{nr}\Gamma^n_{rt}$$

$$= \frac{a\ddot{a} - \dot{a}^2}{a^2(1-kr^2)} + \underline{\Gamma^t_{tt}}\Gamma^t_{rr} + \underline{\Gamma^t_{rt}\Gamma^r_{rr}} \underline{- \Gamma^t_{rr}\Gamma^r_{rt}} - \underline{\Gamma^t_{tr}\Gamma^t_{rt}} = \frac{a\ddot{a} - \dot{a}^2}{a^2(1-kr^2)}$$

$$R^{\theta}_{r\theta r} = \frac{\partial \Gamma^{\theta}_{rr}}{\partial \theta} - \frac{\partial \Gamma^{\theta}_{r\theta}}{\partial r} + \Gamma^{\theta}_{n\theta}\Gamma^{n}_{rr} - \Gamma^{\theta}_{nr}\Gamma^{n}_{r\theta}$$

$$\Gamma^{r}_{rr} = \frac{kr}{1-kr^2}.$$

$$\Gamma^{r}_{\theta\theta} = -r(1-kr^2).$$

$$\Gamma^{\theta}_{r\theta} = \frac{1}{r}.$$

$$= \Gamma^{\theta}_{r\theta}\Gamma^{r}_{rr} + \Gamma^{\theta}_{t\theta}\Gamma^{t}_{rr} - \Gamma^{\theta}_{rr}\Gamma^{r}_{r\theta}$$

$$= \frac{k}{1-kr^2} + \frac{\dot{a}}{a}\frac{\dot{a}}{a}\cdot\frac{1}{1-kr^2} = \frac{1}{1-kr^2}\left(k + \frac{\dot{a}^2}{a^2}\right)$$

$$R^{t}_{\theta t\theta} = \frac{\partial \Gamma^{t}_{\theta\theta}}{\partial t} - \frac{\partial \Gamma^{t}_{\theta t}}{\partial \theta} + \Gamma^{t}_{nt}\Gamma^{n}_{\theta\theta} - \Gamma^{t}_{n\theta}\Gamma^{n}_{\theta t}$$

$$= (\dot{a}^2 + a\ddot{a})r^2$$

$$R^{r}_{\theta r\theta} = \frac{\partial \Gamma^{r}_{\theta\theta}}{\partial r} - \frac{\partial \Gamma^{r}_{\theta r}}{\partial \theta} + \Gamma^{r}_{nr}\Gamma^{n}_{\theta\theta} - \Gamma^{r}_{n\theta}\Gamma^{n}_{\theta r}$$

$$= -(1-3kr^2) + \Gamma^{r}_{rt}\Gamma^{t}_{\theta\theta} + \Gamma^{r}_{rr}\Gamma^{r}_{\theta\theta} - \Gamma^{r}_{\theta\theta}\Gamma^{\theta}_{\theta r} - \Gamma^{r}_{r\theta}\Gamma^{r}_{\theta r}$$

$$= -(1-3kr^2) + \dot{a}^2 r^2 - kr^2 + (1-kr^2) = kr^2 + \dot{a}^2 r^2.$$

$$R^{i}_{klm} = \frac{\partial \Gamma^{i}_{km}}{\partial x^{l}} - \frac{\partial \Gamma^{i}_{kl}}{\partial x^{m}} + \Gamma^{i}_{nl}\Gamma^{n}_{km} - \Gamma^{i}_{nm}\Gamma^{n}_{kl}.$$

$$R^{\theta}_{t\theta t} = \frac{\partial \Gamma^{\theta}_{tt}}{\partial \theta} - \frac{\partial \Gamma^{\theta}_{t\theta}}{\partial t} + \Gamma^{\theta}_{n\theta}\Gamma^{n}_{tt} - \Gamma^{\theta}_{nt}\Gamma^{n}_{t\theta} = \frac{a\ddot{a} - \dot{a}^2}{a^2}$$

$$R^{r}_{trt} = \frac{\partial \Gamma^{r}_{tt}}{\partial r} - \frac{\partial \Gamma^{r}_{tr}}{\partial t} + \Gamma^{r}_{nr}\Gamma^{n}_{tt} - \Gamma^{r}_{nt}\Gamma^{n}_{tr}$$

$$= -\frac{a\ddot{a} - \dot{a}^2}{a^2} - \Gamma^{r}_{rt}\Gamma^{r}_{tr} - \Gamma^{r}_{tt}\Gamma^{t}_{tr} = -\frac{\ddot{a}}{a}.$$

$$R_{ik} = R^{l}_{ilk}.$$

$$R_{tt} = \frac{a\ddot{a} - \dot{a}^2}{a^2} - \frac{\ddot{a}}{a} = -\frac{\dot{a}^2}{a^2}.$$

$$R_{rr} = \frac{1}{1-kr^2}\left(k + \frac{\dot{a}^2}{a^2}\right) + \frac{a\ddot{a} - \dot{a}^2}{a^2(1-kr^2)} = \frac{1}{1-kr^2}\left(k + \frac{\ddot{a}}{a}\right)$$

$$R_{\theta\theta} = kr^2 + \dot{a}^2 r^2 + (\dot{a}^2 + a\ddot{a})r^2 = (k + 2\dot{a}^2 + a\ddot{a})\, r^2.$$

unit not right? ...

Scalar curvature

$$R = g^{\mu\nu} R_{\mu\nu}$$

$$h^{ij} = \frac{1}{a^2}\begin{pmatrix} 1-kr^2 & \\ & \frac{1}{r^2} \end{pmatrix}$$

$$= g^{tt} R_{tt} + g^{rr} R_{rr} + g^{\theta\theta} R_{\theta\theta}$$

$$= \frac{\dot{a}^2}{a^2} + \left(k + \frac{\ddot{a}}{a}\right) + (k + 2\dot{a}^2 + a\ddot{a}) \cdot \frac{1}{a^2}$$

$$= 2k + \frac{3\dot{a}^2}{a^2} + \frac{2\ddot{a}}{a^2}.$$

4. (25 pts) Consider the FRW universe of the previous problem. The 4-velocity of comoving observers (those who sit at fixed x^i) is $U^\mu = (1, 0, 0)$. Let the 4-velocity of a free particle be $V^\mu = dx^\mu/d\lambda$ in terms of an affine parameter along its worldline.

(a) Show that the quantity

$$K^2 = a^2(t)\left[V_\mu V^\mu + (U_\mu V^\mu)^2\right]$$

is constant along the particle's (geodesic) worldline.

(b) Show that the magnitude of the spatial velocity of a massive particle is

$$|\vec{V}| = \frac{K}{a}$$

so that particle motions slow down with respect to a family of comoving observers. As a consequence, an ideal gas of such particles will cool as the universe expands. Similarly, show that for massless particles,

$$U_\mu V^\mu = \frac{K}{a}$$

and show that as a consequence photon frequencies redshift in an expanding universe. This is why for instance the ultraviolet photons, emitted during the epoch when electrons and protons formed bound states (Hydrogen atoms) in the early universe, are 3°K microwave photons today.

(c) Consider an inflationary universe, with $a(t) = e^{Ht}$ for some (Hubble) constant H. Find the radius R such that light signals, emitted from $r > R$ at $t = 0$ in the direction of the origin $r = 0$, never arrive at the origin for any finite time t. The resulting sphere of radius R defines the *horizon*, the surface beyond which the observer at the origin cannot see as long as the universe continues to expand exponentially.

(a) $$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\lambda}\frac{dx^\nu}{d\tau}\frac{dx^\lambda}{d\tau} = 0.$$

$$\frac{DV^\mu}{d\tau} = 0.$$

$$\Gamma^t_{rr} = \frac{\dot a}{a(1-kr^2)}. \quad \Gamma^t_{\theta\theta} = a\dot a r^2. \quad \Gamma^r_{tr} = \frac{\dot a}{a}.$$

$$\Gamma^r_{rr} = \frac{kr}{1-kr^2}. \quad \Gamma^r_{\theta\theta} = -r(1-kr^2). \quad \Gamma^\theta_{r\theta} = \frac{1}{r}.$$

$$\frac{d^2t}{d\tau^2} + \Gamma^t_{\nu\lambda}\frac{dx^\nu}{d\tau}\frac{dx^\lambda}{d\tau} = 0. \quad \cdots \quad \text{etc.}$$

or not quite?

Let's calculate derivative of K^2.

$$K^2 = a^2(t)\left[V_\mu V^\mu + (U_\mu V^\mu)^2\right].$$

$$\Rightarrow \frac{d}{d\lambda}(K^2) = 2\frac{da}{d\lambda}a\left[V_\mu V^\mu + (U_\mu V^\mu)^2\right] + a^2\frac{d}{d\lambda}\left[V_\mu V^\mu + (U_\mu V^\mu)^2\right]$$

worldline $\quad \frac{dV^\mu}{d\lambda} + \Gamma^\mu_{\alpha\beta}V^\alpha V^\beta = 0.$

$$\Rightarrow \frac{d}{d\lambda}(\underbrace{V_\mu V^\mu}_{=-1\,?}) = \frac{d}{d\lambda}(g_{\mu\nu}V^\mu V^\nu) = 2g_{\mu\nu}V^\mu\cdot\left(-\Gamma^\mu_{\alpha\beta}V^\alpha V^\beta\right).$$

$$U_\mu V^\mu = g_{\mu\nu}\overset{(1,0,0,0)}{U^\mu}V^\mu = V_0.$$

$$\Rightarrow \frac{d}{d\lambda}(U_\mu V^\mu)^2 = 2U_\mu V^\mu \underline{\frac{d}{d\lambda}(U_\mu V^\mu)} \qquad U_\mu\frac{dV^\mu}{d\lambda}$$

$$= -2U_\mu V^\mu U_\mu \Gamma^\mu_{\alpha\beta}V^\alpha V^\beta \qquad \times$$

$$= -2U_0V^0U_0\Gamma^0_{\alpha\beta}V^\alpha V^\beta.$$

Looking back,

$$\frac{d}{d\lambda}(K^2) = 2\frac{da}{d\lambda}a\left[V_\mu V^\mu + (U_\mu V^\mu)^2\right] - 2a^2 g_{\mu\nu}V^\mu\Gamma^\mu_{\alpha\beta}V^\alpha V^\beta \quad = 0.$$

$$- 2a^2U_0V^0U_0\Gamma^0_{\alpha\beta}V^\alpha V^\beta$$

...

$$K^2 = a^2(t)\left[V_\mu V^\mu + (U_\mu V^\mu)^2\right].$$

worldline $\quad \dfrac{dV^\mu}{d\lambda} + \Gamma^\mu_{\alpha\beta} V^\alpha V^\beta = 0.$

$V_\mu V^\mu = -1. \qquad U_\mu V^\mu = g_{\mu\nu} \overset{(1,0,0,0)}{U^\mu} V^\mu = V^0.$

$$K^2 = a^2(V^{0\,2} - 1).$$

$$\frac{dK^2}{d\lambda} = 2a\frac{da}{d\lambda}(V^{0\,2}-1) + a^2 \cdot 2V^0 \underbrace{\frac{dV^0}{d\lambda}}_{} \qquad - \Gamma^0_{\alpha\beta} V^\alpha V^\beta$$

$$\Gamma^t_{rr} = \frac{\dot a}{a(1-kr^2)}, \qquad \Gamma^t_{\theta\theta} = a\dot a r^2.$$

$\lambda \leftrightarrow \tau$

$$\Rightarrow \quad 2a\frac{da}{d\lambda}(V^{0\,2}-1) + a^2 \cdot 2V^0\left(-\Gamma^t_{rr} V^r V^r - \Gamma^t_{\theta\theta} V^\theta V^\theta\right).$$

$$2a\dot a(V^{0\,2}-1) - 2a^2 V^0 \left(\frac{\dot a}{a}\frac{1}{1-kr^2} V^{r\,2} + a\dot a r^2 V^{\theta\,2}\right)$$

$$= 2a\dot a\left(V^{0\,2} - V^0 \frac{V^{r\,2}}{1-kr^2} - V^0 a^2 r^2 V^{\theta\,2} - 1\right) \qquad = 0\ ?$$

$V^\mu = (V^0, V^i). \qquad -1 = g_{\mu\nu} V^\mu V^\nu = -V^{0\,2} + \delta_{ij} V^i V^j.$

From $\dot K$ above, let's say $\quad K^2 = a^2 V_\mu V^\mu.$

(b) $$|\vec{V}| = \sqrt{V_i V^i} = \frac{k}{a(t)}.$$

from textbook ↙

$$E = -P_\mu U^\mu = -P_0 U^0 = -P_0.$$

$$P_i = P_\mu V_i^\mu. \qquad \delta^{ij} P_i P_j = \delta^{ij} P_\mu V_i^\mu P_\nu V_j^\nu$$

$$= P_\mu P_\nu \left(-U^\mu U^\nu + \delta^{ij} V_i^\mu V_j^\nu + U^\mu U^\nu\right)$$

$$= P_\mu P_\nu g^{\mu\nu} + E^2 = -m^2 + E^2.$$

$$U_\mu V^\mu = U_0 V^0 = \frac{k}{a(t)}.$$

is inversely proportional to scaling factor $a(t)$.

thus photon energy $E \sim -P_\mu U^\mu$ redshift on expansion.

(c) $$a(t) = e^{Ht}.$$

$$ds^2 = -dt^2 + a^2(t)\left(\frac{dr^2}{1-kr^2} + r^2 d\theta^2\right).$$

$$\int_t^{t_0} \frac{dt}{a(t)} = \int_0^R \frac{dr}{\sqrt{1-kr^2}} \approx R.$$

$$\int e^{-Ht} dt = \frac{1}{H}(1 - e^{-Ht}) \approx R.$$

$$\Rightarrow \quad R = \frac{1}{H}. \qquad \text{horizon.}$$

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1. (20 pts) Recall that the connection coefficient, $\Gamma^{\rho}_{\mu\nu}$ has the transformation property:

$$\Gamma^{\rho'}_{\mu'\nu'} = \left(\frac{\partial x^{\rho'}}{\partial x^{\rho}}\right)\left[\frac{\partial x^{\mu}}{\partial x^{\mu'}}\frac{\partial x^{\nu}}{\partial x^{\nu'}}\Gamma^{\rho}_{\mu\nu} + \frac{\partial^2 x^{\rho}}{\partial x^{\mu'}\partial x^{\nu'}}\right]. \tag{1}$$

a) Show that this transformation properties implies that the covariant derivative the components of a vector:

$$\nabla_{\mu}V^{\nu} = \partial_{\mu}V^{\nu} + \Gamma^{\nu}_{\mu\rho}V^{\rho}. \tag{2}$$

is a tensor of type $(1,1)$.

b) Show that the difference of any two affine connections:

$$S^{\rho}{}_{\mu\nu} \equiv \widetilde{\Gamma}^{\rho}_{\mu\nu} - \Gamma^{\rho}_{\mu\nu}$$

is a tensor of type $(1,2)$. In general, we can always strip out of the connection the Christoffel part, and the leftover bit is a tensor that is independently defined (and not built out of the metric and its derivatives).

(a) To show $\nabla_\mu V^\nu$ (1,1) type tensor, we look into its coordinate transformation properties. $x^\rho \to x^{\rho'}$. $V^{\nu'} = \frac{\partial x^{\nu'}}{\partial x^{\nu}} V^{\nu}$.

$$\partial_{\mu'} V^{\nu'} = \frac{\partial x^{\mu}}{\partial x^{\mu'}}\frac{\partial}{\partial x^{\mu}}\left(\frac{\partial x^{\nu'}}{\partial x^{\nu}} V^{\nu}\right)$$

$$= \frac{\partial x^{\mu}}{\partial x^{\mu'}}\frac{\partial x^{\nu'}}{\partial x^{\nu}}\frac{\partial V^{\nu}}{\partial x^{\mu}} + \frac{\partial x^{\mu}}{\partial x^{\mu'}}\frac{\partial^2 x^{\nu'}}{\partial x^{\mu}\partial x^{\nu}} V^{\nu}$$

$$\nabla_{\mu'} V^{\nu'} = \partial_{\mu'} V^{\nu'} + \Gamma^{\nu'}_{\mu'\rho'} V^{\rho'}$$

$$= \underline{\frac{\partial x^{\mu}}{\partial x^{\mu'}}\frac{\partial x^{\nu'}}{\partial x^{\nu}}}\frac{\partial V^{\nu}}{\partial x^{\mu}} + \frac{\partial x^{\mu}}{\partial x^{\mu'}}\frac{\partial^2 x^{\nu'}}{\partial x^{\mu}\partial x^{\nu}} V^{\nu}$$

$$+ \frac{\partial x^{\nu'}}{\partial x^{\nu}}\left(\frac{\partial x^{\mu}}{\partial x^{\mu'}}\frac{\partial x^{\rho}}{\partial x^{\rho'}}\Gamma^{\nu}_{\mu\rho} + \frac{\partial^2 x^{\nu}}{\partial x^{\mu'}\partial x^{\rho'}}\right)\frac{\partial x^{\rho'}}{\partial x^{\rho}} V^{\rho}$$

$$= \frac{\partial x^{\mu}}{\partial x^{\mu'}} \frac{\partial x^{\nu'}}{\partial x^{\nu}} \frac{\partial V^{\nu}}{\partial x^{\mu}} + \frac{\partial x^{\mu}}{\partial x^{\mu'}} \frac{\partial^2 x^{\nu'}}{\partial x^{\mu} \partial x^{\nu}} V^{\nu}$$

cancell out ?

actually add together

$$+ \frac{\partial x^{\nu'}}{\partial x^{\nu}} \frac{\partial x^{\mu}}{\partial x^{\mu'}} \Gamma^{\nu}_{\mu\rho} V^{\rho} + \frac{\partial x^{\nu'}}{\partial x^{\nu}} \frac{\partial^2 x^{\nu}}{\partial x^{\mu'} \partial x^{\rho'}} \frac{\partial x^{\rho'}}{\partial x^{\rho}} V^{\rho}$$

ν' ν'

$\mu'\nu$ $\mu'\rho$

$$\frac{\partial x^{\mu}}{\partial x^{\mu'}} \left(\frac{\partial^2 x^{\nu'}}{\partial x^{\mu} \partial x^{\nu}} V^{\nu} + \frac{\partial^2 x^{\nu'}}{\partial x^{\mu} \partial x^{\rho}} V^{\rho} \right) = 0 \ ?$$

$$\Rightarrow \nabla_{\mu'} V^{\nu'} = \frac{\partial x^{\mu}}{\partial x^{\mu'}} \frac{\partial x^{\nu'}}{\partial x^{\nu}} \left(\frac{\partial V^{\nu}}{\partial x^{\mu}} + \Gamma^{\nu}_{\mu\rho} V^{\rho} \right).$$

that's how (1, 1) type transforms.

(b) $S^{\rho'}_{\mu'\nu'} = \tilde{\Gamma}^{\rho'}_{\mu'\nu'} - \Gamma^{\rho'}_{\mu'\nu'}$

$$= \frac{\partial x^{\rho'}}{\partial x^{\rho}} \left(\frac{\partial x^{\mu}}{\partial x^{\mu'}} \frac{\partial x^{\nu}}{\partial x^{\nu'}} \tilde{\Gamma}^{\rho}_{\mu\nu} + \frac{\partial^2 x^{\rho}}{\partial x^{\mu'} \partial x^{\nu'}} \right) - \frac{\partial x^{\rho'}}{\partial x^{\rho}} \left(\frac{\partial x^{\mu}}{\partial x^{\mu'}} \frac{\partial x^{\nu}}{\partial x^{\nu'}} \Gamma^{\rho}_{\mu\nu} + \frac{\partial^2 x^{\rho}}{\partial x^{\mu'} \partial x^{\nu'}} \right)$$

$$= \frac{\partial x^{\rho'}}{\partial x^{\rho}} \frac{\partial x^{\mu}}{\partial x^{\mu'}} \frac{\partial x^{\nu}}{\partial x^{\nu'}} S^{\rho}_{\mu\nu}. \quad \underbrace{+ \frac{\partial x^{\rho'}}{\partial x^{\rho}} \frac{\partial^2 x^{\rho}}{\partial x^{\mu'} \partial x^{\nu'}} - \frac{\partial x^{\rho'}}{\partial x^{\rho}} \frac{\partial^2 x^{\rho}}{\partial x^{\mu'} \partial x^{\nu'}}}_{=0}$$

that's tensor type (1, 2).

2. (30 pts) One way of constructing curved spaces is to embed them in a higher dimensional space, *e.g.* the two-sphere embedded in $\mathbb{R}^3$ as the surface of fixed radius. Consider an m-dimensional manifold $\mathcal{M}$ with metric components g_{ij} in some coordinate system x^i, $i = 1, ..., m$. Let $\Sigma \subset \mathcal{M}$ be an n-dimensional submanifold, *i.e.* Σ is itself a manifold embedded in $\mathcal{M}$ by a map $\phi : \Sigma \to \mathcal{M}$. Let y^a, $a = 1, ..., n$ be a system of coordinates on Σ. Then we can use the embedding functions $x^i(y^a)$ to *induce* a metric on Σ via the *pull-back* $\phi^*(g)$, essentially the projection of the metric tensor on $\mathcal{M}$ onto Σ using the embedding:

$$ds^2_\Sigma = (\phi^* g)_{ab}\, dy^a dy^b = \left[g_{ij}(x(y)) \frac{\partial x^i}{\partial y^a} \frac{\partial x^j}{\partial y^b} \right] dy^a dy^b$$

a) A one-form (*i.e.* cotangent vector field) $\tilde{v}$ in $\mathcal{M}$ is an element of the cotangent space for each point in $\mathcal{M}$. The components v_i of the one-form in a coordinate basis $\tilde{e}^i$ can be turned into the components of a one-form v_a on Σ via the *pull-back*

$$(\phi^* v)_a = \frac{\partial x^i}{\partial y^a} v_i$$

Show that v_a indeed transforms as the components of a one-form on Σ.

b) Find the analogous formula for the pullback of an arbitrary $(0, k)$ tensor $\mathcal{T}$.

c) Consider flat $\mathbb{R}^3$ in spherical polar coordinates. Find the induced metric on the sphere of radius R.

d) Consider the hyperboloid $\eta_{\mu\nu} x^\mu x^\nu = a^2$ in 3d Minkowski spacetime. Show that

$$(x^0, x^1, x^2) = a(\text{sinh}t, \text{cosh}t \cos\phi, \text{cosh}t \sin\phi)$$

is a valid parametrization of this hyperboloid. Find the corresponding induced metric on this 2d Lorentzian hyperboloid Σ, known as two-dimensional de Sitter space. Compute the Christoffel symbols, Riemann tensor, Ricci tensor, and Ricci scalar of Σ.

e) Show that the metrics of the two-sphere $\mathbb{S}^2$ and the de Sitter space Σ are related by the analytic continuation $\theta = it + \pi/2$. This relation is sometimes useful in relating analytic computations on the sphere to corresponding calculations in de Sitter space (in particle physics this sort of continuation is known as *Wick rotation*).

(Note: it is by no means necessary to embed a curved space in fa higher-dimensional lat space, in fact sometimes such an embedding is impossible; Riemannian geometry is intrinsic to the space in question and does not refer to any embedding in another space. Rather, the idea here is that, given an embedding, curved geometry can be constructed by projecting the ambient (in this case flat) geometry onto the hypersurface using the embedding.)

(a) Prove $V_a = (\phi^* v)_a = \frac{\partial x^i}{\partial y^a} V_i$ is one-form on Σ.

definition of (0,1) tensor field $T_a = T_\mu (dx^\mu)_a$.

For one-form on M, $\tilde{v} = V_i dx^i$.

$$= \underbrace{\frac{\partial x^i}{\partial y^a} V_i}_{V_a} dy^a$$

$\Rightarrow \tilde{v} = V_a dy^a$

also one-form.

(b) As hinted by $(\phi^* g)_{ab} = g_{ij}(x(y)) \frac{\partial x^i}{\partial y^a} \frac{\partial x^j}{\partial y^b}$. (0,2)

let's write $\phi^* T = \prod_r \frac{\partial x^{i_r}}{\partial y^{a_r}} T_{1,\dots k}$. (0, k)

(c) From higher to lower dimension.

$\mathbb{R}^3$ sphere surface

x^i: $x = R \sin\theta \cos\phi$

$y = R \sin\theta \sin\phi$

$z = R \cos\theta$

y^a: θ, ϕ.

$$\Rightarrow \quad ds_\Sigma^2 = \left(\frac{\partial x_i}{\partial \theta}\right)^2 d\theta^2 + \left(\frac{\partial x_i}{\partial \phi}\right)^2 d\phi^2$$

$$= (R^2\cos^2\theta + R^2 \sin^2\theta)\, d\theta^2 + R^2 \sin^2\theta\, d\phi^2$$

$$= R^2 d\theta^2 + R^2 \sin^2\theta\, d\phi^2.$$

(d) $\eta_{\mu\nu} x^\mu x^\nu = -a^2 \sinh^2 t + a^2 \cosh^2 t \cos^2\phi + a^2 \cosh^2 t \sin^2\phi$

$= a^2$. so is valid.

$(\phi^* g)_{ab} = g_{ij}(x(y)) \frac{\partial x^i}{\partial y^a} \frac{\partial x^j}{\partial y^b}$.

$x^0 = a \sinh t$

$x^1 = a \cosh t \cos\phi$

$x^2 = a \cosh t \sin\phi$.

$y^a : t$

$y^b : \phi$

$g_{tt}^* = -a^2 \cosh^2 t + a^2 \sinh^2 t = -a^2$.

$g_{\phi\phi}^* = a^2 \cosh^2 t$.

$$\Rightarrow \quad ds^2_\Sigma = -a^2 dt^2 + a^2 \cosh^2 t \, d\phi^2 .$$

$$\Gamma^i_{kl} = \frac{1}{2} h^{im} \left(\frac{\partial h_{mk}}{\partial x^l} + \frac{\partial h_{ml}}{\partial x^k} - \frac{\partial h_{kl}}{\partial x^m} \right)$$

$$R^i_{klm} = \frac{\partial \Gamma^i_{km}}{\partial x^l} - \frac{\partial \Gamma^i_{kl}}{\partial x^m} + \Gamma^i_{nl} \Gamma^n_{km} - \Gamma^i_{nm} \Gamma^n_{kl} . \qquad R_{ik} = g^{lm} R_{limk} .$$

$$\Gamma^t_{t\phi} = \frac{1}{2} h^{tt} \frac{\partial h_{tt}}{\partial \phi} = 0 . \qquad \Gamma^\phi_{t\phi} = \frac{1}{2} h^{\phi\phi} \frac{\partial h_{\phi\phi}}{\partial t} = \frac{1}{2} \frac{1}{a^2 \cosh^2 t} 2a^2 \cosh t \sinh t$$

$$= \tanh t .$$

$$\Gamma^t_{\phi\phi} = \frac{1}{2} h^{tt} \cdot - \frac{\partial h_{\phi\phi}}{\partial t} = \ominus \frac{1}{2a^2} \cdot - 2a^2 \cosh t \sinh t = -\cosh t \sinh t .$$

$$R^\phi_{t\phi t} = \frac{-\partial \Gamma^\phi_{t\phi}}{\partial t} + 0 = \frac{1}{\cosh^2 t} . = R_{tt} .$$

$$R^t_{\phi t \phi} = \frac{\partial \Gamma^t_{\phi\phi}}{\partial t} - \Gamma^t_{t\phi} \Gamma^t_{\phi t} = -\cosh^2 t - \sinh^2 t = 1 - 2\cosh^2 t . = R_{\phi\phi} .$$

$$R = g^{tt} R_{tt} + g^{\phi\phi} R_{\phi\phi}$$

$$= -\frac{1}{a^2 \cosh^2 t} + \frac{1}{a^2 \cosh^2 t} (1 - 2\cosh^2 t) = -\frac{2}{a^2} .$$

(e) As we already have

$$S^2 : \quad ds^2_\Sigma = R^2 d\theta^2 + R^2 \sin^2 \theta \, d\phi^2 .$$

$$\text{de Sitter} : \quad ds^2_\Sigma = -a^2 dt^2 + a^2 \cosh^2 t \, d\phi^2 .$$

$$\theta = it + \frac{\pi}{2} \longrightarrow \quad ds^2 = R^2(-dt^2) + R^2 \sin^2\left(it + \frac{\pi}{2}\right) d\phi^2$$

$$\xrightarrow{R=a} \quad = -a^2 dt^2 + a^2 \cos^2(it)\, d\phi^2$$

$$\cos x = \frac{e^{ix} + e^{-ix}}{2}. \qquad \cos(it) = \frac{e^{-t} + e^{t}}{2} = \cosh t.$$

equal.

3. (20 pts) Prove the following variational identities:

$$\begin{aligned}
\delta g^{\alpha\beta} &= -g^{\alpha\mu}\,\delta g_{\mu\nu} g^{\nu\beta} \\
\delta(\det g) &= (\det g)\, g^{\alpha\beta}\delta g_{\alpha\beta} \\
\delta\Gamma^{\alpha}_{\beta\gamma} &= \frac{1}{2} g^{\alpha\mu}\left(\nabla_\beta\,\delta g_{\mu\gamma} + \nabla_\gamma\,\delta g_{\beta\mu} - \nabla_\mu\,\delta g_{\beta\gamma}\right) \\
\delta R^{\alpha}{}_{\beta\mu\nu} &= \nabla_\mu\,\delta\Gamma^{\alpha}{}_{\beta\nu} - \nabla_\nu\,\delta\Gamma^{\alpha}{}_{\beta\mu}
\end{aligned} \tag{3}$$

1. Let's start from $g^{\mu\rho} g_{\rho\nu} = \delta^{\mu}_{\nu}$.

$$\Rightarrow (\delta g^{\mu\rho}) g_{\rho\nu} + g^{\mu\rho}\,\delta(g_{\rho\nu}) = 0.$$

$$\delta g^{\mu\rho} g_{\rho\nu} g^{\nu\lambda} = -g^{\mu\rho}\,\delta g_{\rho\nu}\, g^{\nu\lambda} \longrightarrow \delta g^{\mu\lambda} = -g^{\mu\rho}\,\delta g_{\rho\nu}\, g^{\nu\lambda}.$$

that is $\delta g^{\alpha\beta} = -g^{\alpha\mu}\,\delta g_{\mu\nu}\, g^{\nu\beta}$.

2. $\delta(\det g) = ?$ $\quad \det g = \prod_i \lambda_i$. $\quad \ln \det g = \sum_i \ln\lambda_i = \mathrm{Tr}(\ln g)$.

$$\Rightarrow \delta(\det g) = (\det g)\cdot \delta\,\mathrm{Tr}(\ln g) = \det g \cdot \underline{\mathrm{Tr}\left(\frac{\delta g}{g}\right)}.$$

$$= \det g \cdot \underline{g^{\alpha\beta}\,\delta g_{\alpha\beta}}.$$

3. Start from $\Gamma^{\alpha}_{\beta\gamma} = \frac{1}{2} g^{\alpha m}\left(\frac{\partial g_{m\beta}}{\partial x^\gamma} + \frac{\partial g_{m\gamma}}{\partial x^\beta} - \frac{\partial g_{\beta\gamma}}{\partial x^m}\right)$

$$\delta\Gamma^{\alpha}_{\beta\gamma} = \frac{1}{2}\,\delta g^{\alpha m}\left(\frac{\partial g_{m\beta}}{\partial x^\gamma} + \frac{\partial g_{m\gamma}}{\partial x^\beta} - \frac{\partial g_{\beta\gamma}}{\partial x^m}\right) + \frac{1}{2} g^{\alpha m}\left(\frac{\partial\,\delta g_{m\beta}}{\partial x^\gamma} + \frac{\partial\,\delta g_{m\gamma}}{\partial x^\beta} - \frac{\partial\,\delta g_{\beta\gamma}}{\partial x^m}\right)$$

$\delta g^{\alpha m} = -g^{\alpha\mu}\,\delta g_{\mu\nu}\, g^{\nu m}$. neglected ?

Let's use metric compatibility

$$\nabla_\lambda g_{\mu\nu} = \partial_\lambda g_{\mu\nu} - \Gamma^\rho_{\lambda\mu} g_{\rho\nu} - \Gamma^\rho_{\lambda\nu} g_{\mu\rho} = 0.$$

$$\nabla_\lambda g_{\mu\nu} \longleftrightarrow \partial_\lambda g_{\mu\nu}.$$

We have $$\delta\Gamma^\alpha_{\beta\gamma} = \frac{1}{2} g^{\alpha\mu}\left(\nabla_\beta \delta g_{\mu\gamma} + \nabla_\gamma \delta g_{\beta\mu} - \nabla_\mu \delta g_{\beta\gamma}\right).$$

4. $$R^\alpha_{\beta\mu\nu} = \frac{\partial \Gamma^\alpha_{\beta\nu}}{\partial x^\mu} - \frac{\partial \Gamma^\alpha_{\beta\mu}}{\partial x^\nu} + \Gamma^\alpha_{n\mu}\Gamma^n_{\beta\nu} - \Gamma^\alpha_{n\nu}\Gamma^n_{\beta\mu}.$$

$$\delta R^\alpha_{\beta\mu\nu} = \partial_\mu \underline{\delta\Gamma^\alpha_{\beta\nu}} - \partial_\nu \delta\Gamma^\alpha_{\beta\mu} + \delta\Gamma^\alpha_{n\mu}\underline{\Gamma^n_{\beta\nu}} + \Gamma^\alpha_{n\mu}\underline{\delta\Gamma^n_{\beta\nu}} - \delta\Gamma^\alpha_{n\nu}\Gamma^n_{\beta\mu} - \Gamma^\alpha_{n\nu}\delta\Gamma^n_{\beta\mu}$$

Since $$\nabla_\mu \Gamma^\alpha_{\beta\nu} = \partial_\mu \Gamma^\alpha_{\beta\nu} + \Gamma^\alpha_{\mu\lambda}\Gamma^\lambda_{\beta\nu} - \Gamma^\lambda_{\mu\beta}\Gamma^\alpha_{\lambda\nu} - \Gamma^\lambda_{\mu\nu}\Gamma^\alpha_{\beta\lambda}.$$

and $$\delta\Gamma^\alpha_{\beta\gamma} = \frac{1}{2} g^{\alpha\mu}\left(\nabla_\beta \delta g_{\mu\gamma} + \nabla_\gamma \delta g_{\beta\mu} - \nabla_\mu \delta g_{\beta\gamma}\right).$$

We have $$\nabla_\mu \delta\Gamma^\alpha_{\beta\nu} = \partial_\mu \delta\Gamma^\alpha_{\beta\nu} + \Gamma^\alpha_{\mu\lambda}\delta\Gamma^\lambda_{\beta\nu} - \Gamma^\lambda_{\mu\beta}\delta\Gamma^\alpha_{\lambda\nu} - \Gamma^\lambda_{\mu\nu}\delta\Gamma^\alpha_{\beta\lambda}.$$

$$\delta R^\alpha_{\beta\mu\nu} = \begin{pmatrix} \partial_\mu \delta\Gamma^\alpha_{\beta\nu} + \Gamma^\alpha_{\lambda\mu}\delta\Gamma^\lambda_{\beta\nu} \\ - \Gamma^\lambda_{\mu\beta}\delta\Gamma^\alpha_{\lambda\nu} - \Gamma^\lambda_{\mu\nu}\delta\Gamma^\alpha_{\beta\lambda} \end{pmatrix} - \begin{pmatrix} \partial_\nu \delta\Gamma^\alpha_{\beta\mu} + \Gamma^\alpha_{\nu\lambda}\delta\Gamma^\lambda_{\beta\mu} \\ - \Gamma^\lambda_{\nu\beta}\delta\Gamma^\alpha_{\lambda\mu} - \Gamma^\lambda_{\nu\mu}\delta\Gamma^\alpha_{\beta\lambda} \end{pmatrix}$$

$$+ \Gamma^\lambda_{\mu\beta}\delta\Gamma^\alpha_{\lambda\nu} + \Gamma^\lambda_{\mu\nu}\delta\Gamma^\alpha_{\beta\lambda} + \delta\Gamma^\alpha_{\lambda\mu}\Gamma^\lambda_{\beta\nu} + \Gamma^\alpha_{\lambda\mu}\delta\Gamma^\lambda_{\beta\nu} + \Gamma^\lambda_{\nu\beta}\delta\Gamma^\alpha_{\lambda\mu} + \Gamma^\lambda_{\nu\mu}\delta\Gamma^\alpha_{\beta\lambda} - \delta\Gamma^\alpha_{\lambda\nu}\Gamma^\lambda_{\beta\mu} - \Gamma^\alpha_{\lambda\nu}\delta\Gamma^\lambda_{\beta\mu}$$

cancell out

$$= \nabla_\mu \delta\Gamma^\alpha_{\beta\nu} - \nabla_\nu \delta\Gamma^\alpha_{\beta\mu}.$$

4. (30 pts) A good approximation to the metric outside the surface of the earth is provided by

$$ds^2 = -(1+2\Phi)dt^2 + (1-2\Phi)\left[dx^2 + dy^2 + dz^2\right]$$

where

$$\Phi = -\frac{GM}{r}$$

is the Newtonian gravitational potential and

$$r^2 = x^2 + y^2 + z^2 \quad .$$

(Note: This metric is expressed in slightly different coordinates as compared to HW4 problem 2.)

An astronaut takes a spacewalk untethered from their spaceship. Suppose the spaceship is traveling a circular orbit in the plane of the equator

$$x = r_0 \cos \omega t \ , \qquad y = r_0 \sin \omega t \ .$$

a) Show that this orbit solves the geodesic equations. What is the orbital period ω?

b) Derive the equation of geodesic deviation for the separation

$$\xi^i = x^i_{\text{astronaut}} - x^i_{\text{spaceship}}$$

as a function of time t, working to leading order in the weak field limit $\Phi \ll 1$ and assuming non-relativistic motion – evaluate the Riemann curvature tensor that appears in the geodesic equation, in this approximation.

c) Change coordinates to the frame co-rotating with the spaceship (*i.e.* the coordinate frame in which the spaceship is always located at the origin of coordinates, with coordinate axes pointing in the instantaneous radial, azimuthal, and polar directions). Solve the equations of geodesic deviation for vanishing initial relative velocity $(d\xi/dt)_{t=0} = 0$, and relative displacement $\xi|_{t=0}$ (i) only in the polar direction; and (ii) only in the radial direction. (*Hint: Look for solutions oscillating with period ω, and for secular (non-oscillating) solutions growing at most linearly in t; then match boundary conditions. The intuition here is thatwhile the orbit of the spaceship is a circle, the orbit of the astronaut is an ellipse that, relative to the circle, spends half the orbit at smaller radius and halve the orbit at larger radius, and so in the co-rotating frame is oscillating up and down.*)

d) Verify your result by examining the Keplerian orbit of the astronaut, showing that one obtains the same result in the given approximation.

(a) $$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\rho\sigma}\frac{dx^\rho}{d\tau}\frac{dx^\sigma}{d\tau} = 0 .$$

$$\Gamma^\alpha_{\beta\gamma} = \frac{1}{2}g^{\alpha m}\left(\frac{\partial g_{m\beta}}{\partial x^\gamma} + \frac{\partial g_{m\gamma}}{\partial x^\beta} - \frac{\partial g_{\beta\gamma}}{\partial x^m}\right)$$

$$\Gamma^\alpha_{tt} = \frac{1}{2}g^{\alpha\alpha}\cdot -\frac{\partial g_{tt}}{\partial x^\alpha} .$$

$$\Gamma^x_{tt} = \frac{1}{2} g^{xx} \cdot - \frac{\partial g_{tt}}{\partial x} = \frac{1}{2}(1-2\phi) \cdot 2 \frac{\partial \phi}{\partial x}$$

$$= \left(1 + \frac{2GM}{r}\right) \cdot \frac{-GM}{r_0^3} x .$$

$$\Gamma^y_{tt} = \left(1 + \frac{2GM}{r}\right) \cdot - \frac{GM}{r_0^3} y .$$

Also, $\frac{dt}{d\tau} \approx 1$. $\frac{dx}{d\tau} = -\omega r_0 \sin\omega t$. $\frac{dy}{d\tau} = \omega r_0 \cos\omega t$.

$$\frac{d^2x}{d\tau^2} = -\omega^2 r_0 \cos\omega t. \quad \frac{d^2y}{d\tau^2} = -\omega^2 r_0 \sin\omega t.$$

Back to equation, $\frac{d^2 x^i}{d\tau^2} - \left(1 + \frac{2GM}{r}\right) \frac{GM}{r_0^3} x^i = 0$. $|\Phi| \ll 1$.

$$\sim \frac{dx^i}{d\tau^2} = \frac{GM}{r_0^3} x^i .$$ is our orbit.

period $\omega = \sqrt{\frac{GM}{r_0^3}}$.

(b) general form of geodesic deviation:

$$\frac{D}{d\tau} \frac{D \delta x^\mu}{d\tau} + R^\mu_{\nu\sigma\lambda} \delta x^\sigma \frac{dx^\nu}{d\tau} \frac{dx^\lambda}{d\tau} = 0.$$

$$\Rightarrow \frac{D^2 \delta x^\mu}{d\tau^2} = R^\lambda_{\mu\nu\sigma} U^\mu U^\nu \delta x^\sigma . \quad ?$$

below solution from online reference

$$R^i_{0j0} = \partial_0 \Gamma^i_{0j} - \partial_j \Gamma^i_{00} . = -\partial_j \partial^i \Phi .$$

$$\frac{D^2\xi^i}{D\tau^2} = R^i_{\ 0j0}\,\xi^j = -\partial^i\partial_j\Phi\,\xi^j.$$

$$= -\frac{GM}{r^3}\left(3\frac{x^i x^j}{r^2} - \delta_{ij}\right)\xi^j.$$

(c) $R^r_{\ 0r0} = \frac{2GM}{r^3}$. $\quad R^\theta_{\ 0\theta 0} = -\frac{GM}{r^3}$.

$$\Rightarrow \quad \ddot{\xi}^r = -\frac{2GM}{r^3}\xi^r. \qquad \ddot{\xi}^\theta = \frac{GM}{r^3}\xi^\theta.$$

$$\Rightarrow \xi^r(t) = \xi^r(0)\cos\left(\sqrt{\frac{2GM}{r_0^3}}\,t\right).$$

$$\xi^\theta(t) = A\cos\omega t + B\sin\omega t.$$

(d) let's use $\quad r = \frac{P}{1+e\cos\theta}. \qquad h = r^2\dot{\theta}.$

eqn of motion : $\quad \ddot{r} - r\dot{\theta}^2 = -\frac{GM}{r^2}.$

$r = r_0. \qquad \dot{\theta} = \sqrt{\frac{GM}{r_0^3}}.$

If with perturbation $\quad r = r_0 + \xi^r. \qquad \theta = \theta_0 + \xi^\theta.$

$$\Rightarrow \quad \ddot{\xi}^r = -\frac{2GM}{r^3}\xi^r. \qquad \ddot{\xi}^\theta = \frac{GM}{r^3}\xi^\theta.$$

consistent.

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1. (25 pts) (See the lecture notes, pp. 48-50, *i.e.* Lecture 10)

A *p-form* is a totally antisymmetric tensor of type $(0, p)$. The *exterior* (or *wedge product* of a p-form $\mathcal{T}$ and a k-form $\mathcal{W}$ is the $p + k$ form $\mathcal{T} \wedge \mathcal{W}$ obtained by taking the tensor product and totally antisymmetrizing on all the indices in a basis. Finally, the *exterior derivative* $d\mathcal{W}$ of a k-form $\mathcal{W}$ is a $(k + 1)$-form defined by the completely antisymmetrized covariant derivative of $\mathcal{W}$ (one can check that the choice of a connection Γ drops out of this definition due to the antisymmetry, so d is independent of the choice of metric on $\mathcal{M}$). (See for instance Lecture 10 of the class notes for a few more details.)

The generalization of Stokes' theorem is that the integral of a total differential $(k+1)$-form $d\mathcal{W}$ over a $(k + 1)$-dimensional submanifold Σ is equal to the integral of $\mathcal{W}$ over its boundary $\partial\Sigma$

$$\int_\Sigma d\mathcal{W} = \int_{\partial\Sigma} \mathcal{W}$$

a) Prove this result. Note that you already used the 2d version of this result in the magnetic monopole problem in HW 3, where the integral of the magnetic field strength on either hemisphere reduced to an integral of the vector potential on the equator.

b) Specialize this expression to three dimensions, with $\mathcal{W}$ a one-form. Show that one recovers the familiar result

$$\int (\nabla \times \mathbf{A}) \cdot d\mathbf{S} = \oint \mathbf{A} \cdot d\ell$$

where $d\mathbf{S}$ is the surface element on a two-dimensional submanifold, and the contour integral on the RHS runs over its boundary.

(a) k-form W could be written as:

$$W = \frac{1}{k!} W_{\mu_1 \cdots \mu_k} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_k}.$$

its exterior derivative:

$$dW = \frac{1}{k!} \partial_\nu W_{\mu_1 \cdots \mu_k} dx^\nu \wedge dx^{\mu_1} \cdots \wedge dx^{\mu_k}.$$

$$\int_\Sigma dW = \int_\Sigma \frac{1}{k!} \underline{\partial_\nu W_{\mu_1 \cdots \mu_k}}\ \underline{dx^\nu} \wedge dx^{\mu_1} \cdots \wedge dx^{\mu_k}$$

$$? = \int_{\partial\Sigma} \frac{1}{k!} W_{\mu_1 \cdots \mu_k} dx^{\mu_1} \cdots \wedge dx^{\mu_k} = \int_{\partial\Sigma} W.$$

(b) For 1-form $W = W_\mu dx^\mu$. $\quad dW = \partial_\nu W_\mu \, dx^\nu \wedge dx^\mu$.

$$\Rightarrow \int_\Sigma \partial_\nu W_\mu \, dx^\nu \wedge dx^\mu = \int_{\partial\Sigma} W_\mu dx^\mu$$

Set W_μ as components of 1-form $\vec{A}$. $\quad \partial_\nu W_\mu \to \nabla \times \vec{A}$.

We recover $\iint_\Sigma (\nabla \times \vec{A}) \cdot d\vec{S} = \int_{\partial\Sigma} \vec{A} \cdot d\vec{l}$.

not quite. should be

$$\oint f_\mu dx^\mu = \iint (\partial_\mu f_\nu - \partial_\nu f_\mu) \, d\sigma^{\mu\nu}.$$

should use $\quad d\Sigma_\mu = \sqrt{|g|}\, \varepsilon_{\mu\nu\kappa\lambda} \, dV^{\nu\kappa\lambda}$.

$$dS_{\mu\nu} = \sqrt{|g|}\, \varepsilon_{\mu\nu\kappa\lambda} \, d\sigma^{\kappa\lambda}. \qquad ?$$

2. (25 pts) *Energy bounds*

a) Show that the energy-momentum tensor of the electromagnetic field satisfies the dominant energy condition, and thus also the weak and null energy conditions.

b) Consider a massive scalar field $\phi(x)$ whose action in $n+1$ dimensions is

$$\mathcal{S}_{\text{scalar}} = \int d^{n+1}x\,\sqrt{-g}\left[-\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi - \frac{m^2}{2}\phi^2\right]$$

Derive the energy-momentum tensor $T^{\mu\nu}$ of the scalar. Assuming that the scalar is spatially homogeneous ($\phi = \phi(t)$) and the only source of stress-energy, show that the stress tensor has the form of a perfect fluid, and find the energy density ρ and pressure P. Under what conditions on ϕ, $\partial_t\phi$ does the scalar field stress-energy satisfy (i) the null energy condition? (ii) the weak energy condition? (iii) the strong energy condition?

(a) $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$.

$$T^{\mu\nu} = \frac{1}{\mu_0}\left(F^{\mu}_{\ \sigma} F^{\nu\sigma} - \frac{1}{4}g^{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}\right).$$

dominant energy condition: $T_{\mu\nu}v^\mu v^\nu \geq 0$. $T^{\mu\nu}v_\nu$ timelike.

$$T_{\mu\nu}v^\mu v^\nu = \frac{1}{\mu_0}\left(v^\mu v^\nu F^{\mu}_{\ \sigma}F^{\nu\sigma} - \frac{1}{4}v^\mu v_\mu F_{\rho\sigma}F^{\rho\sigma}\right)$$

use
$$F_{\mu\nu} = \begin{pmatrix} & -\frac{1}{c}E_1 & -\frac{1}{c}E_2 & -\frac{1}{c}E_3 \\ \frac{1}{c}E_1 & & B_3 & -B_2 \\ \frac{1}{c}E_2 & -B_3 & & B_1 \\ \frac{1}{c}E_3 & B_2 & -B_1 & \end{pmatrix}.$$

$F^{\mu} = g^{\mu\rho}g^{\nu\sigma}F_{\rho\sigma}$.

$F^{0i} = -E^i/c$.

$F^{ij} = \varepsilon_{ijk}B_k$.

$$F_{\rho\sigma}F^{\rho\sigma} = 2\left(|\vec{B}|^2 + \frac{|\vec{E}|^2}{c^2}\right).$$

$$F^{\nu}_{\ \sigma}F^{\nu\sigma} = F^{0}_{\ 0}F^{00} + F^{0}_{\ i}F^{0i} + F^{j}_{\ k}F^{jk} + F^{i}_{\ 0}F^{j0}$$

$$= |\vec{E}|^2/c^2 + \varepsilon_{ikn}B_n\,\varepsilon_{jkm}B_m + E^iE^j/c^2$$

$$= \frac{1}{c^2}\left(|\vec{E}|^2 + E^i E^j\right) + \left(\delta^i_j \delta^n_m - \delta^i_m \delta^n_j\right) B_n B_m$$

$$= \frac{1}{c^2}\left(|\vec{E}|^2 + E^i E^j\right) + |\vec{B}|^2 \delta^i_j - B^i B^j$$

$$T^{\mu\nu} = \frac{1}{\mu_0}\left(F^{\mu}_{\ \sigma} F^{\nu\sigma} - \frac{1}{4} g^{\mu\nu} F_{\rho\sigma} F^{\rho\sigma}\right).$$

$$= \frac{1}{\mu_0}\left[\frac{1}{c^2}\left(|\vec{E}|^2 + E^i E^j\right) + |\vec{B}|^2 \delta^i_j - B^i B^j - \frac{1}{2}\left(|\vec{B}|^2 + \frac{|\vec{E}|^2}{c^2}\right)\right]$$

$$T^{00} = \frac{1}{2\mu_0}\left(|\vec{B}|^2 + |\vec{E}|^2/c^2\right).$$

$$T^{0j} = \frac{1}{\mu_0}(\vec{E}\times\vec{B})^j \quad ?$$

$$T^{ij} = \frac{1}{\mu_0}\left(E^i E^j/c^2 - B^i B^j - \frac{1}{2}\delta^{ij}\left(|\vec{E}|^2 + |\vec{B}|^2\right)\right).$$

$$T^{\mu\nu} v_\nu = T^{\mu 0} v_0 + T^{\mu i} v_i$$

$$= \frac{1}{2\mu_0}\left(|\vec{E}|^2/c^2 + |\vec{B}|^2\right) + \frac{1}{\mu_0}(\vec{E}\times\vec{B})^i v_i + \frac{1}{\mu_0}\left(E^i E^j + B^i B^j - \frac{1}{2}\delta^{ij}\left(|\vec{E}|^2 + |\vec{B}|^2\right)\right) v_j$$

$$(T^{\mu\nu} v_\nu)\, g_{\mu\lambda}\, (T^{\lambda\rho} v_\rho) = -(T^{0\nu} v_\nu)^2 + (T^{i\nu} v_\nu)(T^{i\rho} v^\rho) \overset{?}{\leq} 0. \qquad \text{time-like.}$$

to get dominant condition.

should directly calculate $T^{\mu\nu} v_\mu v_\nu$.

$$T^{\mu\nu} v_\mu v_\nu = \frac{1}{\mu_0}\Big(\underbrace{F^{\nu}_{\ \sigma} F^{\mu\sigma} v_\mu v_\nu}_{\downarrow} - \frac{1}{4} g^{\mu\nu} F_{\rho\sigma} F^{\rho\sigma} v_\mu v_\nu\Big).$$

$$v_0 v_0 F^{0\sigma} F^{0}_{\ \sigma} + 2 v_0 v_i F^{0\sigma} F^{i}_{\ \sigma} + v_i v_j F^{i\sigma} F^{j}_{\ \sigma}$$

$\delta^i_j \delta^n_m - \delta^i_m \delta^n_j$

$$= v_0^2 |\vec{E}|^2/c^2 + 2 v_0 v_i \Big(-\frac{E^{\sigma}}{c} \varepsilon^{ij\sigma} B_\sigma\Big) + v_i v_j \big(\varepsilon_{ikn} B_n \varepsilon_{jkm} B_m\big)$$

$$= v_0^2 |\vec{E}|^2/c^2 - 2 v_0 v_i \varepsilon^{ijk} E^k B_k + v_i v_j \big(|\vec{B}|^2 \delta^i_j - B^i B^j\big)$$

$$\frac{1}{4} g^{\mu\nu} F_{\rho\sigma} F^{\rho\sigma} v_\mu v_\nu$$

$$= \frac{1}{2} g^{\mu\nu} \big(|\vec{B}|^2 - |\vec{E}|^2/c^2\big) v_\mu v_\nu = \frac{1}{2}\big(|\vec{B}|^2 - |\vec{E}|^2/c^2\big)\big(v_0^2 - |\vec{v}|^2\big)$$

$\Rightarrow$

$$T^{\mu\nu} v_\mu v_\nu = \frac{1}{\mu_0}\Big[v_0^2 |\vec{E}|^2/c^2 - 2 v_0 v_i \varepsilon^{ijk} E^k B_k + v_i v_j \big(|\vec{B}|^2 \delta^i_j - B^i B^j\big) + \frac{1}{2}\big(|\vec{B}|^2 - |\vec{E}|^2/c^2\big)\big(v_0^2 - |\vec{v}|^2\big)\Big]$$

$$= \frac{1}{\mu_0}\Big[\frac{1}{2}\big(|\vec{E}|^2/c^2 + |\vec{B}|^2\big)\big(v_0^2 + |\vec{v}|^2\big) - (\vec{B}\cdot\vec{v})^2 - 2 v_0 v_i \varepsilon^{ijk} E^k B_k\Big] \overset{?}{\geq} 0$$

maybe rearrange to square form

then we have dominat condition.

with $v^0 > |\vec{v}|$, $T^{\mu\nu} v_\mu v_\nu \geq 0$. to get weak condition.

$v^0 = |\vec{v}|$. $T^{\mu\nu} v_\mu v_\nu \geq 0$. to get null condition.

(b) $\delta\sqrt{-g} = \frac{1}{2}\sqrt{-g}\, g^{\mu\nu}\,\delta g_{\mu\nu}$.

For $S = \int d^{n+1}x\, \mathcal{L}\sqrt{-g}$. $\quad \delta(\mathcal{L}\sqrt{-g}) = \sqrt{-g}\left(\frac{\delta\mathcal{L}}{\delta g_{\mu\nu}} + \frac{1}{2}g^{\mu\nu}\mathcal{L}\right)\delta g_{\mu\nu}$.

$$T^{\mu\nu} = \frac{-2}{\sqrt{-g}}\frac{\delta S}{\delta g_{\mu\nu}} = -2\underset{\sim\sim}{\frac{\partial\mathcal{L}}{\partial g_{\mu\nu}}} - g^{\mu\nu}\mathcal{L}.$$

$$\mathcal{L} = -\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi - \frac{m^2}{2}\phi^2.$$

$$\frac{\partial\mathcal{L}}{\partial g_{\mu\nu}} = -\frac{1}{2}\delta g^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi/\delta g_{\mu\nu} = \frac{1}{2}g^{\mu\alpha}g^{\nu\beta}\partial_\alpha\phi\,\partial_\beta\phi.$$

$$\Rightarrow T^{\mu\nu} = -g^{\mu\alpha}g^{\nu\beta}\partial_\alpha\phi\,\partial_\beta\phi + \frac{1}{2}g^{\mu\nu}\left(g^{\alpha\beta}\partial_\alpha\phi\,\partial_\beta\phi + m^2\phi^2\right).$$

$T^{00} = -\frac{1}{2}\dot{\phi}^2 - \frac{1}{2}m^2\phi^2$. $\quad \rho$

$T^{0i} = 0$. $\quad (\partial_i\phi = 0) \qquad (\rho + P)v^\mu v^\nu + P g^{\mu\nu}$

$T^{ij} = -\frac{1}{2}g_{ij}\dot{\phi}^2 + \frac{1}{2}\delta_{ij}m^2\phi^2 \quad P\delta_{ij}$

$$T^{\mu\nu}v_\mu v_\nu = T^{00}v_0 v_0 + T^{ij}v_i v_j$$

$$= -\frac{1}{2}(\dot{\phi}^2 + m^2\phi^2)\underbrace{v_0^2}_{1+|\vec{v}|^2} - \frac{1}{2}(\dot{\phi}^2 - m^2\phi^2)|\vec{v}|^2$$

$$= -\dot{\phi}^2|\vec{v}|^2 - \frac{1}{2}(\dot{\phi}^2 + m^2\phi^2).$$

sign not right?

with k^μ zero vector.

$$T^{\mu\nu}k_\mu k_\nu = \frac{1}{2}(\dot{\phi}^2+m^2\phi^2) \geq 0.$$ satisfy zero energy condition.

with v^μ time like,

$$T^{\mu\nu}v_\mu v_\nu = \dot{\phi}^2|\vec{v}|^2 + \frac{1}{2}(\dot{\phi}^2+m^2\phi^2) \geq 0.$$

satisfy weak energy condition.

$$\left(T^{\mu\nu}-\frac{1}{2}g_{\mu\nu}T\right)v_\mu v_\nu$$

$$= \dot{\phi}^2|\vec{v}|^2 + \frac{1}{2}(\dot{\phi}^2+m^2\phi^2) - \frac{1}{2}\left(\dot{\phi}^2+\frac{n}{2}(\dot{\phi}^2+m^2\phi^2)\right)$$

$$= \left(\frac{-n}{2}+|\vec{v}|^2\right)\dot{\phi}^2 + \frac{1-n}{2}m^2\phi^2 \geq 0.$$

satisfy strong energy condition when equation hold.

3. (30 pts) Fluid dynamics

Consider a relativistic fluid with the stress tensor

$$T^{\mu\nu} = (\rho + p)u^\mu u^\nu + pg^{\mu\nu} \quad .$$

Here u^μ is the four-velocity of the fluid flow, ρ its energy density and p its pressure density.

(a) Use conservation of the stress tensor $\nabla_\mu T^{\mu\nu} = 0$ together with conservation of particle number $\nabla_\mu(nu^\mu) = 0$ (where n is the number density of particles) to show that

$$\frac{d\rho}{d\tau} - \frac{\rho + p}{n}\frac{dn}{d\tau} = 0 \tag{1}$$

$$(\rho + p)a_\mu + \nabla_\mu p + u_\mu \frac{dp}{d\tau} = 0 \tag{2}$$

where $d/d\tau = u^\mu \nabla_\mu$ is the derivative with respect to proper time along a fluid element's worldline, and a is the 4-acceleration.

(b) Consider steady flows in the absence of gravity, for which there is some Lorentz frame $(t, \mathbf{x})$ for which all the hydrodynamic variables are independent of t. Evaluate the time component of (2) in this frame and combine the result with (1) to show that

$$\frac{d}{d\tau}\left[u^t\left(\frac{\rho + p}{n}\right)\right] = 0$$

Show that the non-relativistic limit of this result is Bernoulli's theorem: That the quantity

$$\frac{1}{2}\mathbf{v}^2 + U + \frac{p}{\rho_M}$$

is conserved along flow lines, where U is the internal energy per unit mass and ρ_M is the mass density [related to ρ and n by $\rho = \rho_M(1 + U)$ and $\rho_M = mn$ for particles of mass m].

(a) Use $T^{\mu\nu} = (\rho+p)u^\mu u^\nu + pg^{\mu\nu}$. $\quad \nabla_\mu T^{\mu\nu} = 0$.

$$\Rightarrow \nabla_\mu T^{\mu\nu} = \nabla_\mu\left[(\rho+p)u^\mu u^\nu\right] + \nabla_\mu p\, g^{\mu\nu} + p\underbrace{\nabla_\mu g^{\mu\nu}}_{\text{consider } = 0} = 0.$$

$* u_\nu$

$$\Rightarrow u_\nu \nabla_\mu T^{\mu\nu} = u_\nu \nabla_\mu\left[(\rho+p)u^\mu u^\nu\right] + u_\nu \nabla^\nu p$$

$$= u_\nu\left(\nabla_\mu(\rho+p)u^\nu\right)u^\mu + u_\nu(\rho+p)u^\nu\nabla_\mu u^\mu + u_\nu\nabla^\nu p$$

$$= \nabla_\mu\left[(\rho+p)u^\nu\right] - (\rho+p)\nabla_\mu u^\mu + u_\nu\nabla^\nu p$$

$$= -\nabla_\mu\left[(\rho+p)u^\mu\right] + \underline{(\rho+p)u^\mu u^\nu\nabla_\mu u_\nu} + u_\nu\nabla^\nu p$$

$$= -u_\nu\nabla^\nu\rho - (\rho+p)\nabla_\mu u^\mu = 0. \qquad (*)$$

Plug back in

$$\nabla_\mu T^{\mu\nu} = \nabla_\mu\left[(\rho+p)\, u^\mu u^\nu\right] + g^{\mu\nu}\nabla_\mu p = 0.$$

$$\nabla_\mu\left[(\rho+p)\, u^\mu u^\nu\right] + g^{\mu\nu}\nabla_\mu p$$

$$= \left(g^{\mu\nu} + u^\mu u^\nu\right)\nabla_\mu p + \underline{u^\mu u^\nu \nabla_\mu \rho} + (\rho+p)\, u^\mu \nabla_\mu u^\nu + (\rho+p)\, u^\nu \nabla_\mu u^\mu$$

$$= \left(g^{\mu\nu} + u^\mu u^\nu\right)\nabla_\mu p \underline{- u^\nu (\rho+p)\nabla_\mu u^\mu} + (\rho+p)\, u^\mu \nabla_\mu u^\nu + \underline{(\rho+p)\, u^\nu \nabla_\mu u^\mu}$$

$$= \left(g^{\mu\nu} + u^\mu u^\nu\right)\nabla_\mu p + (\rho+p)\, u^\mu \nabla_\mu u^\nu = 0.$$

$$\Rightarrow \underbrace{u^\mu \nabla_\mu u^\nu}_{a_\mu} = -\frac{1}{\rho+p}\left(g^{\mu\nu} + u^\mu u^\nu\right)\nabla_\mu p.$$

this is $\quad (\rho+p)\, a_\mu + \left(g^{\mu\nu} + u^\mu u^\nu\right)\nabla_\mu p = 0.$

$$(\rho+p)\, a_\mu + \nabla_\mu p + u_p \frac{dp}{d\tau} = 0. \quad \text{get second eqn.}$$

Also, from $\nabla_\mu (n u^\mu) = 0$, $\quad \nabla_\mu n\, u^\mu + n \nabla_\mu u^\mu = 0.$

$$\Rightarrow \nabla_\mu u^\mu = -\frac{1}{n}\nabla_\mu n = -\frac{1}{n}\frac{dn}{d\tau}.$$

use $\quad -u_\nu \nabla^\nu \rho - (\rho+p)\nabla_\mu u^\mu = 0. \quad (*)$

To have $\quad u_\nu \nabla^\nu \rho = \frac{\rho+p}{n}\frac{dn}{d\tau}. \quad \Rightarrow \quad \frac{d\rho}{d\tau} - \frac{\rho+p}{n}\frac{dn}{d\tau} = 0.$

get first eqn.

(b) $(\rho+p)a_\mu + \nabla_\mu p + u_p \frac{dp}{d\tau} = 0.$

t component $\quad (\rho+p)a^t + \underbrace{\nabla^t p}_{\approx 0} + u^t\frac{dp}{d\tau} = 0.$

for steady flow $\frac{\partial}{\partial t} f = 0. \quad \Rightarrow (\rho+p)u^\nu \nabla_\nu u^t + u^t \frac{dp}{d\tau} = 0.$

since $\frac{d}{d\tau} \to u^\nu \nabla_\nu. \quad \Rightarrow \frac{d}{d\tau} u^t(\rho+p) = 0.$

$$\frac{d}{d\tau}\left[u^t \frac{\rho+p}{n}\right] = 0.$$

is desired equation.

non-relativistic limit:

$u^t = \gamma \approx 1. \qquad u^t \approx 1 + \frac{1}{2}u^i u^i.$

$\frac{1}{2}\vec{v}^2 + U + \frac{p}{\rho_m} \stackrel{?}{=} \text{const}.$

$\rho = \rho_m(1+U). \quad \rho_m = mn. \quad \frac{\rho}{1+U} = mn$

$$\Rightarrow u^t\left(\frac{\rho+p}{n}\right) \approx \frac{\rho+p}{n} + \frac{1}{2}\vec{v}^2\frac{\rho+p}{n} = \text{const}.$$

$$\frac{1}{2}\vec{v}^2 + U + \frac{p}{\rho_m} = \frac{1}{2}\vec{v}^2 + U + \frac{p}{\rho}(1+U)$$

express in ρ_m

$$= \frac{1}{2}\vec{v}^2 + \frac{\rho+p}{\rho}U + \frac{p}{\rho}.$$

$$\frac{1}{2}\vec{v}^2 + U + \frac{p}{\rho_m} = \frac{1}{2}\vec{v}^2 + \frac{\rho}{mn} - 1 + \frac{p}{mn}$$

$$= \frac{1}{2}\vec{v}^2 + \frac{\rho}{mn} - 1 + \frac{k_B T}{m}.$$

$$\frac{\rho+p}{n} = \frac{mn(1+U)+p}{n}$$

?

4. (30 pts) FRW Cosmology

Consider again the spatially homogeneous metric ansatz from HW4 problem 3

$$ds^2 = -dt^2 + a^2(t)\left(\frac{dr^2}{1-kr^2} + r^2 d\Omega_{n-1}^2\right)$$

and the associated Friedmann equations, which are the Einstein equations coupled to a perfect fluid energy-momentum tensor specialized to the assumption of spatial homogeneity (note that the Killing vectors related to this homogeneity allow us to drop the dependence of the metric on the associated coordinates, and allow us to write the field equations in terms of only the scale factor a and its time derivatives)

$$H^2 = \frac{16\pi G}{n(n-1)}\rho - \frac{k}{a^2}$$

$$\partial_t H = -\frac{8\pi G}{n-1}(\rho + P) + \frac{k}{a^2}$$

where n is the spatial dimension, and $H = (\partial_t a)/a$ is the local logarithmic expansion rate of the spatial geometry (HW4 problem 3 specialized to $n = 2$, but the above expressions are generally valid).

(a) Derive these equations for $n = 2$ (you may use the answers to the previous problems).

(b) Find the equation of motion of the homogeneous scalar field of problem 2 above to find a closed system of differential equations for homogeneous cosmology. The homogeneous scalar dynamics should have the form of a damped harmonic oscillator, with the damping proportional to H. This dynamical damping is known as *Hubble friction*.

(c) There are two qualitatively distinct dynamical regimes, underdamped and overdamped motion of the scalar (roughly whether the acceleration or Hubble friction term is the most important time derivative term in the ϕ equation of motion, respectively). Consider first the overdamped motion. Show that in this regime, the scale factor a is changing much faster than the scalar field is evolving, by showing that $H^2 \gg G\dot{\phi}^2$. In this case we can approximate, over time scales set by H (the so-called *Hubble time*), that $\phi \sim$ *const*, and therefore the $m^2\phi^2$ term in the action behaves (to a good approximation) like a cosmological constant over scales of order the Hubble time. Solve the equations of motion in this approximation, taking into account $\dot{\phi}$ but ignoring $\ddot{\phi}$. This regime of cosmological dynamics is known as *slow-roll inflation*.

(d) Now consider the underdamped regime. Show that for $G\phi^2 \ll 1$, the scalar field oscillates on a time scale much smaller than the Hubble time H^{-1}. In this situation, it makes sense to time-average the field motion to get an averaged equation of state, where $\bar{\rho}$ and $\bar{P}$ are the energy density and pressure of the homogeneous scalar, averaged over a few oscillation periods. Show that $\bar{P} \ll \bar{\rho}$, and solve the Friedmann equations to leading order in this locally time-averaged approximation. Such a homogeneous but rapidly oscillating (relative to cosmological time scales) scalar has featured in certain models of dark matter using a scalar field known as the *axion*.

(a) Follow derivation from book

Einstein eqn : $R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$.

Use $T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T = \frac{1}{2}(\rho - p) g_{\mu\nu} + (\rho + p) u_\mu u_\nu = S_{\mu\nu}$.

$$\Rightarrow \quad R_{\mu\nu} = 8\pi G S_{\mu\nu} + \Lambda g_{\mu\nu}. \qquad u^i = 0, \ u^0 = 1.$$

Comoving coordinate system

For FLRW metric

$$ds^2 = -dt^2 + a^2(t)\left(\frac{dr^2}{1-kr^2} + r^2 d\Omega^2\right)$$

$$R_{00} = -n\frac{\ddot{a}}{a}, \qquad R_{ij} = \left(\frac{\ddot{a}}{a} + (n-1)\frac{\dot{a}^2}{a^2} + (n-1)\frac{k}{a^2}\right) g_{ij}.$$

$$R = 2n\frac{\ddot{a}}{a} + n(n-1)\left(\frac{\dot{a}^2}{a^2} + \frac{k}{a^2}\right).$$

For ideal fluid, $T_{\mu\nu} = (\rho + p) u_\mu u_\nu + p g_{\mu\nu}$.

In FRW, $T_{00} = \rho$, $T_{ij} = p g_{ij}$.

$$S_{\mu\nu} = T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T.$$

$$S_{00} = \frac{n(n-1)}{2}\frac{\dot{a}^2}{a^2} + n\frac{k}{a^2} = \frac{8\pi G}{c^4}\rho. \qquad H = \frac{\dot{a}}{a}.$$

$$\Rightarrow \quad H^2 = \frac{16\pi G}{n(n-1)c^4}\rho - \frac{k}{a^2}. \qquad (*)$$

$$S_{ij} = -\left[(n-1)\frac{\ddot{a}}{a} + \frac{n(n-1)}{2}\frac{\dot{a}^2}{a^2} + (n-1)\frac{k}{a^2}\right] g_{ij} = \frac{8\pi G}{c^4} T_{ij}.$$

$$\Rightarrow \quad \partial_t H = \frac{\ddot{a}}{a} = -\frac{8\pi G}{n(n-1)c^4}(\rho + p) + \frac{k}{a^2}. \qquad (**)$$

Specialize to $n=2$.

$$\Rightarrow \quad H^2 = \frac{8\pi G}{c^4}\rho - \frac{k}{a^2}.$$

$$\frac{\ddot{a}}{a} = -\frac{8\pi G}{2c^4}(\rho+p) + \frac{k}{a^2}.$$

(b) Follow standard derivation

$$S = \int d^4x \sqrt{-g}\left[-\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi - V(\phi)\right].$$

FRW $\quad ds^2 = -dt^2 + a^2(t)\,d\vec{x}^2. \quad \rightarrow \quad \sqrt{-g} = a^3.$

$$\mathcal{L} = a^3\left[-\frac{1}{2}(\partial_t\phi)^2 - V(\phi)\right].$$

$$\frac{\partial\mathcal{L}}{\partial\dot{\phi}} = a^3\dot{\phi}. \qquad \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\phi}} = a^3\ddot{\phi} + 3a^2\dot{a}\dot{\phi}.$$

$$\frac{\partial\mathcal{L}}{\partial\phi} = -a^3V'(\phi). \qquad \Rightarrow \quad a^3\ddot{\phi} + 3a^2\dot{a}\dot{\phi} + a^3V'(\phi) = 0.$$

friction term $\quad \ddot{\phi} + \underbrace{3H\dot{\phi}} + V'(\phi) = 0.$

(c) overdamped regime, $|3H\dot{\phi}| \gg |\ddot{\phi}|$.

(From online reference)

$$\Rightarrow \quad \dot{\phi} \approx -\frac{V'(\phi)}{3H}.$$

back to Friedmann eqn

$$H^2 = \frac{8\pi G\rho}{3}. \qquad \rho = \frac{1}{2}\dot{\phi}^2 + V(\phi). \qquad \gg \frac{1}{2}\dot{\phi}^2$$

$$\Rightarrow H^2 \sim \frac{8\pi G V(\phi)}{3} \gg G\dot{\phi}^2.$$

slow-roll inflation

$\phi \sim$ const.

For $V(\phi) \sim m^2\phi^2$. like cosmological constant.

(d) For underdamped regime,

$$\ddot{\phi} + \underbrace{3H\dot{\phi}}_{\text{small}} + V'(\phi) = 0 \quad \rightarrow \quad \ddot{\phi} + m^2\phi = 0. \quad \phi = \phi_0 \cos\omega t.$$

$$\rho = \frac{1}{2}\dot{\phi}^2 + V(\phi). \quad P = \frac{1}{2}\dot{\phi}^2 - V(\phi).$$

$$\Rightarrow \rho = \frac{1}{2}\langle\dot{\phi}^2\rangle + \langle V(\phi)\rangle$$

$$= \frac{1}{4}\omega^2\phi_0^2 + \frac{1}{4}\omega^2\phi_0^2 = \frac{1}{2}\omega^2\phi_0^2.$$

$$P = \frac{1}{2}\langle\dot{\phi}^2\rangle - \langle V(\phi)\rangle = 0.$$

$$\frac{\bar{P}}{\bar{\rho}} \sim 0.$$ cold dark matter.

Frieldmann eqn $\quad H^2 \approx \frac{8\pi G}{3}\frac{1}{2}m^2\phi_0^2.$ dark matter.

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1. (25 pts) *Deflection of light by the Sun*

To high precision, the Sun is static and spherical, and we may approximate the line element outside it by the weak-field form

$$ds^2 = -(1 - 2GM/r)dt^2 + (1 + 2GM/r)(dx^2 + dy^2 + dz^2)$$

A photon moving in the equatorial plan $z = 0$ gets deflected very slightly from the worldline

$$x = t \ , \quad y = b = \text{"}impact\ parameter\text{"} \ , \quad z = 0$$

a) Write down the geodesic equation for the photon's worldline (parametrized by λ), and evaluate the connection coefficients $\Gamma^\mu_{\nu\rho}$ that enter into it.

b) Using the approximation $|p^y| \ll p^t \approx p^x$, show that the geodesic equations can be written

$$\frac{dp^y}{d\lambda} = -\frac{2GMb}{(x^2 + b^2)^{3/2}} p^x \frac{dx}{d\lambda} \tag{1}$$

c) Integrate this equation, with the initial condition $p^y = 0$ at $x = -\infty$, to find the deflection angle $\delta\phi = p^y/p^x$ at $x = +\infty$. Evaluate your result for light deflected by the Sun at grazing incidence $b \approx R_\odot = 7.0 \times 10^{10} cm$, $GM_\odot = 1.5 \times 10^5 cm$.

(a) General form of geodesic equation:

$$\frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu_{\nu\rho} \frac{dx^\nu}{d\lambda}\frac{dx^\rho}{d\lambda} = 0.$$

$$g : \begin{pmatrix} -\left(1-\frac{2GM}{r}\right) & \\ & 1+\frac{2GM}{r} \end{pmatrix}$$

$$\Gamma^i_{kl} = \frac{1}{2} g^{im}\left(\frac{\partial g_{mk}}{\partial x^l} + \frac{\partial g_{ml}}{\partial x^k} - \frac{\partial g_{kl}}{\partial x^m}\right)$$

$$\Gamma^t_{tt} = \frac{1}{2} g^{tt} \frac{\partial g_{tt}}{\partial t} = 0.$$

$$\Gamma^t_{ti} = \frac{1}{2} g^{tt}\left(\frac{\partial g_{tt}}{\partial x^i} + \frac{\partial g_{ti}}{\partial t} - \frac{\partial g_{ti}}{\partial t}\right) = -\left(1-\frac{2GM}{r}\right)^{-1}\frac{GM}{r^3}x^i.$$

$$\Gamma^t_{ij} = \frac{1}{2}g^{tt}\left(\frac{\partial g_{ti}}{\partial x^j} + \frac{\partial g_{tj}}{\partial x^i} - \frac{\partial g_{ij}}{\partial t}\right) = 0.$$

$$\Gamma^i_{tt} = \frac{1}{2} g^{ii} \left(\frac{\partial g_{it}}{\partial t} + \frac{\partial g_{it}}{\partial t} - \frac{\partial g_{tt}}{\partial x^i} \right) = -\left(1 + \frac{2GM}{r}\right)^{-1} \frac{GM}{r^3} x^i .$$

$$\Gamma^i_{tj} = \frac{1}{2} g^{ii} \left(\frac{\partial g_{it}}{\partial x^j} + \frac{\partial g_{ij}}{\partial t} - \frac{\partial g_{tj}}{\partial x^i} \right) = 0 .$$

$$\Gamma^i_{jk} = \frac{1}{2} g^{ii} \left(\frac{\partial g_{ij}}{\partial x^k} + \frac{\partial g_{ik}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^i} \right) = \left(1 + \frac{2GM}{r}\right)^{-1} \cdot \frac{GM}{r^3} \left(\delta_{ij} x^k + \delta_{ik} x^j - \delta_{jk} x^i \right)$$

(b) come back to $\frac{d^2 x^\mu}{d\lambda^2} + \Gamma^\mu_{\nu\rho} \frac{dx^\nu}{d\lambda} \frac{dx^\rho}{d\lambda} = 0$. and use $p^\mu = \frac{dx^\mu}{d\lambda}$.

$$\Rightarrow \quad \frac{d^2 x^i}{d\lambda^2} + \Gamma^i_{tt} \left(\frac{dt}{d\lambda} \right)^2 + \Gamma^i_{jk} \frac{dx^j}{d\lambda} \frac{dx^k}{d\lambda} = 0$$

$$\frac{dP^i}{d\lambda} + \Gamma^i_{tt} p^{t^2} + \Gamma^i_{jk} p^j p^k = 0 .$$

for y direction, $\frac{dp^y}{d\lambda} + \Gamma^y_{tt} P_t^2 + \underbrace{\Gamma^y_{jk} p^j p^k}_{} = 0$. $z = 0$. j.k. y x.

$$\Gamma^y_{tt}: \quad -\left(1 + \frac{2GM}{r}\right)^{-1} \frac{GM}{r^3} y \approx -\frac{GM}{r^3} y$$

$$\Gamma^y_{jk} p^j p^k: \quad \left(1 + \frac{2GM}{r}\right)^{-1} \cdot \frac{GM}{r^3} \left(\delta_{yj} x^k + \delta_{yk} x^j - \delta_{jk} y \right) \approx \frac{GM}{r^3} y \, p^{y^2} + 2 \frac{GM}{r^3} x p^x p^y - \frac{GM}{r^3} y \, p^{x^2}$$

use $|p^y| \ll p^t \approx p^x$.

small

$$\Rightarrow \quad \frac{dp^y}{d\lambda} \underline{- \frac{GM}{r^3} y \, p^{t^2}} + \underline{\frac{GM}{r^3} y \, p^{y^2}} + \underline{2 \frac{GM}{r^3} x p^x p^y} \underline{- \frac{GM}{r^3} y \, p^{x^2}} = 0 .$$

$$\Rightarrow \quad \frac{dp^y}{d\lambda} = \frac{2GM}{r^3} y \, p^{x^2} = -\frac{2GMb}{(x^2 + b^2)^{\frac{3}{2}}} p^x \frac{dx}{d\lambda} .$$

consists.

(c) Find deflection angle $\delta\phi = \frac{p^y}{p^x}$ at $x = +\infty$.

set $\lambda \to x$. $\frac{dp^y}{dx} = -\frac{2GMb}{(x^2+b^2)^{\frac{3}{2}}}$.

$$p^y = \int_{-\infty}^{+\infty} -\frac{2GMb}{(x^2+b^2)^{\frac{3}{2}}}\,dx = ? \qquad x = b\tan\theta. \qquad \cos^3\theta \frac{1}{\cos^2\theta}\,d\theta$$

$$\to \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} -\frac{2GM}{b}\cos\theta\,d\theta = -\frac{4GM}{b}.$$

Plug in numbers $b \approx R_\odot = 7.0\times10^{10}\,cm$.

$GM_\odot = 1.5\times10^5\,cm$.

$$\delta\phi = -\frac{4GM}{b} = \frac{-4\times1.5\times10^5}{7\times10^{10}} = -\frac{6}{7}\times10^{-5} \approx 8.57\times10^{-6}\ \text{rad}.$$

2. (25 pts) *The perihelion precession of Mercury*

Consider the metric of problem 1 as an approximation to the gravitational field experienced by the planet Mercury, and treat Mercury as a small test body orbiting in the equatorial plane $z = 0$ (note: you may find it easiest to work in spherical spatial coordinates). From the property $\underset{\sim}{u} \cdot \underset{\sim}{u} = -1$ of the four-velocity $\underset{\sim}{u}$, and the fact that p_t and p_ϕ are conserved (because the Lagrangian of a test body is independent of the corresponding generalized coordinates), show that the remaining radial dynamics obeys a constraint of the form

$$\mathcal{E} = \frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 + \mathcal{V}_{\text{eff}}(r)$$

where $\mathcal{E}$ and $\mathcal{V}_{\text{eff}}$ are expressed in terms of p_t, p_ϕ and the constants G, $M_\odot$. Compare the effective potential $\mathcal{V}_{\text{eff}}$ to that of Newtonian mechanics for the same problem to find that the two effective potentials differ.

Keeping only the leading term(s) in the expansion for weak fields and non-relativistic velocities, estimate the amount by which the potential has been perturbed by comparing the magnitude of the additional term(s) in $\mathcal{V}_{\text{eff}}$ to the magnitudes of the terms in the Newtonian effective potential. The average orbital radius of Mercury is $r_0 = .387$ times the Earth-Sun distance $1AU = 1.5 \times 10^{13} cm$.

Keplerian planetary motion is such that the elliptical orbits don't precess, *i.e.* the orbit's perihelion (*i.e.* minimum radius of the orbit) occurs at a fixed angle ϕ_0 that doesn't change in time. In other words, for an elliptical Keplerian orbit, the angular frequancy of the radial oscillation equals the angular frequency of the azimuthal motion. The perturbation of the effective potential changes the dynamics; the period of the radial motion shifts, and so during the time it takes for the radial motion to go from perihelion back to perihelion, the change in the azimuthal angle is less than 2π. One can process the estimate of the change in energy to a change in the period of the radial motion, but you need not do this. This particular contribution to the perihelion shift accounts for most of the obsserved "anomalous" 43"/century shift in ϕ_0 for Mercury. A full GR treatment accounts for all of it.

We continue to use weak Schwartz metric

$$ds^2 = -\left(1 - \frac{2GM}{r}\right)dt^2 + \left(1 + \frac{2GM}{r}\right)\left(dx^2 + dy^2 + dz^2\right) \qquad z = 0$$

write in spherical coordinate:

$$ds^2 = -\left(1 - \frac{2GM}{r}\right)dt^2 + \left(1 + \frac{2GM}{r}\right)dr^2 + \underbrace{r^2 d\theta^2}_{=0} + \underbrace{r^2 \sin^2\theta \, d\phi^2}_{r^2 d\phi^2}$$

four velocity $\underset{\sim}{u} = \frac{dx^\mu}{d\tau} = (u^t, u^r, 0, u^\phi)$.

$$p^t = g_{tt} u^t = \underline{-\left(1 - \frac{2GM}{r}\right)u^t} . \quad E\,? \quad \text{not quite}$$

$$p^\phi = g_{\phi\phi} u^\phi = r^2 u^\phi .$$ conserved L.

$$\underset{\sim}{u} \cdot \underset{\sim}{u} = g_{\mu\nu} u^\mu u^\nu$$

$$= g_{tt} {u^t}^2 + g_{rr} {u^r}^2 + g_{\phi\phi} {u^\phi}^2$$

$$= E \cdot \frac{-E}{1-\frac{2GM}{r}} + \left(1+\frac{2GM}{r}\right) {u^r}^2 + L \cdot \frac{L}{r^2} = -1 .$$

$$\Rightarrow \quad \left(1+\frac{2GM}{r}\right) {u^r}^2 = \frac{E}{1-\frac{2GM}{r}} - \frac{L^2}{r^2} - 1 . \quad (*)$$

(*) could actually be applied further approximation ?

compare to $\mathcal{E} = \frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 + V_{eff}(r)$.

$$\frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 = \frac{1}{2} E \left(1-\frac{2GM}{r}\right)^{-1}\left(1+\frac{2GM}{r}\right)^{-1} - \left(\frac{L^2}{2r^2}+\frac{1}{2}\right)\left(1+\frac{2GM}{r}\right)^{-1} = E - V_{eff} .$$

$$\approx \frac{1}{2} E \left(1+\frac{4G^2M^2}{r^2}\right) - \frac{L^2}{2r^2} - \frac{1}{2} + \left(\frac{L^2}{2r^2}+\frac{1}{2}\right)\frac{2GM}{r} = E - V_{eff}$$

$$V_{eff} = \frac{1}{2}E + \frac{2G^2M^2E}{r^2} + \frac{L^2}{2r^2} + \frac{1}{2} - \frac{GML^2}{r^3} - \frac{GM}{r}$$

correction terms

$$\Delta V = \frac{1}{2}E + \frac{2G^2M^2E}{r^2} + \frac{1}{2} - \frac{GML^2}{r^3}$$

$$r_0 = 0.387 \times 1\,AU = 0.387 \times 1.5 \times 10^{13}\ cm .$$

$$GM_\odot = 1.5 \times 10^5\ cm .$$

Let's start over again

We continue to use weak Schwartz metric

$$ds^2 = -\left(1-\frac{2GM}{r}\right)dt^2 + \left(1+\frac{2GM}{r}\right)(dx^2+dy^2+dz^2) \qquad z=0$$

write in spherical coordinate:

$$ds^2 = -\left(1-\frac{2GM}{r}\right)dt^2 + \left(1+\frac{2GM}{r}\right)dr^2 + \underbrace{r^2 d\theta^2}_{=0} + \underbrace{r^2\sin^2\theta\, d\phi^2}_{r^2 d\phi^2}$$

four velocity $\underset{\sim}{u} = \frac{dx^\mu}{d\tau} = (u^t, u^r, 0, u^\phi)$.

$$p^t = g_{tt}u^t = -\underline{\left(1-\frac{2GM}{r}\right)u^t}. \qquad E\ ? \quad \text{not quite}$$

$$p^\phi = g_{\phi\phi}u^\phi = r^2 u^\phi. \qquad \text{conserved } L.$$

$$\underset{\sim}{u}\cdot\underset{\sim}{u} = g_{\mu\nu}u^\mu u^\nu$$

$$= g_{tt}{u^t}^2 + g_{rr}{u^r}^2 + g_{\phi\phi}{u^\phi}^2$$

$$= E\cdot\frac{-E}{1-\frac{2GM}{r}} + \left(1+\frac{2GM}{r}\right){u^r}^2 + L\cdot\frac{L}{r^2} = -1.$$

$$\Rightarrow \left(1+\frac{2GM}{r}\right){u^r}^2 = \frac{E}{1-\frac{2GM}{r}} - \frac{L^2}{r^2} - 1.$$

notice $\dot{r} = \frac{dr}{d\tau} = \frac{d\varphi}{d\tau}\frac{dr}{d\varphi} = \frac{L}{r^2}\frac{dr}{d\varphi}$. maybe previous result is right

$$\Delta V = \frac{2G^2M^2E}{r^2}.$$

$$\frac{\Delta V}{-\frac{GM}{r}} = \frac{2GME}{r}$$

$r_\circ = 0.387 \times 1.5 \times 10^{13}$ cm.

$$\approx \frac{2GM}{rc^2} = 5.75 \times 10^{-25}.$$

$GM_\odot = 1.5 \times 10^{5}$ cm.

change in energy $\rightarrow$ change in period of radial motion ?

GR account for all of it.

3. (25 pts) *Gravitational analogues of electromagnetic fields*

a) Derive equation 4.4.24 of Wald: The nonrelativistic, weak field approximation to the gravitational dynamics of a test body is given by

$$\mathbf{a} = -\mathbf{E} - 4\mathbf{v} \times \mathbf{B}$$

where $\mathbf{a}$ is the acceleration of the test body, and $\mathbf{E}, \mathbf{B}$ are expressions analogous to the electric and magnetic fields in electromagnetism, but with the vector potential A_μ replaced by $\bar{\gamma}_{0\mu}$.

b) Show that the "gravitational electric and magnetic fields" $\mathbf{E}$ and $\mathbf{B}$ inside a spherical shell of mass M and radius R (with $GM \ll R$), which is slowly rotating with angular velocity $\boldsymbol{\omega}$, are given by

$$\mathbf{E} = 0 \ , \qquad \mathbf{B} = \frac{2GM}{3R}\boldsymbol{\omega}$$

c) An observer at the center of this shell parallel propagates along their geodesic a vector $\underset{\sim}{v}$, with $\underset{\sim}{v} \cdot \underset{\sim}{u} = 0$; here $\underset{\sim}{u}$ is the tangent vector to their worldline. Show the the spatial components $\mathbf{v}$ precess according to

$$\frac{d\mathbf{v}}{dt} = \boldsymbol{\Omega} \times \mathbf{v}$$

where $\boldsymbol{\Omega} = 2\mathbf{B}$. This effect, first analyzed by Lense and Thirring in 1918, can be interpreted as a "dragging of inertial frames" by the rotating shell, *i.e.* the inertial frames tend to get swept along by the rotation. At the center of the shell, the local standard of "nonrotating", defined by parallel propagation along a geodesic, differs from what observers at infinity would regard as "nonrotating".

(a) Consistent with Wald, we use post-Newtonian approximation and linearized gravity. write gauge as

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}. \qquad |h_{\mu\nu}| \ll 1. \qquad \partial_t h_{\mu\nu} \approx 0.$$

$$h_{00} = -\frac{2\Phi}{c^2}. \qquad h_{0i} \sim A_i.$$

$$\Box h_{\mu\nu} - \partial_\nu \partial^\alpha h_{\alpha\mu} - \partial_\mu \partial^\alpha h_{\alpha\nu} + \eta_{\mu\nu}\partial^\alpha\partial^\beta h_{\alpha\beta} = -16\pi G T_{\mu\nu}.$$

introduce $\bar{\gamma}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h. \qquad h = \eta^{\alpha\beta} h_{\alpha\beta}.$

we have $\Box \bar{\gamma}_{\mu\nu} = -16\pi T_{\mu\nu}.$ $\sim$ Maxwell eqn

in Newtonian limit, $v/c \ll 1$.

$T_{00} \approx \rho, \quad T_{0i} \approx \rho v^i, \quad T_{ij} \approx 0.$

$\Rightarrow \nabla^2 \bar{\gamma}_{00} = -16\pi G\rho.$ compare to $\nabla^2 \Phi = 4\pi G\rho.$

$\Rightarrow \bar{\gamma}_{00} = -4\Phi.$

$\nabla^2 \bar{\gamma}_{0i} = -16\pi G T_{0i}.$ gives $\bar{\gamma}_{0i} = -4A_i^g.$

compare to definition $E_i = -\frac{1}{2}\frac{\partial h_{00}}{\partial x^i} = -\nabla\Phi.$

$B_i = \varepsilon^{ijk}\frac{\partial h_{0k}}{\partial x^j} = \nabla \times \vec{A}_g.$

look back on geodesic equation:

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} = 0.$$

for $v/c \ll 1.$ $\Gamma^i_{00} = \underbrace{\frac{1}{2}\frac{\partial h_{00}}{\partial x^i}}_{}.$ $-E_i$ $\quad \Gamma^i_{0j} = \underbrace{\frac{1}{2}\left(\frac{\partial h_{0i}}{\partial x^j} + \frac{\partial h_{0j}}{\partial x^i} - \frac{\partial h_{ij}}{\partial t}\right)}_{}.$ $\frac{1}{2}B_i$

$$\Rightarrow \frac{d^2x^i}{dt^2} = -\Gamma^i_{00} - 2\Gamma^i_{0j}\frac{dx^j}{d\tau}.$$

this is $\vec{a} = -\vec{E} - 4\vec{v}\times\vec{B}.$

(b) For the rotating sphere

$\nabla^2 \bar{\gamma}_{0i} = -16\pi G T_{0i}.$ $\qquad E_i = -\frac{1}{2}\frac{\partial h_{00}}{\partial x^i} = 0.$

$T^{0i} = \rho v^i \approx \frac{J^i}{R^3} = \frac{2}{3}M\frac{\omega^i}{R}.$ $\qquad B_i = \varepsilon^{ijk}\frac{\partial h_{0k}}{\partial x^j}.$

from equivalence, there will be a magnetic field inside sphere:

$$\bar{\gamma}_{0i} = -4G\frac{J}{R^3}\varepsilon_{ijk}x^j.$$

$$\Rightarrow B^i = \varepsilon^{ijk}\frac{\partial\bar{\gamma}_{0k}}{\partial x^j} = \varepsilon^{ijk}\partial_j\left(-4G\cdot\frac{2}{3}M\frac{w^j}{R}\varepsilon_{klm}x^l\right)$$

$$= \frac{2GM}{3R}\omega^i.$$

(c) For observer

$$\frac{dv^i}{dt} + \Gamma^i_{0j}v^j = 0. \qquad \Gamma^i_{0j} = \varepsilon^{ijk}B^k.$$

$$\frac{dv^i}{dt} = -\Gamma^i_{0j}v^j = \varepsilon^{ijk}B^kv^j. \qquad \vec{B} = \frac{2GM}{3R}\vec{\omega}.$$

$$\Rightarrow \frac{d\vec{v}}{dt} = 2\vec{B}\times\vec{v}. \quad ?$$

4. (25 pts) *Falling into a black hole*

Consider an observer sitting at constant spatial Schwarzschild coordinates (r_0, θ_0, ϕ_0) near a Schwarzschild black hole of mass M, with metric

$$ds^2 = -f_0(r)dt^2 + \frac{dr^2}{f_0(r)} + r^2 d\Omega_2^2 \ , \qquad f_0 = \left(1 - \frac{2GM}{r}\right)$$

The observer drops a probe of mass m from rest, which emits electromagnetic radiation at a constant wavelength λ_{emit} (in the probe's rest frame).

a) Calculate the coordinate speed dr/dt of the probe, as a function of r. For this you will need to solve the geodesic equations; use as many conservation laws as possible to simplify your life.

b) Show that the probe reaches the singularity at $r = 0$ in finite proper time.

c) Calculate the wavelength λ_{obs} measured by the observer at r_0, as a function of the radius r_{emit} at which the radiation is emitted.

d) Show that at late times, the redshift of the observed radiation grows exponentially:

$$\frac{\lambda_{\text{obs}}}{\lambda_{\text{emit}}} \propto \exp\left[t_{\text{obs}}/T\right]$$

Find an expression for the time constant T in terms of the black hole mass M.

Note that all the received radiation comes from the probe before it crosses the black hole's event horizon; after the probe crosses the horizon, even the radially outward directed radiation falls into the singularity along with the probe.

(a) conservation of energy gives $E = -g_{tt}\dot{t} = f_0(r)\dot{t}$.

also $\underbrace{-f_0(r)\dot{t}^2}_{E^2/f_0(r)} + \frac{\dot{r}^2}{f_0(r)} = -1$.

$\Rightarrow \dot{r} = E^2 - f_0(r)$.

since $\dot{r}(r_0) = 0$. $E = \sqrt{f_0(r_0)} = \sqrt{1 - \frac{2GM}{r_0}}$.

$$\dot{r} = -\sqrt{\frac{2GM}{r} - \frac{2GM}{r_0}}.$$

$$\dot{t} = \frac{E}{f_0(r)} = \sqrt{1 - \frac{2GM}{r_0}} \Big/ \left(1 - \frac{2GM}{r}\right).$$

$$\Rightarrow \quad \frac{dr}{dt} = \frac{\dot r}{\dot t} = -\sqrt{\frac{2GM}{r} - \frac{2GM}{r_0}} \cdot \left(1 - \frac{2GM}{r}\right) \Big/ \sqrt{1 - \frac{2GM}{r_0}} .$$

(b) $d\tau^2 = -ds^2 = -f_0(r)\,dt^2 + \dfrac{dr^2}{f_0(r)}$.

$$d\tau^2 = -f_0(r)\frac{\dot t^2}{\dot r^2}dr^2 + \frac{dr^2}{f_0(r)} = \frac{dr^2}{\dot r^2}\frac{1}{f_0(r)}\left(1 - E^2\right) = \frac{dr^2}{\dot r^2}\frac{r}{r_0} .$$

let's keep go with $\quad d\tau = \dfrac{dr}{\dot r} = -\dfrac{dr}{\sqrt{\frac{2GM}{r} - \frac{2GM}{r_0}}}$.

$$\tau \approx \int_{r_0}^{0} \frac{-dr}{\sqrt{\frac{2GM}{r}}} = \frac{\sqrt{2}}{3}\frac{r_0^{\frac{3}{2}}}{\sqrt{GM}} .$$

singular at $r=0$.

(c) Look into 4-velocity of source and observer.

$$U_e^\mu = \left(\frac{E}{f_0(r)}, \dot r\right) . \qquad U_o^\mu = \left(\frac{1}{\sqrt{f_0(r_0)}}, 0\right) .$$

4-vector of photon $\quad k^\mu = (\omega, k^r)$.

$$k^\mu k_\mu = 0 \quad \Rightarrow \quad \omega = \sqrt{f_0(r)}\,k^r .$$

$$(U^\mu k_\mu)_o = g_{tt}\,U_o^t k^t = -f_0(r_0)\frac{1}{\sqrt{f_0(r_0)}}\,\omega_0 .$$

$$(U^\mu k_\mu)_e = g_{tt}\,U_e^t k^t + g_{rr}\,U_e^r k^r$$

$$= -f_0(r_e)\frac{E}{f_0(r_e)}\omega_e + \frac{1}{f_0(r_e)}\dot r k^r = -E\omega_e + \frac{\dot r}{f_0(r_e)}k^r .$$

$$\Rightarrow \quad \frac{\lambda_o}{\lambda_e} = \frac{(u^\mu k_\mu)_e}{(u^\mu k_\mu)_o} = \frac{-E\,\omega_e + \frac{\dot r}{f_o(r_e)}\frac{\omega_e}{\sqrt{f_o(r)}}}{-f_o(r_o)\frac{1}{\sqrt{f_o(r_o)}}\,\omega_o}$$

$$= \frac{E - \frac{\dot r}{f_o(r_e)}}{\sqrt{f_o(r_o)}} = 1 - \frac{\dot r}{f_o(r_e)\sqrt{f_o(r_o)}}.$$

$$= 1 + \frac{\sqrt{\frac{2GM}{r_e} - \frac{2GM}{r_o}}}{f_o(r_e)\sqrt{f_o(r_o)}}.$$

(d) $r_e \to 2GM + \varepsilon$.

$$\dot r \approx -\sqrt{\frac{r}{2GM} - 1} \approx -\sqrt{\frac{\varepsilon}{2GM}}.$$

$$\Rightarrow \int \frac{d\varepsilon}{\sqrt{\varepsilon}} = -\sqrt{\frac{1}{2GM}}\,t + C.$$

$$\Rightarrow \sqrt{\varepsilon} = \sqrt{\varepsilon_o} - \frac{t}{2\sqrt{2GM}}.$$ is the expansion of $\varepsilon \propto e^{-\frac{t}{4GM}}$.

$$\frac{dr}{dt} \approx f_o(r) \approx \frac{\varepsilon}{2GM}. \qquad \Rightarrow \quad t_o - t_e = \int_{r_e}^{r_o} \frac{2GM}{\varepsilon}\,d\varepsilon$$

$$= 4GM \ln \frac{r_e - 2GM}{2GM}.$$

$$t_o \propto 4GM \ln t_e.$$

$$\frac{\lambda_o}{\lambda_e} \propto \frac{1}{r_e - 2GM} \propto e^{\frac{t_o}{4GM}}.$$ exponential redshift.

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1. (25 pts) Background Field Method and Gravitational Radiation

Consider the split of the metric $\mathbf{g}$ into a background $\hat{\mathbf{g}}$ and a small fluctuation $\delta\mathbf{g} = \kappa\mathbf{h}$

$$g_{\mu\nu} = \hat{g}_{\mu\nu} + \kappa h_{\mu\nu}$$

where $\kappa^2 = 32\pi G$.

One can use the above split into background and perturbation to set up a systematic procedure to solve the field equations order by order in κ. To begin, one can plug the above metric into the Einstein-Hilbert action and expand in powers of κ up to and including second order. At zeroth order in the action, one has the Einstein-Hilbert action evaluated at $\hat{g}$. At first order, one can think of $h_{\mu\nu}$ as an arbitrary variation of the metric in the vicinity of $\hat{g}_{\mu\nu}$: $\kappa h_{\mu\nu}$ multiplies the Einstein equations, and one can then ask that the background $\hat{\mathbf{g}}$ satisfies the Einstein field equations. The action at second order in κ has the structure of a kinetic term for the perturbation $\mathbf{h}$

$$\mathcal{S}_{(2)} = \frac{\kappa^2}{16\pi G}\int \sqrt{-\hat{g}}\, h_{\mu\nu}\Delta^{\mu\nu\alpha\beta}h_{\alpha\beta}$$

where $\Delta^{\mu\nu\alpha\beta}$ is a second-order differential operator on (0,2) tensors. If one varies this action with respect to $\mathbf{h}$, one gets the (wave) equation of motion satisfied by small fluctuations (such as gravitational waves) around the background metric $\hat{\mathbf{g}}$. If one varies it with respect to $\hat{\mathbf{g}}$, one gets (after appropriate integrations by parts to write things in terms of first dervatives of $\mathbf{h}$ only) what can be thought of as the "stress-energy tensor" of gravitational perturbations.

(As an aside, the general structure of the expansion allows one to set up a perturbation theory for a solution of the Einstein equations in a power series in κ, starting from an existing solution (the one with the metric $\hat{\mathbf{g}}$). At lowest order, one solves the linearized field equations. At next order there will be second-order, $O(\kappa)$ corrections to $\mathbf{h}$, as well as terms quadratic in the first-order perturbation found already; one solves the resulting linear equation for the former with the latter as a source. At order $n+1$, there is an $O(\kappa^n)$ correction to $\mathbf{h}$ that appears linearly, sourced by a polynomial of terms in already found lower corrections. At each order one is solving a linear equation with a specified source, a problem much easier that solving a nonlinear PDE.)

Consider now this second variation of the Einstein-Hilbert action in the perturbation $h_{\mu\nu}$. (HW5 considered the first variation of the connection and curvature, etc, which were used in class to find the equations of motion) Assume as above that at zeroth order, the background $\hat{\mathbf{g}}$ satisfies the vacuum Einstein equations. The explicit form of the action expanded to this order can be written

$$\mathcal{S}_{(2)} = \frac{1}{2}\int \sqrt{-\hat{g}}\Big[(\hat{\nabla}_\alpha h_{\mu\nu})(\hat{\nabla}^\alpha h^{\mu\nu}) - (\hat{\nabla}_\alpha h)(\hat{\nabla}^\alpha h) + 2(\hat{\nabla}^\nu h)(\hat{\nabla}^\beta h_{\beta\nu}) - 2(\hat{\nabla}_\nu h_{\alpha\beta})(\hat{\nabla}^\alpha h^{\nu\beta})\Big]$$

where indices are raised and lowered with respect to the background metric, $\hat{\nabla}$ is the covariant derivative with respect to the background, $h = h_{\alpha\beta}\hat{g}^{\alpha\beta}$ is the trace of the perturbation, etc.

One can show (but you need not) that varying this action with respect to the perturbation $\mathbf{h}$, the result is the LHS of the Einstein equations linearized around the background $\hat{\mathbf{g}}$; so indeed it is the action that describes gravitational waves in the geometry $\hat{g}_{\mu\nu}$.

a) Vary with respect to the background metric to derive what one might think of as the "energy-momentum tensor" $t^{\mu\nu}_{(h)}$ of the gravitational fluctuation (please note that the interpretation of this quantity as an energy-momentum tensor is fraught with subtleties; see Carroll section 7.6 or Wald section 4.4; nevertheless, to the order we are working it's fine to do so). To simplify the analysis, after making the variation of $\hat{\mathbf{g}}$, set this background equal to the Lorentz metric $\boldsymbol{\eta}$ (so *e.g.* the covariant derivative becomes the ordinary derivative), and impose the DeDonder gauge.

b) Evaluate this energy-momentum tensor on gravitational quadrupole radiation from the binary star system discussed in class (two stars of equal mass M and orbital radius R). Integrate the flux of gravitational radiation energy over a sphere at asymptotically large radius, and average your result over an orbital period of the source to find the average power radiated by the binary system. To simplify your life, first show that asymptotically, the terms contributing to the energy flux come from the oscillating quadrupole and not the static $\Phi = 2GM/r$ monopole gravitational field of the source.

c) The first concrete evidence for gravitational radiation came from observations in the 1970s of orbital decay of the Hulse-Taylor binary pulsar. Approximate it as the binary system above, with two solar mass stars at an orbital radius of $R = 10^6 km$, and estimate the rate of orbital decay $\partial_t R$ using conservation of energy.

(a) We calculate $\quad t^{\mu\nu}_{(h)} = -\frac{2}{\sqrt{-g}} \frac{\delta S_{(2)}}{\delta g_{\mu\nu}}$.

$$g^{\mu\alpha} g_{\alpha\nu} = \delta^\mu_\nu$$

$$\delta g^{\mu\alpha} g_{\alpha\nu} + g^{\mu\alpha} \delta g_{\alpha\nu} = 0.$$

$$\delta g^{\mu\nu} = -g^{\mu\alpha} g^{\nu\beta} \delta g_{\alpha\beta}.$$

$$\delta\sqrt{-g} = \frac{1}{2}\sqrt{-g}\, g^{\mu\nu} \delta g_{\mu\nu}.$$

$$\delta(\sqrt{-g}\mathcal{L}) = \sqrt{-g}\left(\delta\mathcal{L} + \frac{1}{2}\mathcal{L} g^{\mu\nu}\delta g_{\mu\nu}\right).$$

use approximation $\quad \delta(\nabla_\alpha h_{\mu\nu}) \approx 0$.

$$h^{\mu\nu} = g^{\mu\alpha} g^{\nu\beta} h_{\alpha\beta}. \qquad \delta h^{\mu\nu} = \delta g^{\mu\alpha} g^{\nu\beta} h_{\alpha\beta} + g^{\mu\alpha}\delta g^{\nu\beta} h_{\alpha\beta}$$

$$= -g^{\mu\lambda} g^{\alpha\sigma} \delta g_{\lambda\sigma} g^{\nu\beta} h_{\alpha\beta} - g^{\mu\alpha} g^{\nu\lambda} g^{\beta\sigma} \delta g_{\lambda\sigma} h_{\alpha\beta}$$

$$= -h^{\mu\alpha} g^{\nu\beta} \delta g_{\alpha\beta} - h^{\nu\beta} g^{\mu\alpha} \delta g_{\alpha\beta}.$$

Set to Lorentz metric η. $\quad g_{\mu\nu} \approx \eta_{\mu\nu}$.

$$\delta(\nabla_\alpha h^{\mu\nu}) = -\partial_\alpha(h^{\mu\lambda} g^{\beta\nu}\delta g_{\lambda\beta}) - \partial_\alpha(h^{\nu\lambda} g^{\mu\alpha}\delta g_{\lambda\beta}).$$

$$\delta\left[(\nabla_\alpha h_{\mu\nu})(\nabla^\alpha h^{\mu\nu})\right] = 2(\nabla_\alpha h_{\mu\nu})(\nabla^\alpha \delta h^{\mu\nu}).$$

$$\delta\left[(\nabla_\alpha h)(\nabla^\alpha h)\right] = 2(\nabla_\alpha h)(\nabla^\alpha \delta h).$$

$$\delta\left[2(\nabla^\nu h)(\nabla^\beta h_{\nu\beta})\right] = 2\nabla^\nu h\, \nabla^\beta \delta h_{\nu\beta} + 2\nabla^\nu \delta h\, \nabla^\beta h_{\nu\beta}$$

$$\delta\left[-2(\nabla^\nu h_{\alpha\beta})(\nabla^\alpha h^\beta_\nu)\right] = -2\nabla^\nu h_{\alpha\beta}\, \nabla^\alpha \delta h^\beta_\nu - 2\nabla^\nu \delta h_{\alpha\beta}\, \nabla^\alpha h^\beta_\nu.$$

$$\delta S_{(2)} = \frac{1}{2}\int \sqrt{-g}\left(\delta\mathcal{L} + \frac{1}{2}\mathcal{L} g^{\mu\nu}\delta g_{\mu\nu}\right) d^4x$$

I myself can't do the calculation...

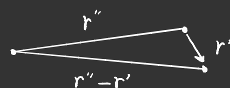
...

$$\delta S_{(2)} = \frac{1}{2}\int \sqrt{-g}\left[\nabla^\mu h_{\alpha\beta}\nabla^\nu h^{\alpha\beta} - \frac{1}{2} g^{\mu\nu}\nabla_\lambda h_{\alpha\beta}\nabla^\lambda h^{\alpha\beta}\right]\delta g_{\mu\nu}\, d^4x.$$

$$t^{\mu\nu}_{(h)} = \frac{1}{16\pi G}\left(\nabla^\mu h_{\alpha\beta}\nabla^\nu h^{\alpha\beta} - \frac{1}{2} g^{\mu\nu}\nabla_\lambda h_{\alpha\beta}\nabla^\lambda h^{\alpha\beta}\right)$$

$$= \frac{1}{16\pi G}\left(\partial^\mu h_{\alpha\beta}\partial^\nu h^{\alpha\beta} - \frac{1}{2}\eta^{\mu\nu}\partial_\lambda h_{\alpha\beta}\partial^\lambda h^{\alpha\beta}\right)$$

(b) from textbook,



$$\bar{h}_{\mu\nu} \approx \frac{4G}{r}\int T^*_{\mu\nu}\, d^3x'$$

$$= \frac{2G}{r}\left(\frac{\partial^2}{\partial t^2}\int x^i x^j T_{00}\, d^3x\right)^*.$$

$\sim$ second derivative of $Q^{ik} = \int \rho x^i x^k d^3x$.

$$D^{ij} = 3Q^{ij} - \delta^{ij} Q^k_k.$$

traceless part of $\bar{h}_{ij}(x) = \frac{2G}{3r}\underline{\left(\frac{d^2 D_{ij}}{dt^2}\right)}^*$.

radiation only has to do with

second derivative of qudrapole

for rotating system

$\propto MR^2\omega^2\cos 2\omega t.$

$$-\frac{dE}{dt} = \frac{32G}{5}\omega^6(I_{11}-I_{22})^2 = \frac{32G}{5c^5}\omega^6 I^2 e^2.$$

(*) derivation in supplement

for two star system

$$-\frac{dE}{dt} = \frac{64G^4}{5c^5R^5}M^5.$$

(c) For two star system $E = -\frac{GM^2}{2R}.$

$$\Rightarrow \frac{dE}{dt} = \frac{GM^2}{2R^2}\frac{dR}{dt}.$$

$$\frac{dR}{dt} = \frac{2R^2}{GM^2}\frac{dE}{dt} = \frac{2R^2}{GM^2}\frac{64G^4}{5c^5R^5}M^5 = \frac{128G^3M^3}{5c^5R^3}.$$

put in $M = 1.989\times 10^{30}$ kg.

$R = 10^6$ km.

$\partial_t R = 2.47\times 10^{-8}$ m/s. ?

(*)

$$\begin{cases} x^0 = y^0 = t \\ x^1 = y^1\cos\omega t - y^2\sin\omega t. \\ x^2 = y^1\sin\omega t + y^2\cos\omega t. \\ x^3 = y^3. \end{cases}$$

$$I_{ij} = \iiint \rho y^i y^j d^3y.$$

$$Q^{ik} = \int \rho x^i x^k d^3x.$$

$$dF = \frac{G}{36\pi}\left[\left(\frac{\dddot{D}^{11}-\dddot{D}^{22}}{2}\right)^2 + \left(\dddot{D}^{12}\right)^2\right]^* d\Omega = \frac{G}{4\pi}\left[\left(\frac{\dddot{Q}^{11}-\dddot{Q}^{22}}{2}\right)^2 + \left(\dddot{Q}^{12}\right)^2\right]^* d\Omega.$$

$$-\frac{dE}{dt} = 4\pi\overline{\frac{dF}{d\Omega}} = \frac{G}{45}\sum_{i,j}\dddot{D}^{ij}\dddot{D}^{ij} = \frac{G}{5}\left(\dddot{Q}^{ij}\dddot{Q}_{ij} - \frac{1}{3}\left(\dddot{Q}^i_i\right)^2\right).$$

$$Q^{11} = \iiint x^1x^1\rho\, d^3x = I_{11}\cos^2\omega t + I_{22}\sin^2\omega t.$$

$$Q^{12} = \frac{1}{2}(I_{11}-I_{22})\sin 2\omega t.$$

$$Q^{22} = \frac{1}{2}(I_{11}+I_{22}) + \frac{1}{2}(I_{11}-I_{22})\cos 2\omega t.$$

$$Q^{13} = 0, \quad Q^{23} = 0. \quad Q^{33} = I_{13}.$$

$$\left(\dddot{Q}^{11}\right)^2 = 16\omega^6(I_{11}-I_{22})^2\sin^2 2\omega t.$$

$$\left(\dddot{Q}^{12}\right)^2 = 16\omega^6(I_{11}-I_{22})^2\cos^2 2\omega t.$$

$$\left(\dddot{Q}^{22}\right)^2 = 16\omega^6(I_{22}-I_{11})^2\sin^2 2\omega t.$$

$$\sum_{i,j}\left(\dddot{Q}^{ij}\dddot{Q}^{ij} - \frac{1}{3}\dddot{Q}^i_i\dddot{Q}^i_i\right) = 32\omega^6(I_{11}-I_{22})^2.$$

$$-\frac{dE}{dt} = \frac{32}{5}G\omega^6(I_{11}-I_{22})^2 = \frac{32G}{5c^5}\omega^6 I^2 e^2.$$

2. (25 pts) The Chandrasekhar limit

A "main sequence" star such as the Sun maintains equilibrium via the balance of gravitational attraction and thermal pressure. The latter is fueled by nuclear burning in the star's core; eventually though, the fuel runs out and the star collapses. What is its fate? The answer depends on the star's mass. Stars such as the sun form *white dwarfs*, where the pressure supporting the star is degeneracy pressure of the Fermi repulsion of electrons. This problem explores the physics of white dwarf stars.

The total energy can be written $E = K + U$, where K is the total kinetic energy of the degenerate electrons and U is the gravitational potential energy. Let us make two simplifying assumptions: (1) the density of the star is uniform; and (2) the linearized (Newtonian) approximation to gravity.

a) Show that

$$U = -\frac{3}{5}\frac{GM^2}{R}$$

by assembling the star by depositing a sequence of radial shells. Here M is the mass of the star and R is its radius.

b) The constant phase-space density of a degenerate Fermi gas means that

$$dN = \frac{3V}{\Lambda^3}\, p^2 dp \ , \qquad \Lambda = (3\pi^2)^{1/3}\hbar$$

where $V = \frac{4\pi}{3}R^3$ is the volume of the star. This leads to

$$p_F = \Lambda\Big(\frac{N}{V}\Big)^{1/3}$$

Show that, assuming that the electron gas is non-relativistic, the kinetic energy of the star is

$$K = \frac{3}{5}N\frac{\Lambda^2}{2m}\Big(\frac{N}{V}\Big)^{2/3}$$

(a) $\rho = \dfrac{M}{\frac{4}{3}\pi R^3}$.

$$U = \int_0^R -\frac{GM\cdot\frac{r^3}{R^3}\cdot\rho\cdot 4\pi r^2 dr}{r}$$

$$= -GM\cdot 4\pi\rho\cdot\frac{R^5}{5R^3} = -\frac{3GM^2}{5R}.$$

(b) $$K = \int_0^{p_F}\frac{p^2}{2m}\,dN$$

$$= \int_0^{\Lambda\left(\frac{N}{V}\right)^{\frac{1}{3}}}\frac{p^2}{2m}\,\frac{3V}{\Lambda^3}\,p^2 dp$$

$$= \frac{3V}{2m\Lambda^3} \cdot \frac{1}{5} \Lambda^5 \left(\frac{N}{V}\right)^{\frac{5}{3}}$$

$$= \frac{3}{2m} \cdot \frac{1}{5} \Lambda^2 \left(\frac{N}{V}\right)^{\frac{2}{3}} \cdot N$$

$$= \frac{3}{5} N \frac{\Lambda^2}{2m} \left(\frac{N}{V}\right)^{\frac{2}{3}}.$$

(c) $$E = \frac{A}{R^2} - \frac{B}{R}.$$

$$\frac{dE}{dR} = -\frac{2A}{R^3} + \frac{B}{R^2}. \qquad \Rightarrow R_c = \frac{2A}{B}.$$

$$R_c = \frac{2 \times \frac{3}{20} \left(\frac{9\pi}{8}\right)^{\frac{2}{3}} \frac{\hbar^2}{m} \left(\frac{M}{m_p}\right)^{\frac{5}{3}}}{\frac{3}{5} G M^2}$$

$$= \frac{1}{2} G^{-1} \left(\frac{9\pi}{8}\right)^{\frac{2}{3}} \frac{\hbar^2}{m} \left(\frac{M}{m_p}\right)^{\frac{5}{3}}.$$

(d) $$E = K + U$$

$$\approx \frac{3}{4} \frac{Vc}{\Lambda^3} \left(p_F^4 + m^2c^2p_F^2\right) - \frac{3}{5} \frac{GM^2}{R}$$

$$= \frac{3}{4} \frac{Vc}{\Lambda^3} \left(p_F^2 + m^2c^2\right) \Lambda^2 \left(\frac{N}{V}\right)^{\frac{2}{3}} - \frac{3}{5} \frac{GM^2}{R}$$

$$= \frac{3}{4} \frac{c}{\Lambda} \left(\underline{p_F^2} + m^2c^2\right) N^{\frac{2}{3}} \underline{V^{-\frac{1}{3}}} - \frac{3}{5} \frac{GM^2}{R}$$

$$\Lambda^2 \left(\frac{N}{V}\right)^{\frac{2}{3}} \qquad \left(\frac{4}{3}\pi R^3\right)^{-\frac{1}{3}}$$

$$\Lambda^2 N^{\frac{2}{3}} \left(\frac{4}{3}\pi R^3\right)^{-\frac{2}{3}}$$

c) The atomic nuclei in a white dwarf such as C^{12} and O^{16} have twice as many nucleons as electrons; thus $M = 2Nm_p$ where m_p is the proton mass. Show that in the non-relativistic approximation

$$E = \frac{A}{R^2} - \frac{B}{R}$$

where

$$A = \frac{3}{20}\left(\frac{9\pi}{8}\right)^{2/3}\frac{\hbar^2}{m}\left(\frac{M}{m_p}\right)^{5/3}, \qquad B = \frac{3}{5}GM^2 .$$

Find the value R_* of the star's radius that minimizes the total energy E.

d) If the mass is too large, however, the kinetic energy needed to generate the pressure supporting the star is such that the electrons become relativistic. Now use the relativistic kinetic energy of the electron to show that

$$K \approx \frac{3}{4}\frac{Vc}{\Lambda^3}\left[p_F^4 + m^2c^2p_F^2 + \dots\right]$$

so that the energy now has the form

$$E \approx \frac{C}{R} + DR \ ;$$

find the constant C. Show that for

$$M > M_C = \frac{15}{64}\frac{\left[5\pi(\hbar c/G)^3\right]^{1/2}}{m_p^2} \approx 1.72 M_\odot \ ,$$

the constant C becomes negative and there is no stable solution. The mass M_C is known as the *Chandrasekhar limit*, after the UChicago faculty member who derived it in 1931. A more accurate calculation using a more realistic equation of state leads to the Chandrasekhar mass $M_C = 1.4M_\odot$. For stars with mass not too much larger than M_C, the endpoint of collapse is a neutron star, with the electrons and protons combining into neutrons (and neutrinos which escape the star), and the pressure being supplied by the degenerate Fermi gas of neutrons. Eventually, when the mass exceeds about $3M_\odot$, the neutron degeneracy pressure fails to support the star, and it collapses into a black hole.

$$= \frac{3}{4}\frac{c}{\Lambda}\left(\Lambda^2 N^{\frac{2}{3}}\left(\frac{4}{3}\pi R^3\right)^{-\frac{2}{3}} + m_e^2c^2\right) N^{\frac{2}{3}}\left(\frac{4}{3}\pi R^3\right)^{-\frac{1}{3}} - \frac{3}{5}\frac{GM^2}{R}$$

not of form $\frac{C}{R} + DR$?

the term $\propto R^{-1}$ is $\frac{3}{4}\frac{m_e^2c^3}{\Lambda}N^{\frac{2}{3}}\left(\frac{4}{3}\pi\right)^{-\frac{1}{3}}\frac{1}{R} - \frac{3}{5}\frac{GM^2}{R}$

$$C = \frac{3}{4}\frac{m_e^2c^3}{\Lambda}N^{\frac{2}{3}}\left(\frac{4}{3}\pi\right)^{-\frac{1}{3}} - \frac{3}{5}GM^2 .$$

running out of time to verify this ...

for $M > M_c = 1.72\, M_\odot$.

plug back E to check positivity.