 yuxiang@uchicago.edu

1. The Ising model defines a spin variable on a site i as $\sigma_i = \pm 1$, with the Hamiltonian

$$H = \frac{1}{2}\sum_{i,j=1}^{N} J_{ij}\sigma_i\sigma_j - h\sum_i \sigma_i \tag{1}$$

where h is an external magnetic field and J_{ij} the interaction energy. In the *infinite range* model, we set $J_{ij} = -J/N$, and assume N large.

(a) Show that the energy of any spin configuration can be written as a function of the magnetization $m = \sum_i \sigma_i/N = M/N$ and the field h

(b) Show that the partition function can be written in the form

$$Z(h,T) = \sum_M \exp[-\beta F(m,h)] \tag{2}$$

where the summation extends over configurations labelled by their magnetization M. Determine $F(m,h)$ as a power series to fourth order in m.

(c) Show that the free energy is given by

$$F(h,T) = min[F(m,h)]_m$$

(d) Find the critical temperature T_c, and the spontaneous magnetization $\bar{m}(T)$ for $h = 0$

(e) Calculate the singular behavior near T_c of the specific heat and of the magnetic susceptibility.

(a) $H = -\sum_{i,j} \frac{J}{N}\sigma_i\sigma_j - h\sum_i \sigma_i$ missing $\frac{1}{2}$!

In the context of this problem, we should be talking about the fully-connected Ising model

field of all other spins

$$E = -\sum_i \frac{J}{N}\sigma_i \underbrace{\frac{(N-1)m}{N}}_{} - h\sum_i \sigma_i$$

$$= -Jm^2\left(1-\frac{1}{N}\right) - hNm$$

(b) $Z = \sum_i e^{-\beta E_i}$ (i: all spin configuration)

M

$\uparrow \ \downarrow \ \Uparrow \ \downarrow$

$\cdots\cdots\ N$

$\uparrow \frac{N+M}{2} \quad \frac{N-M}{2} \downarrow$

$$= \sum_M C_N^{\frac{N-M}{2}} e^{-\beta\left(-\frac{JM^2}{N^2}+\frac{JM^2}{N^3}-hM\right)}$$

$$= \sum_M \frac{N!}{\left(\frac{N-M}{2}\right)!\left(\frac{N+M}{2}\right)!} e^{-\beta\left(-\frac{JM^2}{N^2}+\frac{JM^2}{N^3}-hM\right)}$$

$$= \sum_M e^{-\beta\left(-\frac{JM^2}{N^2}+\frac{JM^2}{N^3}-hM\right)+\ln\frac{N!}{\left(\frac{N-M}{2}\right)!\left(\frac{N+M}{2}\right)!}}$$

$$F(m,h) = -\frac{JM^2}{N^2}+\frac{JM^2}{N^3}-hM-\frac{1}{\beta}\ln\frac{N!}{\left(\frac{N-M}{2}\right)!\left(\frac{N+M}{2}\right)!}$$

$$= -Jm^2 + J\frac{m^2}{N} - Nhm - \frac{1}{\beta}\underbrace{\left[\ln N! - \ln\left(\tfrac{N-M}{2}\right)! - \ln\left(\tfrac{N+M}{2}\right)!\right]}$$

$$N(\ln N - 1) - \frac{N-M}{2}\left(\ln\frac{N-M}{2}-1\right) - \frac{N+M}{2}\left(\ln\frac{N+M}{2}-1\right)$$

$$= N\ln N - \frac{N}{2}\ln\frac{N-M}{N+M} + \frac{M}{2}\ln\frac{N-M}{N+M}$$

$\ln\frac{1-m}{1+m} \approx -2m - \frac{2m^3}{3}$

$$= N\ln N - \frac{N}{2}\ln\frac{1-m}{1+m} + \frac{Nm}{2}\ln\frac{1-m}{1+m}$$

$$\sim N\ln N - \frac{N}{2}(1-m)\left(\frac{-2m}{1-m^2} + \frac{-4m}{(1-m^2)^2}\cdot\frac{m^2}{2}\right) \qquad -2m - \frac{2m^3}{3}$$

$$\sim N\ln N + Nm(1-m) + N\,2m^3(1+m^2)(1-m)$$

$$\sim N\ln N + N\underbrace{(m - m^2 + 2m^3 - 2m^4)} \qquad \frac{m^4}{3} - \frac{m^3}{3} + m^2 - m$$

$$F(m,h) \simeq -Jm^2 + J\frac{m^2}{N} - Nhm - \frac{1}{\beta}\left[N\ln N + N\underbrace{(m-m^2+2m^3-2m^4)}\right] \qquad \frac{m^4}{3} - \frac{m^3}{3} + m^2 - m$$

$$= -\frac{1}{\beta}N\ln N - N\left(h - \frac{1}{\beta}\right)m + \left(\frac{J}{N} - \frac{N}{\beta}\right)m^2 + \left(-J - \frac{2N}{\beta}\right)m^3 + \frac{2}{\beta}m^4.$$

(c) $$F(h,T) = -\frac{\ln Z}{\beta} = -\frac{\ln\left(\sum_M e^{-\beta F(m,h)}\right)}{\beta}$$

$$\frac{\partial F(m,h)}{\partial m} = \frac{8}{\beta} m^3 - 3\left(J + \frac{2N}{\beta}\right) m^2 + 2\left(\frac{J}{N} + \frac{N}{\beta}\right) m - N\left(h + \frac{1}{\beta}\right)$$

$$F(h,T) = -\frac{\ln\left(\sum_m e^{-\beta F(m,h)}\right)}{\beta} \overset{?}{=} \min\left[F(m,h)\right]_m \ ?$$

(d) $h = 0$.

$$\frac{\partial F(m,h)}{\partial m} = \frac{8}{\beta} m^3 - 3\left(J + \frac{2N}{\beta}\right) m^2 + 2\left(\overset{\approx 0}{\frac{J}{N}} + \frac{N}{\beta}\right) m - \frac{N}{\beta}$$

```
ClearAll["Global`*"]
FullSimplify[Solve[8 m^3 - 3 (+2 N) m^2 + 2 (+N) m - N == 0, m]];

expr = 1/12 (3 N + (3^(2/3) N (-4 + 3 N))/(9 N (4 + (-2 + N) N) + 2 Sqrt[3] Sqrt[N^2 (108 + N (-92 + 45 N))])^(1/3) + (27 N (4 + (-2 + N) N) + 6 Sqrt[3] Sqrt[N^2 (108 + N (-92 + 45 N))])^(1/3));

approx = Series[expr, {N, 0, 1}] // Normal;

Simplify[N/4 - N^(2/3)/(3×2^(2/3) ((N+N)/N)^(1/3)) + (N^(1/3) ((N+N)/N)^(1/3))/(2×2^(1/3))]

1/12 (6 N^(1/3) - 2 N^(2/3) + 3 N)
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$$m_0 \approx \frac{N}{4} - \frac{N^{\frac{2}{3}}}{6} + \frac{N^{\frac{1}{3}}}{2} \approx \frac{N}{4}.$$

$$F(h,T) = \min\left\{F(m,h)\right\}_m = F(m,h)\big|_{m=m_0}$$

$$= -\frac{1}{\beta} N \ln N - \frac{32N^2 - 8N^3 + 3N^4}{128\beta} + \frac{JN}{16} - \overset{=0}{\frac{hN^2}{4}} - \frac{JN^3}{64}.$$

$$= \left(\frac{JN}{16} - \frac{JN^3}{64}\right) - \left(N \ln N + \frac{32N^2 - 8N^3 + 3N^4}{128}\right) kT.$$

$T_c =$

Let's calculate over again.

(a) $H = -\frac{1}{2}\sum_{i,j} \frac{J}{N}\sigma_i\sigma_j - h\sum_i \sigma_i$

In the context of this problem, we should be talking about the fully-connected Ising model

field of all other spins

$$E = -\frac{1}{2}\sum_i \frac{J}{N}\sigma_i \underbrace{\frac{(N-1)m}{N}}_{} - h\sum_i \sigma_i$$

$$= -\frac{1}{2}JNm^2\left(1-\frac{1}{N}\right) - hNm$$

(b) $Z = \sum_i e^{-\beta E_i}$ (i: all spin configuration)

M

$\uparrow \downarrow \uparrow \downarrow$ …… N

$\uparrow \frac{N+M}{2}$ $\frac{N-M}{2} \downarrow$

$$= \sum_M C_N^{\frac{N-M}{2}} e^{-\beta\left(-\frac{JM^2}{2N} + \frac{JM^2}{2N^2} - hM\right)}$$

$$= \sum_M \frac{N!}{\left(\frac{N-M}{2}\right)!\left(\frac{N+M}{2}\right)!} e^{-\beta\left(-\frac{JM^2}{2N} + \frac{JM^2}{2N^2} - hM\right)}$$

$$= \sum_M e^{-\beta\left(-\frac{JM^2}{2N} + \frac{JM^2}{2N^2} - hM\right) + \ln\frac{N!}{\left(\frac{N-M}{2}\right)!\left(\frac{N+M}{2}\right)!}}$$

$$F(m,h) = -\frac{JM^2}{2N} + \frac{JM^2}{2N^2} - hM - \frac{1}{\beta}\ln\frac{N!}{\left(\frac{N-M}{2}\right)!\left(\frac{N+M}{2}\right)!}$$

$$= -\frac{1}{2}NJm^2 + \frac{1}{2}Jm^2 - Nhm - \frac{1}{\beta}\underbrace{\left[\ln N! - \ln\left(\frac{N-M}{2}\right)! - \ln\left(\frac{N+M}{2}\right)!\right]}_{}$$

$$N(\ln N - 1) - \frac{N-M}{2}\left(\ln\frac{N-M}{2} - 1\right) - \frac{N+M}{2}\left(\ln\frac{N+M}{2} - 1\right)$$

$$= N\ln N - \frac{N}{2}\ln\frac{N^2-M^2}{4} + \frac{M}{2}\ln\frac{N-M}{N+M}$$

$$\approx N\ln N - N\ln N + N\ln 2 + \frac{Nm}{2}\left(-2m - \frac{2}{3}m^3\right)$$

$$\approx N\ln 2 - Nm^2 - \frac{N}{3}m^4$$

$$F(m,h) \approx -\frac{1}{2}NJm^2 + \frac{1}{2}Jm^2 - Nhm - \frac{1}{\beta}\left(N\ln 2 - Nm^2 - \frac{N}{3}m^4\right)$$

$$= \frac{N}{3\beta}m^4 + \left[\frac{N}{\beta} - \frac{1}{2}(N-1)J\right]m^2 - Nhm - \frac{N}{\beta}\ln 2$$

(c) $$F(h,T) = -\frac{\ln Z}{\beta} = -\frac{\ln\left(\sum_M e^{-\beta F(m,h)}\right)}{\beta}$$

$$F(h,T) = -\frac{\ln\left(\sum_M e^{-\beta F(m,h)}\right)}{\beta} \overset{?}{=} \min\left[F(m,h)\right]_m ?$$

$$F(m,h) \approx \frac{N}{3\beta}m^4 + \left(\frac{N}{\beta} - \frac{NJ}{2}\right)m^2 - Nhm - \frac{N}{\beta}\ln 2.$$

$$\frac{\partial F(m,h)}{\partial m} = \frac{1}{\beta}\left[\frac{4}{3}Nm^3 + (2N - N\beta J)m \underbrace{-\,Nh}_{\simeq 0}\right] = 0$$

$$\Rightarrow \boxed{m_0 = \sqrt{\frac{3(2-\beta J)}{4}}}.$$

$$F(h,T) = F(m,h)\big|_{m=m_0}$$

$$= \frac{N}{3\beta}m_0^4 + \left[\frac{N}{\beta} - \frac{1}{2}(N-1)J\right]m_0^2 - \frac{N}{\beta}\ln 2$$

$$= \frac{9N}{4\beta} + \frac{3}{4}J - \frac{9}{4}JN - \frac{3}{8}J^2\beta + \frac{9}{16}J^2N\beta - \frac{N\ln 2}{\beta}.$$

(d) $\frac{\partial F(h,T)}{\partial \beta} = \frac{1}{16}\left[J^2(9N-6) + \frac{4N(4\ln 2-9)}{\beta^2}\right]$

$$\Rightarrow \beta_0 = \frac{1}{kT_0} = \frac{2\sqrt{N(9-4\ln 2)}}{\sqrt{9N-6}\,J}.$$

$$T_0 = \frac{\sqrt{9N-6}\,J}{2\sqrt{N(9-4\ln 2)}\,k} \approx \frac{3\sqrt{N-\frac{2}{3}}}{2\sqrt{6.23}\sqrt{N}}\frac{J}{k} \approx \boxed{0.6\,\frac{J}{k}}.$$

$$\bar{m} = -\frac{1}{N}\frac{\partial F(h,T)}{\partial h} = m_0 = \sqrt{\frac{3(2-\beta J)}{4}}$$

(e) $C = -T\frac{\partial^2 F}{\partial T^2}$

$$= -T\frac{\partial^2}{\partial T^2}\left(\frac{1}{16}\left[J^2(9N-6) + 4N(4\ln 2-9)k^2T^2\right]\right)$$

$$= -T\cdot\frac{\partial}{\partial T}\left(2\left(\ln 2-\frac{9}{4}\right)Nk^2T\right)$$

$$= 2\left(\frac{9}{4}-\ln 2\right)Nk^2T \approx 3.1\,Nk^2T.$$

$$\chi = \frac{\partial M}{\partial h}, \quad \frac{4}{3}Nm^3 + (2N-N\beta J)m - Nh = 0.$$

$$\Rightarrow h = \frac{4}{3}m^3 + (2-\beta J)m.$$

$$\frac{\partial h}{\partial m} = 4m^2 + 2-\beta J = 4(2-\beta J).$$

$$\chi = \frac{\partial h}{\partial m} = \frac{1}{4(2-\beta J)}.$$

2. Use the following Landau theory for the Free energy density (per unit volume) of a ferroelectric material with electrical polarization P in an electric field E

$$\mathcal{F} = \frac{1}{2}aP^2 + \frac{1}{4}bP^4 + \frac{1}{6}cP^6 - PE \tag{3}$$

where the coefficient $a = a' \times (T - T_o)$ is temperature dependent, and all the other coefficients are constant. Although the polarisation P is of course a vector, we assume that it can point only in a symmetry direction of the crystal, and so it is replaced by a scalar.

(a) Assume that $b > 0$ and $c = 0$. Use Eq.3 to determine the form for the equilibrium $P(T)$ when $E = 0$

(b) Again for the case $b > 0$ and $c = 0$, determine the dielectric susceptibility $\chi = \frac{\partial P}{\partial E}$ both above and below the critical temperature.

(c) Sketch curves for $P(T)$, $\chi^{-1}(T)$, and $\chi(T)$.

(d) The electric field E is increased slowly from zero to large positive values, reversed to large negative values, ind then increased back to zero again. Sketch the form of the hysteresis loop in the P, E plane for $T < T_o$.

(e) In a different material, the free energy is described by a similar form to Eq.3, but with $b < 0$ and $c > 0$. By sketching $\mathcal{F}$ at different temperatures, discuss the behaviour of the equilibrium polarisation and the linear susceptibility, contrasting the results with those found in (c).

(f) Using the model in (e) sketch the P-E hysteresis curves in three cases: $T < T_o$, $T_c > T > T_o$, and $T > T_c$, where T_c is the equilibrium transition temperature at zero electric field.

(a) $F = \frac{1}{2}aP^2 + \frac{1}{4}bP^4 - PE$.

$\frac{\partial F}{\partial P} = aP + bP^3 - E$. $E = 0$, $\frac{\partial F}{\partial P} = 0$. $\Rightarrow bP^3 = -a'(T-T_0)P$.

$P_1(T) = \sqrt{\frac{-a'}{b}(T-T_0)}$. $P_2(T) = 0$. $(a'<0)$ $P_1(T) = \sqrt{\frac{a'}{b}(T_0-T)}$. $P_2(T) = 0$. $(a'>0)$

(b) $a\frac{\partial P}{\partial E} + 3bP^2\frac{\partial P}{\partial E} - 1 = 0$. $\chi = \frac{\partial P}{\partial E} = \frac{1}{a+3bP^2}$. expression see below.

(c)

In[85]:= ClearAll["Global`*"]

Simplify[Solve[a P + b P^3 − EG == 0, P]]

Out[86]=
$$\left\{\left\{P\to \frac{-2\times 3^{1/3}\, a\, b + 2^{1/3}\left(9\, b^2\, EG+\sqrt{3}\,\sqrt{b^3\,(4\, a^3+27\, b\, EG^2)}\right)^{2/3}}{6^{2/3}\, b \left(9\, b^2\, EG+\sqrt{3}\,\sqrt{b^3\,(4\, a^3+27\, b\, EG^2)}\right)^{1/3}}\right\}, \left\{P\to \frac{2\times 2^{1/3}\times 3^{1/6}\,(3\, i+\sqrt{3})\, a\, b + i\, 3^{1/3}\,(i+\sqrt{3})\left(18\, b^2\, EG + 2\sqrt{3}\,\sqrt{b^3\,(4\, a^3+27\, b\, EG^2)}\right)^{2/3}}{12\, b\left(9\, b^2\, EG+\sqrt{3}\,\sqrt{b^3\,(4\, a^3+27\, b\, EG^2)}\right)^{1/3}}\right\},$$
$$\left\{P\to \frac{2\times 2^{1/3}\times 3^{1/6}\,(-3\, i+\sqrt{3})\, a\, b + 3^{1/3}\,(-1-i\sqrt{3})\left(18\, b^2\, EG + 2\sqrt{3}\,\sqrt{b^3\,(4\, a^3+27\, b\, EG^2)}\right)^{2/3}}{12\, b\left(9\, b^2\, EG+\sqrt{3}\,\sqrt{b^3\,(4\, a^3+27\, b\, EG^2)}\right)^{1/3}}\right\}\right\}$$

In[89]:=
$$P0 = \frac{-2\times 3^{1/3}\, a\, b + 2^{1/3}\left(9\, b^2\, EG+\sqrt{3}\,\sqrt{b^3\,(4\, a^3+27\, b\, EG^2)}\right)^{2/3}}{6^{2/3}\, b \left(9\, b^2\, EG+\sqrt{3}\,\sqrt{b^3\,(4\, a^3+27\, b\, EG^2)}\right)^{1/3}};$$

$$\chi = \text{FullSimplify}\left[\frac{1}{a + 3\, b\, P0^2}\right];$$

In[93]:= χ = {χ} /. a → a' (T − T0)

Out[93]=
$$\left\{\frac{1}{(T-T0)\, a' + \dfrac{\left(-2\times 3^{1/3}\,(T-T0)\, a' + 2^{1/3}\left(9+\sqrt{3}\,\sqrt{27+4\,(T-T0)^3\,(a')^3}\right)^{2/3}\right)^2}{2\times 6^{1/3}\left(9+\sqrt{3}\,\sqrt{27+4\,(T-T0)^3\,(a')^3}\right)^{2/3}}}\right\}$$

expression of χ

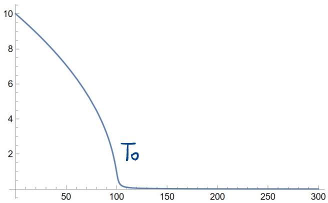
In[95]:= a' = 1; b = 1; EG = 1; χ

Out[95]=
$$\left\{\frac{1}{T + \dfrac{\left(2^{1/3}\left(9+\sqrt{3}\,\sqrt{27+4\,(T-T0)^3}\right)^{2/3}-2\times 3^{1/3}\,(T-T0)\right)^2}{2\times 6^{1/3}\left(9+\sqrt{3}\,\sqrt{27+4\,(T-T0)^3}\right)^{2/3}} - T0}\right\}$$

In[98]:= T0 = 100; a = a' (T − T0);

$$\text{Plot}\left[\left\{\frac{-2\times 3^{1/3}\, a\, b + 2^{1/3}\left(9\, b^2\, EG+\sqrt{3}\,\sqrt{b^3\,(4\, a^3+27\, b\, EG^2)}\right)^{2/3}}{6^{2/3}\, b \left(9\, b^2\, EG+\sqrt{3}\,\sqrt{b^3\,(4\, a^3+27\, b\, EG^2)}\right)^{1/3}}\right\}, \{T, 0, 300\}\right]$$

Out[99]=



a' = −1; b = 1; EG = 1;

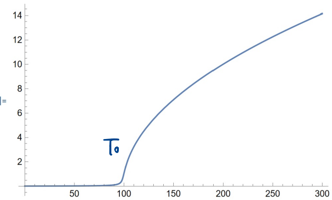
χ

T0 = 100; a = a' (T − T0);

$$\text{Plot}\left[\left\{\frac{-2\times 3^{1/3}\, a\, b + 2^{1/3}\left(9\, b^2\, EG+\sqrt{3}\,\sqrt{b^3\,(4\, a^3+27\, b\, EG^2)}\right)^{2/3}}{6^{2/3}\, b \left(9\, b^2\, EG+\sqrt{3}\,\sqrt{b^3\,(4\, a^3+27\, b\, EG^2)}\right)^{1/3}}\right\}, \{T, 0, 300\}\right]$$

Out[156]=
$$\left\{\frac{1}{T + \dfrac{\left(2^{1/3}\left(9+\sqrt{3}\,\sqrt{27+4\,(T-T0)^3}\right)^{2/3}-2\times 3^{1/3}\,(T-T0)\right)^2}{2\times 6^{1/3}\left(9+\sqrt{3}\,\sqrt{27+4\,(T-T0)^3}\right)^{2/3}} - T0}\right\}$$

Out[158]=

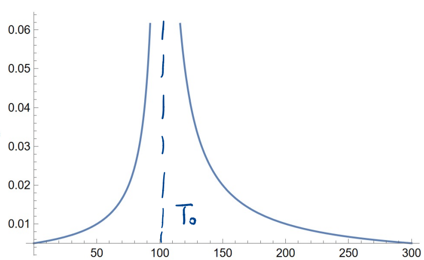


P−T (a' > 0)

P−T (a' < 0)

In[100]:=
$$\text{Plot}\left[\left\{\frac{1}{T + \dfrac{\left(2^{1/3}\left(9+\sqrt{3}\,\sqrt{27+4\,(T-T0)^3}\right)^{2/3}-2\times 3^{1/3}\,(T-T0)\right)^2}{2\times 6^{1/3}\left(9+\sqrt{3}\,\sqrt{27+4\,(T-T0)^3}\right)^{2/3}} - T0}\right\}, \{T, 0, 300\}\right]$$

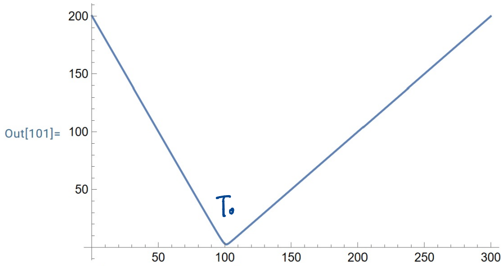
Out[100]=



χ − T

In[101]:= $\text{Plot}\left[\left\{T + \dfrac{\left(2^{1/3}\left(9+\sqrt{3}\sqrt{27+4(T-T0)^3}\right)^{2/3} - 2\times 3^{1/3}(T-T0)\right)^2}{2\times 6^{1/3}\left(9+\sqrt{3}\sqrt{27+4(T-T0)^3}\right)^{2/3}} - T0\right\}, \{T, 0, 300\}\right]$

Out[101]=



$\chi^{-1} - T$

(d)

In[312]:= ClearAll["Global`*"]

$$P0 = \frac{-2\times 3^{1/3}\,a\,b + 2^{1/3}\left(9\,b^2\,EG + \sqrt{3}\sqrt{b^3\left(4\,a^3 + 27\,b\,EG^2\right)}\right)^{2/3}}{6^{2/3}\,b\left(9\,b^2\,EG + \sqrt{3}\sqrt{b^3\left(4\,a^3 + 27\,b\,EG^2\right)}\right)^{1/3}};$$

a = (T - 100); b = 1;

T = 90;

$$\text{plt1} = \text{Plot}\left[\left\{\frac{-2\times 3^{1/3}\,a\,b + 2^{1/3}\left(9\,b^2\,EG + \sqrt{3}\sqrt{b^3\left(4\,a^3 + 27\,b\,EG^2\right)}\right)^{2/3}}{6^{2/3}\,b\left(9\,b^2\,EG + \sqrt{3}\sqrt{b^3\left(4\,a^3 + 27\,b\,EG^2\right)}\right)^{1/3}}\right\}, \{EG, -100, 100\}\right];$$

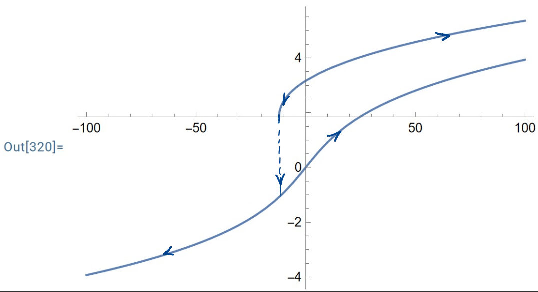
a = - (T - 100); b = 1;

T = 90;

$$\text{plt2} = \text{Plot}\left[\left\{\frac{-2\times 3^{1/3}\,a\,b + 2^{1/3}\left(9\,b^2\,EG + \sqrt{3}\sqrt{b^3\left(4\,a^3 + 27\,b\,EG^2\right)}\right)^{2/3}}{6^{2/3}\,b\left(9\,b^2\,EG + \sqrt{3}\sqrt{b^3\left(4\,a^3 + 27\,b\,EG^2\right)}\right)^{1/3}}\right\}, \{EG, -100, 100\}\right];$$

Show[plt1, plt2, PlotRange → All]

Out[320]=



P - E

(e) $F = \frac{1}{2}aP^2 + \frac{1}{4}bP^4 + \frac{1}{6}cP^6 - PE.$ $\quad b<0,\ c>0$

$a'>0$: $T<T_0$. 2 equilibrium P. $\quad T>T_0$. $P_{\text{equilibrium}} = 0$.

$a'<0$: $T<T_0$. $P(T)=0$. $\quad T>T_0$. 2 $P(T)$.

In[461]:= ClearAll["Global`*"]

$F = \frac{1}{2} a P^2 + \frac{1}{4} b P^4 + \frac{1}{6} c P^6 - P\ EG;$

a = (T - 100); b = -1; c = 1; EG = 1; F

Out[463]= $-P - \frac{P^4}{4} + \frac{P^6}{6} + \frac{1}{2} P^2 (-100 + T)$

In[464]:= T = 90;

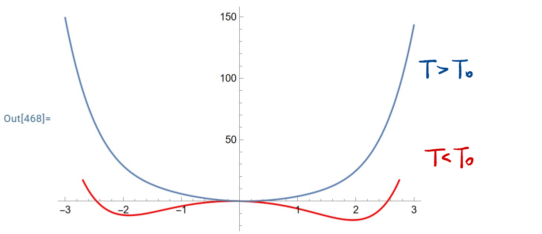
plt1 = Plot$\left[-P - \frac{P^4}{4} + \frac{P^6}{6} + \frac{1}{2} P^2 (-100 + T), \{P, -3, 3\}, \text{PlotStyle} \to \text{Red}\right]$;

In[466]:= T = 110;

plt2 = Plot$\left[-P - \frac{P^4}{4} + \frac{P^6}{6} + \frac{1}{2} P^2 (-100 + T), \{P, -3, 3\}\right]$;

Show[plt1, plt2, PlotRange → All]

Out[468]=



F-P. $a' > 0$. $b < 0$, $c > 0$.

$P(T) = 0$ for $T > T_0$.

2 $P(T)$ for $T < T_0$.

In[469]:= ClearAll["Global`*"]

$F = \frac{1}{2} a P^2 + \frac{1}{4} b P^4 + \frac{1}{6} c P^6 - P\ EG;$

a = - (T - 100); b = -1; c = 1; EG = 1; F

T = 90;

plt1 = Plot$\left[-P - \frac{P^4}{4} + \frac{P^6}{6} + \frac{1}{2} P^2 (100 - T), \{P, -3, 3\}, \text{PlotStyle} \to \text{Red}\right]$;

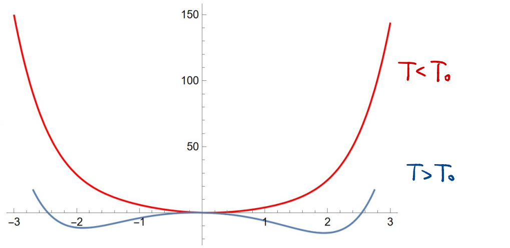
T = 110;

plt2 = Plot$\left[-P - \frac{P^4}{4} + \frac{P^6}{6} + \frac{1}{2} P^2 (100 - T), \{P, -3, 3\}\right]$;

Show[plt1, plt2, PlotRange → All]

Out[471]= $-P - \frac{P^4}{4} + \frac{P^6}{6} + \frac{1}{2} P^2 (100 - T)$

Out[476]=



F-P. $a' < 0$. $b < 0$. $c > 0$.

$P(T) = 0$ for $T < T_0$.

2 $P(T)$ for $T > T_0$.

In[485]:= ClearAll["Global`*"]

$F = \frac{1}{2} a P^2 + \frac{1}{4} b P^4 + \frac{1}{6} c P^6 - P\ EG;$

a = (T - 100); b = 1; c = 0; EG = 1; F

T = 90;

plt1 = Plot$\left[-P + \frac{P^4}{4} + \frac{1}{2} P^2 (-100 + T), \{P, -3, 3\}, \text{PlotStyle} \to \text{Red}\right]$;

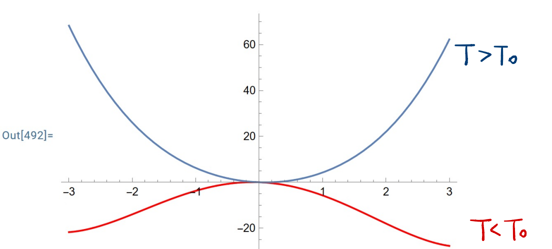
T = 110;

plt2 = Plot$\left[-P + \frac{P^4}{4} + \frac{1}{2} P^2 (-100 + T), \{P, -3, 3\}\right]$;

Show[plt1, plt2, PlotRange → All]

Out[487]= $-P + \frac{P^4}{4} + \frac{1}{2} P^2 (-100 + T)$

Out[492]=



F-P. $a' > 0$. $b > 0$. $c = 0$.

$P(T) = 0$ for $T > T_0$.

1 $P(T)$ for $T < T_0$.

In[501]:= ClearAll["Global`*"]

$F = \frac{1}{2} a P^2 + \frac{1}{4} b P^4 + \frac{1}{6} c P^6 - P\ EG;$

a = - (T - 100); b = 1; c = 0; EG = 1; F

T = 90;

plt1 = Plot$\left[-P + \frac{P^4}{4} + \frac{1}{2} P^2 (100 - T), \{P, -3, 3\}, \text{PlotStyle} \to \text{Red}\right]$;

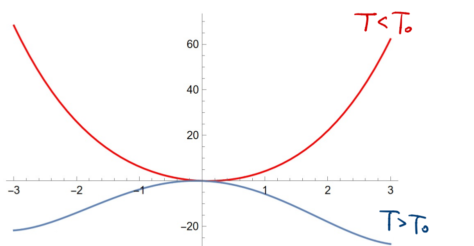
T = 110;

plt2 = Plot$\left[-P + \frac{P^4}{4} + \frac{1}{2} P^2 (100 - T), \{P, -3, 3\}\right]$;

Show[plt1, plt2, PlotRange → All]

Out[503]= $-P + \frac{P^4}{4} + \frac{1}{2} P^2 (100 - T)$

Out[508]=



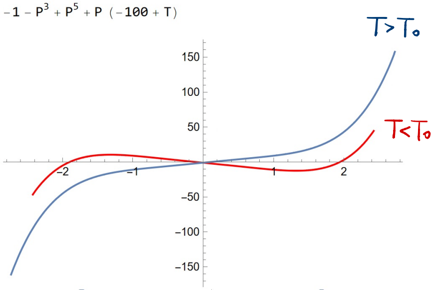
F-P. $a' < 0$. $b > 0$. $c = 0$

$P(T) = 0$ for $T < T_0$.

1 $P(T)$ for $T > T_0$

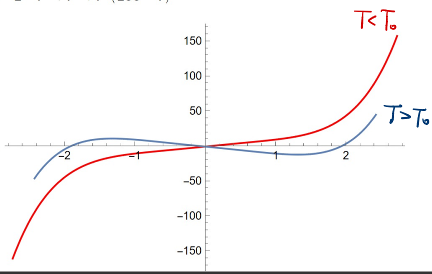
```
In[541]:= ClearAll["Global`*"]
          dF = a P + b P^3 + c P^5 - EG;
          a = (T - 100); b = -1; c = 1; EG = 1; dF
          T = 90;
          plt1 = Plot[-1 - P^3 + P^5 + P (-100 + T), {P, -3, 3}, PlotStyle → Red];
          T = 110;
          plt2 = Plot[-1 - P^3 + P^5 + P (-100 + T), {P, -3, 3}];
          Show[plt1, plt2, PlotRange → All]
Out[543]= -1 - P^3 + P^5 + P (-100 + T)
```

Out[548]=




```
In[549]:= ClearAll["Global`*"]
          dF = a P + b P^3 + c P^5 - EG;
          a = -(T - 100); b = -1; c = 1; EG = 1; dF
          T = 90;
          plt1 = Plot[-1 - P^3 + P^5 + P (100 - T), {P, -3, 3}, PlotStyle → Red];
          T = 110;
          plt2 = Plot[-1 - P^3 + P^5 + P (100 - T), {P, -3, 3}];
          Show[plt1, plt2, PlotRange → All]
Out[551]= -1 - P^3 + P^5 + P (100 - T)
```

Out[556]=



$dF - P$. $a' > 0$. $b < 0$, $c > 0$.

1 root for $T > T_0$.

3 root for $T < T_0$.

(f) $F = \frac{1}{2} a P^2 + \frac{1}{4} b P^4 + \frac{1}{6} c P^6 - PE$.

$$\frac{\partial F}{\partial P} = aP + bP^3 + cP^5 - E = 0.$$

$$c(P^2)^2 + bP^2 + a - \frac{E}{P} = 0. \qquad \text{with } E = 0. \quad P_0 = \left(\frac{-b \pm \sqrt{b^2 - 4ac}}{2c}\right)^{\frac{1}{2}},$$

$$\chi = \frac{\partial P}{\partial E} = \frac{1}{a + 3bP^2 + 5cP^4}. \qquad \text{Assume } P = P_0 + f(E).$$

$$a f(E) + b \cdot 3P_0^2 f(E) + c \cdot 5P_0^4 f(E) - E = 0.$$

$$f(E) = E / \left(a + 3bP_0^2 + 5cP_0^4\right).$$

$$= E / \left(-4a + \frac{b\left(b + \sqrt{b^2 - 4ac}\right)}{c}\right).$$

In[29]:= ClearAll["Global`*"]

$$P = \frac{\sqrt{-\frac{b+\sqrt{b^2-4ac}}{c}}}{\sqrt{2}} + \frac{EG}{-4a + \frac{b\left(b+\sqrt{b^2-4ac}\right)}{c}};$$

$$\chi = \text{Simplify}\left[\frac{1}{a + 3bP^2 + 5cP^4}\right];$$

In[28]:= a = T - 100; b = -1; c = 1; χ;

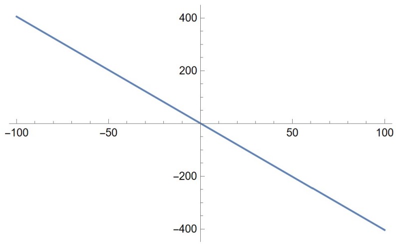
In[8]:= $$P = \frac{\sqrt{-\frac{b+\sqrt{b^2-4ac}}{c}}}{\sqrt{2}} + \frac{EG}{-4a + \frac{b\left(b+\sqrt{b^2-4ac}\right)}{c}}$$

Out[8]= $$\frac{\sqrt{1-\sqrt{1-4(-100+T)}}}{\sqrt{2}} + \frac{EG}{1-\sqrt{1-4(-100+T)}-4(-100+T)}$$

In[44]:= T = 100.2;

$$\text{plt1} = \text{Plot}\left[\left\{\frac{\sqrt{1-\sqrt{1-4(-100+T)}}}{\sqrt{2}} + \frac{EG}{1-\sqrt{1-4(-100+T)}-4(-100+T)}\right\}, \{EG, -100, 100\}\right]$$

Out[45]=



this should be wrong, but do not have more time to spend on this problem.

3. (a) Making use of the spin commutation relation, $[\hat{S}_m^\alpha, \hat{S}_n^\beta] = i\delta_{mn}\epsilon^{\alpha\beta\gamma}\hat{S}_i^\gamma$, apply the identity $i\dot{\hat{\mathbf{S}}}_i = [\hat{\mathbf{S}}_i, \hat{H}]$, to express the equation of motion of a spin in a nearest neighbour spin S one-dimensional Heisenberg ferromagnet, $\hat{H} = -J\sum_m \hat{\mathbf{S}}_m \cdot \hat{\mathbf{S}}_{m+1}$.

(b) Interpreting the spins as *classical* vectors, and taking the continuum limit, show that the equation of motion of the *hydrodynamic modes* takes the form

$$\dot{\mathbf{S}} = Ja^2\mathbf{S} \times \partial^2\mathbf{S},$$

where a denotes the lattice spacing. [Hint: in transferring to the continuum limit, apply a Taylor expansion to the spins viz. $S_{m+1} = S_m + a\partial S_m + \frac{a^2}{2}\partial^2 S_m + \cdots$.]

(c) Confirm that the equation of motion is solved by the *Ansatz*, $\mathbf{S}(x,t) = (c\cos(kx - \omega t), c\sin(kx - \omega t), \sqrt{S^2 - c^2})$, and determine the dispersion. Sketch a 'snapshot' configuration of the spins in the chain.

S_i^x, S_z

(a) $i\dot{S}_i = [S_i, H] = [S_i, -J\sum_m S_m \cdot S_{m+1}]$

$i\dot{S}_{ix} = [S_i, -J\sum_m S_m \cdot S_{m+1}]_x$

$[S_{ix}, S_i \cdot S_{i+1} + S_{i-1} \cdot S_i] = [S_{ix}, \vec{S}_i]\vec{S}_{i+1} + \vec{S}_{i-1}[S_{ix}, \vec{S}_i]$

$= (iS_{iz}\hat{y} - iS_{iy}\hat{z})\vec{S}_{i+1} + \vec{S}_{i-1}(iS_{iz}\hat{y} - iS_{iy}\hat{z})$

$= iS_{iz}S_{i+1\,y} - iS_{iy}S_{i+1\,z} + iS_{i-1\,y}S_{iz} - iS_{i-1\,z}S_{iy}$

$\Rightarrow \dot{S}_{ix} = -J(S_{iz}S_{i+1\,y} - S_{iy}S_{i+1\,z} + S_{i-1\,y}S_{iz} - S_{i-1\,z}S_{iy})$

similarly, could write other components. ($\dot{S}_{iy}$, $\dot{S}_{iz}$)

We have $\dot{\vec{S}}_i = J(\vec{S}_i \times \vec{S}_{i+1} + \vec{S}_i \times \vec{S}_{i-1})$

(b) $S_{m+1} = S_m + a\partial S_m + \frac{a^2}{2}\partial^2 S_m + \cdots$

$$\vec S_i = J\left(\vec S_i \times \left(\vec S_i + a\partial \vec S_i + \frac{a^2}{2}\partial^2 \vec S_i\right) + \left(\vec S_{i-1} + a\partial \vec S_{i-1} + \frac{a^2}{2}\partial^2 \vec S_{i-1}\right)\times \vec S_{i-1}\right)$$

$$= J\left(a\vec S_i\times\partial\vec S_i + \underline{\frac{a^2}{2}\vec S_i\times\partial^2\vec S_i} + a\partial\vec S_{i-1}\times\vec S_{i-1} + \underline{\frac{a^2}{2}\partial^2\vec S_{i-1}\times\vec S_{i-1}}\right)$$

$\vec S_i \longrightarrow \vec S(x)$, we have

vanish for FM chain

$$\dot{\vec S} = J\left(\underline{a\vec S\times\partial\vec S} + \frac{a^2}{2}\vec S\times\partial^2\vec S + \underline{a\partial\vec S\times\vec S} + \frac{a^2}{2}\partial^2\vec S\times\vec S\right)$$

$$= J a^2 \vec S\times\partial^2\vec S .$$

(c) $\dot{\vec S}_i = J\left(\vec S_i\times\vec S_{i+1} + \vec S_i\times\vec S_{i-1}\right).$ set $S_z = S_0$.

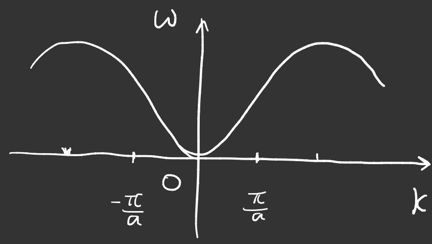
$$\Rightarrow \begin{cases} \dot S_{ix} = J\left[2S_0 S_i^y - S_0\left(S_{i-1}^y + S_{i+1}^y\right)\right] \\ \dot S_{iy} = -J\left[2S_0 S_i^x - S_0\left(S_{i-1}^x + S_{i+1}^x\right)\right] \\ \dot S_{iz} = 0 \end{cases}$$

We have solution of form $\vec S(x,t) = \left(c\cos(kx-wt),\ c\sin(kx-wt),\ \sqrt{S^2-c^2}\right)$.

$$\begin{cases} -wc\cos(kx-wt) = JS_0\left[2c\sin(kx-wt) - c\sin(kx+ka-wt) - c\sin(kx-ka-wt)\right] \\ -wc\sin(kx-wt) = -JS_0\left[2c\cos(kx-wt) - c\cos(kx-ka-wt) - c\cos(kx+ka-wt)\right]. \end{cases}$$

$$\Rightarrow \begin{cases} -w\cos(kx-wt) = JS_0\left(2-2\sin ka\right)\sin(kx-wt). \\ -w\sin(kx-wt) = -JS_0\left(2-2\cos ka\right)\cos(kx-wt). \end{cases}$$

$$\Rightarrow \quad \omega = 2J(1-\cos ka).$$



4. Consider the antiferromagnetic Heisenberg spin chain, $\hat{H} = --J\sum_m \hat{\mathbf{S}}_m \cdot \hat{\mathbf{S}}_{m+1}$, where $J > 0$

As an exercise, fill out the algebra sketched in the lecture.

(a) Make a spin flip on the sublattice, and then apply a Holstein-Primakoff transformation to generate an effective Hamiltonian to leading quadratic order in the bosons.

(b) Diagonalise the resulting Hamiltonian to produce the dispersion of spin waves

(c) Compute the zero-point corrections to the average sublattice magnetization.

(a) Let's flip even site of spins.

$$\tilde{\vec{S}}_i = \begin{cases} \vec{S}_i . & \text{odd } i . \\ -\vec{S}_i . & \text{even } i . \end{cases}$$

$$\tilde{S}_i^+ = \sqrt{2S - a_i^\dagger a_i}\, a_i .$$

$$\tilde{S}_i^- = a_i^\dagger \sqrt{2S - a_i^\dagger a_i} .$$

$$\tilde{S}_i^z = S - a_i^\dagger a_i .$$

$$H = J \sum_m \vec{S}_m \cdot \vec{S}_{m+1} .$$

I finish this problem with help of internet.

$$H = J\left[\sum_m S_m^z S_{m+1}^z + \frac{1}{2}\left(S_m^+ S_{m+1}^- + S_m^- S_{m+1}^+\right)\right]$$

$$= J\sum_m \Big[(S - a_m^\dagger a_m)(S - a_{m+1}^\dagger a_{m+1}) +$$

$$\frac{1}{2}\left(\sqrt{2S - a_m^\dagger a_m}\, a_m a_{m+1}^\dagger \sqrt{2S - a_{m+1}^\dagger a_{m+1}} + a_m^\dagger \sqrt{2S - a_m^\dagger a_m}\sqrt{2S - a_{m+1}^\dagger a_{m+1}}\, a_{m+1}\right)\Big]$$

when numbers of bosons are small. neglect terms $\sim (a^\dagger a)^2$.

$$H \approx J\sum_m \left[\left(S^2 - S(a_m^\dagger a_m + a_{m+1}^\dagger a_m)\right) + \frac{1}{2}\sqrt{4S^2 - 2S(a_m^\dagger a_m + a_{m+1}^\dagger a_{m+1})}\,\left(a_m a_{m+1}^\dagger + a_m^\dagger a_{m+1}\right)\right]$$

$$\approx J\sum_m \left[\left(S^2 - S(a_m^\dagger a_m + a_{m+1}^\dagger a_m)\right) + \Big(S - \underbrace{\frac{1}{4}(a_m^\dagger a_m + a_{m+1}^\dagger a_{m+1})}_{\text{neglect}}\Big)\left(a_m a_{m+1}^\dagger + a_m^\dagger a_{m+1}\right)\right]$$

$$\approx J \sum_m \left[\left(S^2 - S(a_m^\dagger a_m + a_{m+1}^\dagger a_m) \right) + S\left(a_m a_{m+1}^\dagger + a_m^\dagger a_{m+1} \right) \right]$$

$$= NJS^2 - JS \sum_m \left(a_m^\dagger a_m + \underline{a_{m+1}^\dagger a_m} - a_m^\dagger a_{m+1} - \underline{a_m a_{m+1}^\dagger} \right)$$

$$\approx NJS^2 - 2JS \sum_m \left(a_m^\dagger a_m - a_m^\dagger a_{m+1} \right).$$

(b) Introduce Fourier transform

$$a_i = \frac{1}{\sqrt{N}} \sum_k a_k e^{ikx_i} . \qquad a_i^\dagger = \frac{1}{\sqrt{N}} \sum_k a_k^\dagger e^{-ikx_i} .$$

$$\sum_m a_m^\dagger a_m = \sum_m \sum_{kk'} \frac{1}{N} a_k^\dagger a_{k'} \underline{e^{i(k-k')x_m}} \qquad \delta_{kk'}$$

$$= \sum_k a_k^\dagger a_k .$$

$$\sum_m a_m^\dagger a_{m+1} = \sum_m \sum_{kk'} \frac{1}{N} a_k^\dagger a_{k'} \underline{e^{i(kx_m - k'x_{m+1})}} \qquad e^{i(k-k')x_m} e^{-ik'a} \qquad \text{sublattice?}$$

$$= \sum_k a_k^\dagger a_k \left(e^{-ika} + e^{ika} \right)/2 \qquad \frac{N}{2} . \qquad e^{-ika} \quad e^{ika} .$$

$$= \sum_k a_k^\dagger a_k \cdot \cos ka .$$

$$H = NJS^2 - 2JS \sum_k a_k^\dagger a_k (1 - \cos ka) .$$

$$= NJS^2 - 4JS \sum_k a_k^\dagger a_k \underline{\sin^2 \frac{ka}{2}} . \qquad \text{dispersion}$$

(c) $S_i^z = S - \underline{a_i^\dagger a_i}$.

$$\Delta M = -\langle a_i^\dagger a_i \rangle = -\frac{1}{N} \sum_k \langle a_k^\dagger a_k \rangle$$

I know from internet $\langle a_k^\dagger a_k \rangle = \frac{1}{2} \left(\frac{1}{\varepsilon(k)} - 1 \right)$

$$\Delta M = -\frac{1}{N}\sum_{k}\frac{1}{2}\left(\frac{1}{\varepsilon(k)}-1\right)$$

$$= -\frac{1}{2N}\int_{-\pi}^{\pi}\frac{dk}{2\pi}\left(\frac{1}{\sin^2\left(\frac{ka}{2}\right)}-1\right) =$$

In[6]:= $-\frac{1}{2\,\mathrm{N0}}\int \frac{1}{2\pi}\left(\frac{1}{\mathrm{Sin}\left[\frac{k\,a}{2}\right]^2}-1\right)dk$

Out[6]= $\dfrac{\mathrm{Cot}\left[\frac{k}{2}\right]\,\mathrm{Hypergeometric2F1}\left[-\frac{1}{2},1,\frac{1}{2},-\mathrm{Tan}\left[\frac{k}{2}\right]^2\right]}{2\,\mathrm{N0}\,\pi}$

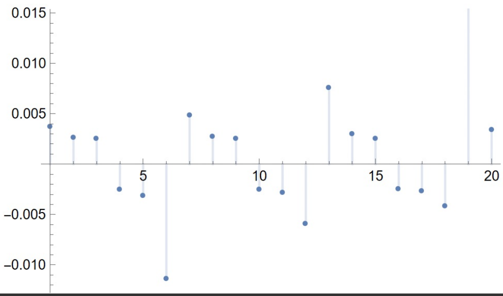
In[7]:= a = 1; N0 = 100;

$\dfrac{\mathrm{Cot}\left[\frac{k}{2}\right]\,\mathrm{Hypergeometric2F1}\left[-\frac{1}{2},1,\frac{1}{2},-\mathrm{Tan}\left[\frac{k}{2}\right]^2\right]}{2\,\mathrm{N0}\,\pi}$

Out[8]= $\dfrac{\mathrm{Cot}\left[\frac{k}{2}\right]\,\mathrm{Hypergeometric2F1}\left[-\frac{1}{2},1,\frac{1}{2},-\mathrm{Tan}\left[\frac{k}{2}\right]^2\right]}{200\,\pi}$

In[9]:= DiscretePlot$\left[\dfrac{\mathrm{Cot}\left[\frac{k}{2}\right]\,\mathrm{Hypergeometric2F1}\left[-\frac{1}{2},1,\frac{1}{2},-\mathrm{Tan}\left[\frac{k}{2}\right]^2\right]}{200\,\pi}, \{k, 1, 20\}\right]$

Out[9]=



5. Applying the Euler-Lagrange equation, obtain the equation of motion associated with the Lagrangian densities:

1. $$\mathcal{L}[\phi, \dot{\phi}, \partial_x\phi] = \frac{m\dot{\phi}^2}{2} - \frac{k_s a^2}{2}(\partial_x\phi)^2 - \frac{m}{2}\omega^2\phi^2$$
2. $$\mathcal{L}[\phi, \dot{\phi}, \partial_x^2\phi] = \frac{m\dot{\phi}^2}{2} - \frac{\kappa}{2}\left(\partial_x^2\phi\right)^2$$
3. $$\mathcal{L}[\phi, \dot{\phi}] = \frac{m\dot{\phi}^2}{2} - \frac{m}{2}\omega^2\phi^2 - \frac{\eta}{4}\phi^4$$
4. $$\mathcal{L}[\{\dot{\phi}_i\}, \{\partial_x\phi_i\}] = \sum_{i=1}^{n}\left[\frac{m}{2}\dot{\phi}_i^2 - \frac{1}{2}k_s a^2(\partial_x\phi_i)^2\right]$$
5. $$\mathcal{L}[\dot{\phi}, \partial_x\phi] = \frac{m}{2}|\dot{\phi}|^2 - \frac{1}{2}k_s a^2|\partial_x\phi|^2$$

[Note that in 5. the field ϕ is complex.] Suggest a physical significance of the last term in 1. What is the effect of this term on the excitation spectrum of the corresponding quantum Hamiltonian? Starting with the Lagrangian 2., obtain the Hamiltonian density.

E-L eqn: $\partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi)}\right) - \frac{\partial\mathcal{L}}{\partial\phi} = 0.$

1. $\partial_0(m\dot{\phi}) + \partial_x(-k_s a^2\partial_x\phi) + m\omega^2\phi = 0.$

$\Rightarrow m\ddot{\phi} - k_s a^2\partial_x^2\phi + m\omega^2\phi = 0.$

2. $\frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right) + \partial_\mu\partial_\nu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\partial_\nu\phi)}\right) = 0.$

$-\partial_0(m\dot{\phi}) + \partial_x^2(\kappa\partial_x^2\phi) = 0.$

$\Rightarrow \kappa\partial_x^4\phi - m\ddot{\phi} = 0.$

3. $\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right) - \frac{\partial\mathcal{L}}{\partial\phi} = 0.$

$\partial_0(m\dot{\phi}) + m\omega^2\phi + \eta\phi^3 = 0. \Rightarrow \ddot{\phi} + \omega^2\phi + \frac{\eta}{m}\phi^3 = 0.$

4. $$\sum_{i=1}^{n}\left[\partial_0(m\dot{\phi}_i) + \partial_x\left(-k_s a^2(\partial_x\phi_i)\right)\right] = 0$$

$$\Rightarrow \quad \sum_{i=1}^{n}\left(m\ddot{\phi}_i - k_s a^2 \partial_x^2 \phi_i\right) = 0.$$

5. $$\mathcal{L} = \frac{m}{2}\dot{\phi}^*\dot{\phi} - \frac{1}{2}k_s a^2 \partial_x\phi^*\partial_x\phi.$$

$$\frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right) = 0, \quad \frac{\partial\mathcal{L}}{\partial\phi^*} - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^*)}\right) = 0$$

$$\partial_0(m\dot{\phi}^*) - k_s a^2 \partial_x(\partial_x\phi^*) = 0, \quad \partial_0(m\dot{\phi}) - k_s a^2 \partial_x(\partial_x\phi) = 0.$$

$$\Rightarrow \begin{cases} m\dot{\phi}^* - k_s a^2 \partial_x^2 \phi^* = 0 \\ m\dot{\phi} - k_s a^2 \partial_x^2 \phi = 0. \end{cases}$$

Significance of $-\frac{m}{2}\omega^2\phi^2$: harmonic potential term could be used in describing many interactions.

It effects the mass of the excitations of field.

$$\mathcal{L} = \frac{m}{2}\dot{\phi}^2 - \frac{k}{2}(\partial_x^2\phi)^2. \qquad \pi = \frac{\partial\mathcal{L}}{\partial(\partial_0\phi)} = m\dot{\phi}.$$

$$\mathcal{H} = \pi\dot{\phi} - \mathcal{L} = \frac{m}{2}\dot{\phi}^2 + \frac{k}{2}(\partial_x^2\phi)^2.$$

6. In an isotropic fluid, we define a linear response function by the current density J induced by a force F:

$$J_i(\mathbf{q},\omega) = \chi_{ij}(\mathbf{q},\omega)F_j(\mathbf{q},\omega) \tag{4}$$

where the subscripts i, j refer to Cartesian components, and we are considering long-wavelength (small q) and low frequency response. We impose mass conservation, i.e. $\partial_\tau \rho + \nabla \cdot \mathbf{J} = 0$.

(a) Explain why the most general response function for an isotropic system is of the form

$$\chi_{ij}(\mathbf{q} \to 0, \omega = 0) = \chi_L \frac{q_i q_j}{q^2} + \chi_T \left(\delta_{ij} - \frac{q_i q_j}{q^2} \right) \tag{5}$$

and describe the response to longitudinal and transverse forces (parallel or perpendicular to the momentum).

(b) Explain what happens if χ_L and χ_T are not the same.

(c) Discuss why this motivates the response function of a superfluid to be of the form, identifying the meaning of c_s,

$$\chi^S_{ij}(\mathbf{q},\omega) = c_s^2 q_i q_j / (c_s^2 q^2 - \omega^2)$$

(a) isotropic system : χ dependent on $\vec{q}$.

Set $\vec{q}$ along i direction, we have

$$\vec{q} = \begin{pmatrix} q \\ 0 \\ 0 \end{pmatrix} \begin{array}{c} i \\ j \\ k \end{array} . \qquad \chi_{ij} = \begin{pmatrix} \chi_L & & \\ & \chi_T & \\ & & \chi_T \end{pmatrix} .$$

we could see $\chi_{xx} = \chi_L$ in longitudial direction,

$\chi_{yy} = \chi_{zz} = \chi_T$ in transverse direction. meet requirement.

More generally, we have

$$\vec{F}_L = \frac{\vec{q}\cdot\vec{F}}{q^2}\vec{q} \qquad \vec{F}_T = \vec{F} - \vec{F}_L .$$

$$\vec{J}_L = \chi \vec{F}_L = \left(\chi_L \frac{q_i q_j}{q^2} + \chi_T\left(\delta_{ij} - \frac{q_i q_j}{q^2}\right)\right)\frac{\vec{q}\cdot\vec{F}}{q^2}\vec{q}$$

$$\stackrel{?}{=} \chi_L \vec{F}_L .$$

$$\vec{J}_T \stackrel{?}{=} \chi_T \vec{F}_T .$$

Each force initiate response in corresponding direction.

(b) Will generate anisotropy in response.

For fluid, it might induce different phyiscal properties in longitudial and trasverse direction. (e.g. refraction index, viscosity)

(c) Superfluid does not have viscosity and will not induce transverse response. We have $\chi_T = 0$.

$$\chi^S_{ij}(\vec{q}\to 0, \omega=0) = \chi_L \frac{q_i q_j}{q^2} . \qquad \text{for } \omega \neq 0 ?$$

$$\chi^S_{ij}(\vec{q}\to 0, \omega=0) = c_s^2 \frac{q_i q_j}{c_s^2 q^2 - \omega^2} .$$

c_s^2 represents longitudial response coefficient and it includes ω-dependence.

7. In conventional bulk superconductors, it appears that on cooling through the transition temperature, the electrical resistivity jumps discontinuously from a finite value, to zero . Also, above the transition temperature, the specific heat is barely changed. (See Fig.1). What is the important physical property of a weak-coupling superconductor that explains why fluctuations are not important? Expressed in terms of a microscopic length scale in a Ginzburg-Landau theory, what is the region of proximity to the critical point where fluctuations are important?

superconducting gap Δ. fluctuations not larger than this gap will be suppressed.

correlation length: $\xi_T \sim \frac{\xi_0}{\sqrt{1-\frac{T_c}{T}}}$.

below info is from internet.

Ginzburg - Landau free energy density $f = \alpha|\psi|^2 + \beta|\psi|^4 + \frac{\hbar^2}{2m}|\nabla\psi|^2$.

around T_c, we have $|\psi| \sim \frac{\alpha^2}{\beta}$.

$\delta\psi \sim \frac{k_B T}{V} \sim \frac{k_B T}{\xi^3}$.

$\alpha \sim \alpha_0(T-T_c)$.

set $|\psi| \sim \delta\psi$, we have

$$\frac{\alpha_0^2(T-T_c)^2}{\beta} = \frac{k_B T}{\xi_0^3}\left(1-\frac{T_c}{T}\right)^{\frac{3}{2}}$$

$$\Rightarrow \Delta T = |T-T_c| \sim T_c\left(\frac{a}{\xi_0}\right)^4.$$

Advanced Statistical Mechanics Pset 2 yuxiang@uchicago.edu

1. *Phase diagram near a Lifshits Point. This is a reminder that ordered phases can be periodic in space*

The free energy functional F has the following form :

$$F = \int d\mathbf{r} \left\{ A\eta^2 + B\eta^4 + \Delta \left(\frac{\partial \eta}{\partial x}\right)^2 + \Gamma \left(\frac{\partial^2 \eta}{\partial x^2}\right)^2 \right\}$$

The Lifshits point is determined by two conditions:

$$A = 0 \qquad \Delta = 0$$

Assuming that $B > 0$ and $\Gamma > 0$, consider periodic modulation of the order parameter $\eta(x) = \eta_0 \cos qx$.

(a) Find the equilibrium pitch q_0 of modulation.

(b) Draw qualitatively the phase diagram in (A, Δ) plane. What are the properties of the phases ?

(c) Find the shape of the line of the second order phase transition into uniform and modulated phases. Comparing the total free energies in both phases, find the shape of line of the first order transition.

(a) $F = \int d\vec{r} \left(A\eta_0^2 \cos^2 qx + B\eta_0^4 \cos^4 qx + \Delta q^2 \eta_0^2 \sin^2 qx + \Gamma q^4 \eta_0^2 \cos^2 qx \right)$

average over x $\longrightarrow$

$$\frac{q}{2\pi}\int_0^{\frac{2\pi}{q}} \cos^2 qx \, dx = \frac{q}{2\pi}\int_0^{\frac{2\pi}{q}} \frac{1-\cos 2qx}{2} dx = \frac{1}{2}.$$

$$\frac{q}{2\pi}\int_0^{\frac{2\pi}{q}} \cos^4 qx \, dx = \frac{q}{2\pi}\int_0^{\frac{2\pi}{q}} \left(\frac{3}{8} + \frac{1}{2}\cos 2x + \frac{1}{8}\cos 4x\right) dx = \frac{3}{8}.$$

$$\Rightarrow \langle F \rangle = \frac{1}{2}A\eta_0^2 + \frac{3}{8}B\eta_0^4 + \frac{1}{2}\Delta q^2\eta_0^2 + \frac{1}{2}\Gamma q^4 \eta_0^2$$

Set $\frac{\partial \langle F \rangle}{\partial q} = 0.$ $\Rightarrow$ $\Delta q + 2\Gamma q^3 = 0.$

$\Rightarrow \quad q = 0 \quad \text{or} \quad q = \sqrt{\frac{-\Delta}{2\Gamma}}$. (exist when $\Delta < 0$ since $\Gamma > 0$)

(b) To gain phase diagram $\sim A, \Delta$. (heat map of F_{min})

It will be convenient to confine η based on F.

$$\frac{\partial f}{\partial \eta} = 2A\eta + 4B\eta^3 - \Delta\left(\frac{\partial \eta}{\partial x}\right)^2 + \Gamma\left(\frac{\partial \eta}{\partial x}\right)^4 = 0. \qquad \frac{\partial^2 f}{\partial \eta^2} = 2A + 12B\eta^2.$$

We have ① $\eta = 0$, $A > 0$. (stable $\frac{\partial^2 f}{\partial \eta^2} > 0$)

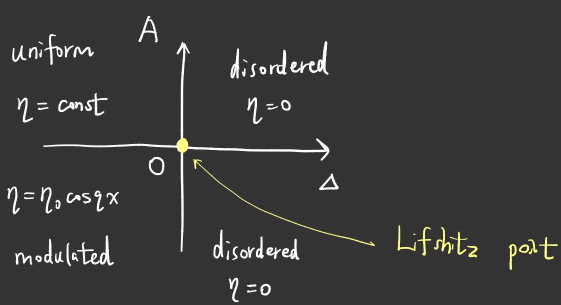
② $\eta = \text{const}$, $\frac{\partial f}{\partial \eta} = 2A\eta_0 + 4B\eta_0^3 = 0. \Rightarrow \eta_0 = 0 \text{ or } \pm\sqrt{\frac{-A}{2B}}$.

$\Rightarrow A < 0$. $\Delta > 0$ to ensure stability.

③ $\eta = \eta_0 \cos qx$. As discussed in (a), we have modulated state

at $\Delta < 0$, $q = \sqrt{-\frac{\Delta}{2\Gamma}}$.

Overall,



(c) As discussed above, for uniform phases,

$$F_1 = A\eta_0^2 + B\eta_0^4 = A \cdot \frac{-A}{2B} + B \cdot \left(\frac{-A}{2B}\right)^2 = -\frac{A^2}{B}.$$

for modulated phases,

$q = \sqrt{-\frac{\Delta}{2\Gamma}}$

$$F_2 = \frac{1}{2} A \eta_0^2 + \frac{3}{8} B \eta_0^4 + \frac{1}{2} \Delta q^2 \eta_0^2 + \frac{1}{2} \Gamma q^4 \eta_0^2$$

$$= \frac{1}{2} A \eta_0^2 + \frac{3}{8} B \eta_0^4 - \frac{\Delta^2}{8\Gamma} \eta_0^2 .$$

```
In[29]:= x = - 2 (A - Δ^2/(8 Γ)) / (3 B0);
         Simplify[1/2 (A - Δ^2/(8 Γ)) x + 3/8 B0 x^2]
Out[30]= - (A - Δ^2/(8 Γ))^2 / (6 B0)
```

has minimum at $\eta_0 = \sqrt{-\frac{2}{3B}\left(A - \frac{\Delta^2}{4\Gamma}\right)}$.

$$F_2 \rightarrow -\frac{\left(A - \frac{\Delta^2}{4\Gamma}\right)^2}{6B} .$$

Compare F_1 and F_2.

Set $F_1 = F_2$, we get $-\frac{A^2}{B} = -\frac{\left(A - \frac{\Delta^2}{4\Gamma}\right)^2}{6B}$

$$\Rightarrow \pm\sqrt{6} A = A - \frac{\Delta^2}{4\Gamma} . \quad \Rightarrow \quad \frac{\Delta^2}{4\Gamma} = (1 \pm \sqrt{6}) A .$$

shape : parabolic curve in phase diagram.

2. *Gaussian model*

As derived in lectures, the free energy density of the Gaussian model of an n-component spin becomes

$$f(t,h) = -\frac{h^2}{t} + n\int_0^\Lambda \frac{d^dq}{(2\pi)^d}\ln(t + Kq^2 + Lq^4) \tag{1}$$

Here Λ is a microscopic cutoff, of order an inverse lattice constant.

Either directly from the above, or by computing the second derivative $\partial^2 f/\partial t^2$, find the singular contribution to the specific heat, and thereby show that the free energy satisfies the scaling form

$$f_{sing} \sim t^{2-\alpha} g_f(h/t^\Delta)$$

Determine the values of α and Δ.

Explain why your calculation shows that the parameter L is *irrelevant*.

Let's derive again and understand each term.

For n-component spin Hamiltonian

$$\mathcal{H}[\phi] = \int d^dr\left[L|\nabla^2\phi(r)|^2 + K|\nabla\phi(r)|^2 + t|\phi(r)|^2 - h\phi(r)\right].$$

$$\phi(r) \to \int \frac{d^dq}{(2\pi)^d}\,\phi(q)\,e^{iq\cdot r}.$$

$$\Rightarrow \quad \mathcal{H}[\phi] = \int \frac{d^dq}{(2\pi)^d}\left[(Lq^4 + Kq^2 + t)|\phi(q)|^2 - h\delta(q)\phi(q)\right].$$

$$Z = \int D[\phi]\,e^{-\mathcal{H}[\phi]}. \qquad f \sim -\ln Z.$$

$$Z = \underset{(q=0)}{\int d\phi\, e^{-t\phi^2 + h\phi}} \cdot \underset{(q\neq 0)}{\prod_q \int d\phi\, e^{-(Lq^4+Kq^2+t)\phi^2}}$$

$$\sim e^{-\frac{h^2}{t}} \cdot e^{n\int \frac{d^dq}{(2\pi)^d}\ln(Lq^4+Kq^2+t)}$$

$$\Rightarrow \quad f(t,h) = -\frac{h^2}{t} + n\int_0^{\Lambda} \frac{d^dq}{(2\pi)^d} \ln\left(Lq^4 + Kq^2 + t\right).$$

$$C \sim -t\frac{\partial^2 f}{\partial t^2}. \qquad \frac{\partial f}{\partial t} = \frac{h^2}{t^2} + n\int_0^{\Lambda} \frac{d^dq}{(2\pi)^d} \frac{1}{Lq^4+Kq^2+t}$$

$$C \sim \frac{2h^2}{t^2} + \underline{nt\int_0^{\Lambda} \frac{d^dq}{(2\pi)^d} \frac{1}{(Lq^4+Kq^2+t)^2}}$$ origin of singularity (*)

$$d^dq \to dq\, q^{d-1}\,\Omega_d.$$

$$(*) \to nt\int_0^{\Lambda} \frac{\Omega_d}{(2\pi)^d} \frac{q^{d-1}}{(Lq^4+Kq^2+t)^2}$$

so above derivation is not neccessary..

No, we really care about f rather than exact form of C.

Still focus on $f(t,h) = -\frac{h^2}{t} + n\int_0^{\Lambda} \frac{d^dq}{(2\pi)^d} \ln\left(Lq^4+Kq^2+t\right).$

$$d^dq \to dq\, q^{d-1}\,\Omega_d. \qquad f(t,h) = -\frac{h^2}{t} + n\int_0^{\Lambda} dq\, q^{d-1} \frac{\Omega_d}{(2\pi)^d} \ln\left(Lq^4+Kq^2+t\right)$$

$$x = \sqrt{\frac{K}{t}}\,q. \quad \Rightarrow f(t,h) = -\frac{h^2}{t} + n\int_0^{\sqrt{\frac{K}{t}}\Lambda} \sqrt{\frac{t}{K}}\sqrt{\frac{t}{K}}^{\,d-1} x^{d-1}\,dx\, \frac{\Omega_d}{(2\pi)^d} \ln\left(L\frac{t^2}{K^2}x^4 + K\frac{t}{K}x^2 + t\right)$$

$$= -\frac{h^2}{t} + n\underline{\int_0^{\sqrt{\frac{K}{t}}\Lambda} \left(\frac{t}{K}\right)^{\frac{d}{2}} x^{d-1}\,dx\, \frac{\Omega_d}{(2\pi)^d}\left[\ln\left(L\frac{t}{K^2}x^4 + x^2 + 1\right) + \ln t\right]} \quad (*)$$

$$(*) \sim \underbrace{\int_0^{\sqrt{\frac{K}{t}}\Lambda} dx\, t^{\frac{d}{2}} x^{d-1} \ln\left(L\frac{t}{K^2}x^4 + x^2 + 1\right)}_{A} + \underbrace{\int_0^{\sqrt{\frac{K}{t}}\Lambda} dx\, t^{\frac{d}{2}} x^{d-1} \ln t}_{B}$$

For singular contribution consider $q \ll 1$. $\left(x \ll \sqrt{\frac{K}{t}}\right)$

$$A = t^{\frac{d}{2}} \int_0^{\sqrt{\frac{K}{t}}\Lambda} dx\, x^{d-1} \ln\left(L\frac{t}{K^2}x^4 + x^2 + 1\right) \qquad B = \ln t \cdot t^{\frac{d}{2}} \int_0^{\sqrt{\frac{K}{t}}\Lambda} dx\, x^{d-1}.$$

$$\approx t^{\frac{d}{2}} \int_0^{\sqrt{\frac{K}{t}}\Lambda} dx\, x^{d-1} \left(L\frac{t}{K^2}x^4 + x^2\right) \qquad = \ln t \cdot t^{\frac{d}{2}} \cdot \frac{1}{d}\left(\sqrt{\frac{K}{t}}\Lambda\right)^d.$$

$$= t^{\frac{d}{2}} \cdot \left[\frac{1}{d+2}\left(\sqrt{\frac{K}{t}}\Lambda\right)^{d+2} + \frac{1}{d+4} L\frac{t}{K^2}\left(\sqrt{\frac{K}{t}}\Lambda\right)^{d+4}\right]$$

combine : $$t^{\frac{d}{2}}\left[\frac{1}{d+2}\left(\sqrt{\frac{K}{t}}\Lambda\right)^{d+2} + \frac{1}{d+4} L\frac{t}{K^2}\left(\sqrt{\frac{K}{t}}\Lambda\right)^{d+4} + \ln t \cdot \frac{1}{d}\left(\sqrt{\frac{K}{t}}\Lambda\right)^d\right]$$

$$\propto t^{-\frac{d+2}{2}} \qquad \propto t^{-\frac{d+2}{2}} \qquad \propto t^{-\frac{d}{2}} \ln t$$

$\uparrow$

note this L term is "absorbed" as previous term, with same dependence of t.

we combine them to be $C t^{-\frac{d+2}{2}}$.

these becomes $A t^{-\frac{d}{2}}$ term on Kardar page 68.

Overall $$f(h,t) = t^{\frac{d}{2}}\left[C t^{-\frac{d+2}{2}} + \ln t \cdot \frac{1}{d}\left(\sqrt{\frac{K}{t}}\Lambda\right)^d\right] - \frac{h^2}{t}$$

$$= t^{\frac{d}{2}} \underbrace{\left[C t^{-\frac{d+2}{2}} + \ln t \cdot \frac{1}{d}\left(\sqrt{\frac{K}{t}}\Lambda\right)^d - \left(\frac{h}{t^{\frac{d}{4}+\frac{1}{2}}}\right)^2\right]}_{g_f}$$

compare with $t^{2-\alpha} g_f(h/t^{\Delta})$.

we have $\alpha = 2 - \frac{d}{2}$, $\Delta = \frac{d}{4} + \frac{1}{2}$.

3. *Dangerously irrelevant variables*

(a) Explain briefly what is meant by the statement that 'mean field theory violates hyperscaling'. Hyperscaling is the Josephson relation $d\nu = 2 - \alpha$, where ν is the correlation length exponent, and α for the specific heat.

(b) Consider the singular part of the free energy density

$$f_s(t, h, L) = t^{d/y_t} f_s(1, ht^{-y_h/y_t}, Lt^{-y_L/y_t})$$

where L is an irrelevant variable, so $y_L < 0$. Hence as $t \to 0$, we expect

$$f_s(t, h, L) = t^{d/y_t} g_f(ht^{-y_h/y_t})$$

Suppose however that

$$\lim_{z \to 0} f_s(x, y, z) = z^{-\mu} \bar{f}(x, y), \quad \mu > 0.$$

Show that this leads to a violation of the Josephson hyperscaling law

(c) Show that near the Gaussian fixed point for Ginzburg-Landau theory in $d > 4$, the quartic coupling u is formally irrelevant and that the crossover exponent $y_u/y_t = -(d-4)/2$.

(d) Compute the Landau (mean field) energy, and show that it scales as $f_0 \sim t^2/u$.

(e) Explain how this resolves the issue.

(a) Josephson theory : $d\nu = 2 - \alpha$. $\quad (\xi \sim t^{-\nu},\ c \sim |t|^{-\alpha})$

↑

long-range correlation around critical point

Mean-field theory suppress the correlation & fluctuation $(d \to \infty)$

↑

spread of correlation is faster and quicker suppressed

distributed by more dimensions and diluted

Thus mean-field theory effectively raises dimensionality and violate the hyperscaling which works at finite dimensionality.

(b) $f_s(t,h,L) = t^{d/y_t} f_s(1, ht^{-y_h/y_t}, Lt^{-y_L/y_t})$

$t\to 0,\quad f_s(t,h,L) = t^{d/y_t}\, g_f(ht^{-y_h/y_t})$. $\qquad (h=0,\ L\to\infty)$

$\Rightarrow \dfrac{\partial f_s}{\partial t} = \dfrac{d}{y_t} t^{d/y_t - 1} f_s'$. $\qquad$ since $C \sim -\dfrac{\partial^2 f_s}{\partial t^2}$.

$\dfrac{\partial^2 f_s}{\partial t^2} = \dfrac{d}{y_t}\left(\dfrac{d}{y_t}-1\right) t^{d/y_t - 2} f_s''$. $\qquad \Rightarrow C \sim t^{d/y_t - 2}$. $-\alpha$

$\alpha = 2 - \dfrac{d}{y_t}$. $\qquad$ Josephson hyperscaling holds.

For $\lim_{z\to 0} f_s(x,y,z) = z^{-\mu}\bar{f}(x,y)$, $\mu>0$.

z

We have $f_s(t,h,L) = t^{d/y_t} f_s(1, ht^{-y_h/y_t}, \underline{Lt^{-y_L/y_t}})$

$t\to 0,\quad f_s(t,h,L) = t^{d/y_t} L^{-\mu} t^{\mu y_L/y_t} \bar{f}_s(ht^{-y_h/y_t})$

$\Rightarrow \dfrac{\partial f_s}{\partial t} \sim \dfrac{d+\mu y_L}{y_t} t^{\frac{d+\mu y_L}{y_t}-1}$.

$\dfrac{\partial^2 f_s}{\partial t^2} \sim \left(\dfrac{d+\mu y_L}{y_t}\right)\left(\dfrac{d+\mu y_L}{y_t}-1\right) t^{\frac{d+\mu y_L}{y_t}-2}$.

$\Rightarrow C \sim t^{\frac{d+\mu y_L}{y_t}-2}$. $\qquad \alpha = 2 - \dfrac{d+\mu y_L}{y_t}$. $\quad \neq d\nu$. $\quad$ violated.

(c) Landau - Ginzburg Hamiltonian :

$$\beta \mathcal{H} = \beta F_0 + \int d^d \vec{x} \left[\frac{t}{2} m^2(\vec{x}) + u m^4(\vec{x}) + \frac{K}{2}(\nabla m)^2 + \cdots - \vec{h} \cdot \vec{m}(\vec{x}) \right].$$

fixed point : $u = 0$. look into scaling behavior of F.

Let's keep go with $F[\phi] = \int d^d x \left[\frac{t}{2}\phi^2 + \frac{K}{2}(\nabla\phi)^2 + u\phi^4 - h\phi \right]$

$x \to bx$.

$d^d x \to d^d x \, b^{-d}$. thus

$\phi \to b^{\Delta_\phi}\phi$.

$$F[\phi] = \int d^d x \left[\frac{t}{2}\phi^2 + \frac{K}{2}(\nabla\phi)^2 + u\phi^4 - h\phi \right]$$

$$\rightarrow \quad d^d x \, b^{-d} \left[t b^{2\Delta_\phi}\phi^2 + \underbrace{K b^{2\Delta_\phi - 2}\phi^2}_{} + u b^{4\Delta_\phi}\phi^4 - h b^{\Delta_\phi}\phi \right]$$

$$\Downarrow$$

$$-d + 2\Delta_\phi - 2 = 0 \quad \Rightarrow \quad \Delta_\phi = \frac{d-2}{2}.$$

$$\rightarrow \quad d^d x \left[t b^{-2}\phi^2 + \phi^2 + u b^{d-4}\phi^4 - h b^{\frac{-d-2}{2}}\phi \right]$$

$$\Rightarrow \quad t' = t b^2, \quad u' = u b^{4-d}, \quad h' = h b^{\frac{d+2}{2}}.$$

For $d > 4$, $u' = b^{4-d}$, $4 - d < 0$. irrelevant. (gets smaller under scaling)

And $y_u / y_t = \frac{4-d}{2}$.

(d) $F[\phi] = \int d^dx \left[\frac{t}{2}\phi^2 + \frac{k}{2}(\nabla\phi)^2 + u\phi^4 - h\phi\right]$

$$f = \frac{t}{2}\phi^2 + \frac{k}{2}(\nabla\phi)^2 + u\phi^4 - h\phi.$$

$$\frac{\partial f}{\partial \phi} = t\phi + 4u\phi^3 - h = 0.$$

$$\Rightarrow f_{min} = \frac{t}{2}\phi_0^2 + \underbrace{\frac{k}{2}(\nabla\phi)^2}_{=0 \text{ for mean field}} + u\phi_0^4 - (t\phi_0 + 4u\phi_0^3)\phi_0$$

$$= -\frac{t}{2}\phi_0^2 - 3u\phi_0^4.$$

From $t\phi_0 + 4u\phi_0^3 - h = 0.$

$$\phi_0 \sim \sqrt{\frac{-t}{u}}$$

$$f_{min} = -\frac{t}{2}\phi_0^2 - 3u\phi_0^4$$

$$\sim -t\cdot\frac{-t}{u} - 3u\cdot\frac{t^2}{u^2} \sim \frac{t^2}{u}.$$

(e) maybe the $e^{-\mu}$ is from quadratic coupling and is made irrelevant around Gaussian fixed point for $d>4$. thus correction is suppressed and Josephson hyperscaling still holds.

For the mean-field approach y_u/y_t is essentially $-1/2$.

Also holds.

4. *Meissner effect*

A superconductor or superfluid is described by a complex order parameter $\psi = |\psi(\mathbf{r}| \exp(i\phi(\mathbf{r}|)$. Consider the low temperature phase, where the modulus of the order parameter is a constant, and only phase fluctuations are relevant.

The Hamiltonian of a charged superfluid coupled to the electromagnetic vector potential $\mathbf{A}$ takes the form

$$\beta H[\mathbf{A}, \phi] = \frac{\beta}{2} \int d^d\mathbf{r} \left[\frac{\rho_0}{m} (\nabla\phi - \mathbf{A})^2 + (\nabla \wedge \mathbf{A})^2 \right] .$$

By transforming into Fourier modes, and integrating out the phase variables, show that the effective Hamiltonian can be re-expressed as

$$e^{-\beta H_{eff}[\mathbf{A}]} = \int D\phi \, e^{-\beta H[\mathbf{A},\phi]} ,$$

where

$$\beta H_{eff}[\mathbf{A}] = \frac{\beta}{2} \sum_q \left[\frac{\rho_0}{m} \left(\mathbf{A}_q \cdot \mathbf{A}_{-q} - \frac{(\mathbf{q} \cdot \mathbf{A_q})(\mathbf{q} \cdot \mathbf{A_{-q}})}{q^2} \right) + (\mathbf{q} \wedge \mathbf{A_q}) \cdot (\mathbf{q} \wedge \mathbf{A_{-q}}) \right]$$

Explain the consequences of this result for a transverse (and thus physical) vector potential, namely for $\mathbf{q} \cdot \mathbf{A_q} = 0$.

$$\beta H[\vec{A}, \phi] = \frac{\beta}{2} \int d^d\vec{r} \left[\frac{\rho_0}{m} (\nabla\phi - \vec{A})^2 + (\nabla \wedge \vec{A})^2 \right].$$

$$\phi(\vec{r}) = |\phi(\vec{r})| e^{i\phi(\vec{r})}$$

Fourier mode: $\phi(\vec{r}) = \int \frac{d^dq}{(2\pi)^d} \phi(\vec{q}) e^{i\vec{q}\cdot\vec{r}}$ $\quad \nabla\phi \to iq\phi.$

$$\vec{A}(\vec{r}) = \int \frac{d^dq}{(2\pi)^d} \vec{A}(\vec{q}) e^{i\vec{q}\cdot\vec{r}} \qquad \nabla \wedge \vec{A} \to i\vec{q} \wedge \vec{A}$$

Focus on $(\nabla\phi - \vec{A})^2 = (\nabla\phi)^2 - 2\nabla\phi\cdot\vec{A} + \vec{A}^2.$

$$\Rightarrow \int \frac{d^dq}{(2\pi)^d} q^2 |\phi(q)|^2 - 2\int \frac{d^dq}{(2\pi)^d} \int \frac{d^dq'}{(2\pi)^d} i\vec{q}\cdot\phi(\vec{q}) \vec{A}(\vec{q}') e^{i(\vec{q}+\vec{q}')\vec{r}} + \int \frac{d^dq}{(2\pi)^d} |\vec{A}(\vec{q})|^2$$

$$= \int \frac{d^dq}{(2\pi)^q} \left(q^2 |\phi(\vec{q})|^2 - 2i\vec{q}\cdot\phi(\vec{q})\vec{A}(-\vec{q}) + |\vec{A}(\vec{q})|^2 \right).$$

Focus on $(\nabla \wedge \vec{A})^2$.

$$\Rightarrow \int \frac{d^dq}{(2\pi)^d} \left| i\,\vec{q} \wedge \vec{A}(\vec{q}) \right|^2 = \int \frac{d^dq}{(2\pi)^d} \left(\vec{q}^{\,2} \left|\vec{A}(\vec{q})\right|^2 - \left(\vec{q}\cdot\vec{A}(\vec{q})\right)^2 \right).$$

Overall $$\beta H_{eff} = \frac{\beta}{2} \int \frac{d^dq}{(2\pi)^d} \left[\frac{\rho_0}{m} \left(\underbrace{q^2 |\phi(\vec{q})|^2}_{\text{phase variable integrated out}} - 2i\,\vec{q}\cdot\phi(\vec{q})\vec{A}(-\vec{q}) + \left|\vec{A}(\vec{q})\right|^2 \right) + \left|\vec{q} \wedge \vec{A}(\vec{q})\right|^2 \right]$$

express in path integral form, the ϕ dependence is included in $D\phi$.

$$\Rightarrow \beta H_{eff}[\vec{A}] = \int D\phi \; e^{-\beta H[A,\phi]}.$$

with $$H[\vec{A},\phi] = \frac{\beta}{2} \sum_{\vec{q}} \left[\frac{\rho_0}{m} \left(\left|\vec{A}(\vec{q})\right|^2 - 2i\,\vec{q}\cdot\vec{A}(-\vec{q}) \right) + \left|\vec{q} \wedge \vec{A}(\vec{q})\right|^2 \right]$$

$$= \frac{\beta}{2} \sum_{q} \left[\frac{\rho_0}{m} \left(\vec{A}_q \cdot \vec{A}_{-q} - \frac{(\vec{q}\cdot\vec{A}_q)(\vec{q}\cdot\vec{A}_{-q})}{q^2} \right) + (\vec{q} \wedge \vec{A}_{\vec{q}})(\vec{q} \wedge \vec{A}_{-\vec{q}}) \right].$$

For a transverse vector potential, $\vec{q}\cdot\vec{A}_q = 0$.

We have $$H[\vec{A},\phi] = \frac{\beta}{2} \sum_{q} \left(\underline{\frac{\rho_0}{m} \vec{A}_q \cdot \vec{A}_{-q}} + \underline{(q A_\perp)^2} \right)$$

The superfluid/superconductor will choose direction of A so that to minimize the $qA_\perp$ term. meaning screening $\vec{B}$ field (Meissener effect)

5. *Landau-Ginzburg model near four dimensions*

This model is defined by an effective Hamiltonian for an $n-$component magnetization $\mathbf{m}$, viz.

$$\beta H[\mathbf{m}] = \int d^d\mathbf{x} \left(\frac{t}{2}|\mathbf{m}|^2 + \frac{K}{2}|\nabla m|^2 + u|\mathbf{m}|^4 - \mathbf{m}\cdot\mathbf{h} \right)$$

. The RG flows to second order in u are given by

$$\frac{dT}{dl} = 2t + \frac{4u(n+2)}{(t+K\Lambda^2)}K_d\Lambda^d - Au^2 \tag{2}$$

$$\frac{du}{dl} = (4-d)u - \frac{4u^2(n+8)}{(t+K\Lambda^2)^2}K_d\Lambda^d \tag{3}$$

where Λ is a cutoff and $K_d = S_d/(2\pi)^d$, with S_d the solid angle of a d-dimensional hypersphere.

Sketch the RG flows in the (t, u) plane in the cases of $d > 4$, and $d < 4$, showing that in the former case the Gaussian fixed point is recovered.

Determine the position of a new fixed point (t^*, h^*) to leading order in $\epsilon = 4-d.$, and linearize near the fixed point to determine to determine the anomalous dimensions y_t, y_u.

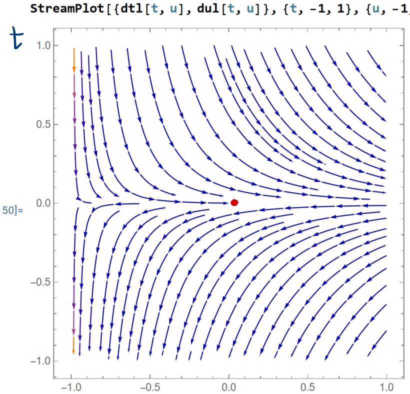
Use a scaling relation to determine the anomalous dimension of the magnetic field y_h.

Hence determine the critical exponents α, β, γ, and ν.

Would higher order terms in the effective Hamiltonian be relevant, if ϵ is small?

```
In[144]:= ClearAll["Global`*"]
n = 1; K = 1; Λ = 1; A = 1;
Kd[d_] := 2 π^(d/2) / Gamma[d / 2];
d = 8;
dtl[t_, u_] := 2 t + (4 u (n + 2))/(t + K Λ^2) Kd[d] Λ^d - A u^2;
dul[t_, u_] := (4 - d) u - (4 u^2 (n + 8))/((t + K Λ^2)^2) Kd[d] Λ^d;
StreamPlot[{dtl[t, u], dul[t, u]}, {t, -1, 1}, {u, -1, 1}]
```

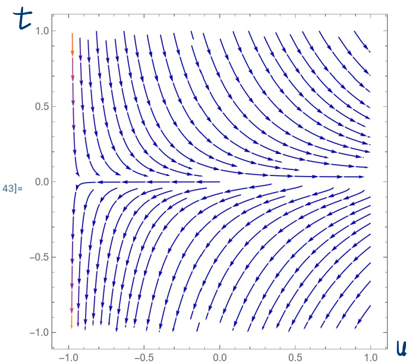
Out[150]=



d=8

```
In[137]:= ClearAll["Global`*"]
n = 1; K = 1; Λ = 1; A = 1;
Kd[d_] := 2 π^(d/2) / Gamma[d / 2];
d = 2;
dtl[t_, u_] := 2 t + (4 u (n + 2))/(t + K Λ^2) Kd[d] Λ^d - A u^2;
dul[t_, u_] := (4 - d) u - (4 u^2 (n + 8))/((t + K Λ^2)^2) Kd[d] Λ^d;
StreamPlot[{dtl[t, u], dul[t, u]}, {t, -1, 1}, {u, -1, 1}]
```

Out[143]=



d=2

Gaussian fixed point : $u = 0$.

$$\frac{dt}{dl} = 2t, \quad \frac{du}{dl} = 0. \quad \Rightarrow \quad (0,0) \text{ recovered for } d = 8.$$

New fixed point :

$$\frac{du}{dl} = 0, \quad \frac{dt}{dl} = 0. \quad \Rightarrow \quad u = \frac{\overset{\varepsilon}{(4-d)}\left(t+K\Lambda^2\right)^2}{4(n+8)K_d\Lambda^d}.$$

In[197]:= `ClearAll["Global`*"]`

$$u = \frac{(4-d)\left(t + K0\,\Lambda^2\right)^2}{4\,(n+8)\,Kd\,\Lambda^d}$$

Out[198]= $$\frac{(4-d)\,\Lambda^{-d}\left(t + K0\,\Lambda^2\right)^2}{4\,Kd\,(8+n)}$$

ε^2 neglected

In[201]:= $$\text{Simplify}\left[2\,t + \frac{4\,u\,(n+2)}{t + K0\,\Lambda^2}\,Kd\,\Lambda^d - A\,u^2\right]$$

Out[201]= $$2\,t - \frac{(-4+d)\,(2+n)\left(t + K0\,\Lambda^2\right)}{8+n} - \frac{A\,(-4+d)^2\,\Lambda^{-2d}\left(t + K0\,\Lambda^2\right)^4}{16\,Kd^2\,(8+n)^2}$$

$$\Rightarrow \quad t = \frac{-\varepsilon(t+K\Lambda^2)(2+n)}{2(8+n)}. \qquad 2(8+n)t = -\varepsilon(2+n)t - \varepsilon K\Lambda^2(2+n)$$

$$\Rightarrow \quad t = \frac{-\varepsilon K\Lambda^2(n+2)}{2(n+8)+\varepsilon(n+2)} \approx \frac{-\varepsilon K\Lambda^2(n+2)}{2(n+8)}.$$

$$u = \frac{\varepsilon\left(t+K\Lambda^2\right)^2}{4(n+8)K_d\Lambda^d} = \frac{\varepsilon K^2\Lambda^{4-d}}{4(n+8)K_d}.$$

$$(t^*, u^*) = \left(\frac{-\varepsilon K\Lambda^2(n+2)}{2(n+8)}, \frac{\varepsilon K^2\Lambda^{4-d}}{4(n+8)K_d}\right).$$

Come back to the equations and linearize around fixed point.

$$t = t^* + \delta t, \quad u = u^* + \delta u.$$

$$\Rightarrow \quad \frac{d}{dl}\begin{pmatrix}\delta t\\ \delta u\end{pmatrix} = \begin{pmatrix} J \end{pmatrix}\begin{pmatrix}\delta t\\ \delta u\end{pmatrix}.$$

$$J = \begin{pmatrix} 2 - \dfrac{4u^*(n+2)}{(t^*+K\Lambda^2)^2} K_d \Lambda^d & -2Au^* \\[2ex] \dfrac{8t^* u^{*2}(n+8)}{(t^*+K\Lambda^2)^3} K_d \Lambda^d & 4-d - \dfrac{8u^*(n+8)}{(t^*+K\Lambda^2)^2} K_d \Lambda^d \end{pmatrix}.$$

In[227]:= `ClearAll["Global`*"]`

$$t = \frac{-\epsilon\,(2+n)\,(K0\,\Lambda^2)}{2\,(8+n)};\quad u = \frac{\epsilon\,(K0\,\Lambda^2)^2}{4\,(n+8)\,Kd\,\Lambda^d};$$

$$\text{Simplify}\left[2 - \frac{4\,u\,(n+2)}{(t+K0\,\Lambda^2)^2}\,Kd\,\Lambda^d\right]$$

$$\text{Simplify}\left[\frac{8\,u^2\,(n+8)}{(t+K0\,\Lambda^2)^3}\,Kd\,\Lambda^d\right]$$

$$\text{Simplify}\left[\epsilon - \frac{8\,u\,(n+8)}{(t+K0\,\Lambda^2)^2}\,Kd\,\Lambda^d\right]$$

Out[229]= $2 - \dfrac{4\,(2+n)\,(8+n)\,\epsilon}{(2\,(-8+\epsilon)+n\,(-2+\epsilon))^2}$

Out[230]= $-\dfrac{4\,K0\,(8+n)^2\,\epsilon^2\,\Lambda^{2-d}}{Kd\,(2\,(-8+\epsilon)+n\,(-2+\epsilon))^3}$

Out[231]= $\epsilon - \dfrac{8\,(8+n)^2\,\epsilon}{(2\,(-8+\epsilon)+n\,(-2+\epsilon))^2}$

$$J = \begin{pmatrix} 2 - \dfrac{n+2}{n+8}\,\varepsilon & \dfrac{\varepsilon A K^2 \Lambda^{4-d}}{2(n+8)K_d} \\[2ex] \dfrac{-K\varepsilon^2\Lambda^{2-d}}{2K_d(n+8)} & -\varepsilon \end{pmatrix}$$

In[267]:= `ClearAll["Global`*"]`

$$J = \begin{pmatrix} 2 - \dfrac{4\,(2+n)\,(8+n)\,\epsilon}{(2\,(-8+\epsilon)+n\,(-2+\epsilon))^2} & \dfrac{A\,\epsilon\,(K0\,\Lambda^2)^2}{4\,(n+8)\,Kd\,\Lambda^d} \\[2ex] -\dfrac{4\,K0\,(8+n)^2\,\epsilon^2\,\Lambda^{2-d}}{Kd\,(2\,(-8+\epsilon)+n\,(-2+\epsilon))^3} & \epsilon - \dfrac{8\,(8+n)^2\,\epsilon}{(2\,(-8+\epsilon)+n\,(-2+\epsilon))^2} \end{pmatrix};$$

$$J = \begin{pmatrix} 2 - \dfrac{(2+n)\,\epsilon}{(8+n)} & \dfrac{A\,\epsilon\,(K0\,\Lambda^2)^2}{2\,(n+8)\,Kd\,\Lambda^d} \\[2ex] -\dfrac{K0\,\epsilon^2\,\Lambda^{2-d}}{2\,Kd\,((8+n))} & -\epsilon \end{pmatrix}$$

higher order term

Out[269]= $\left\{\left\{2 - \dfrac{(2+n)\,\epsilon}{8+n},\ \dfrac{A\,K0^2\,\epsilon\,\Lambda^{4-d}}{2\,Kd\,(8+n)}\right\},\ \left\{-\dfrac{K0\,\epsilon^2\,\Lambda^{2-d}}{2\,Kd\,(8+n)},\ -\epsilon\right\}\right\}$

In[270]:= `Eigenvalues[J]`

Out[270]= $\Big\{\dfrac{1}{2\,Kd\,(8+n)}\Lambda^{-d}\Big(16\,Kd\,\Lambda^d + 2\,Kd\,n\,\Lambda^d - 10\,Kd\,\epsilon\,\Lambda^d - 2\,Kd\,n\,\epsilon\,\Lambda^d - \sqrt{(-A\,K0^3\,\epsilon^3\,\Lambda^6 + 256\,Kd^2\,\Lambda^{2d} + 64\,Kd^2\,n\,\Lambda^{2d} + 4\,Kd^2\,n^2\,\Lambda^{2d} + 192\,Kd^2\,\epsilon\,\Lambda^{2d} + 24\,Kd^2\,n\,\epsilon\,\Lambda^{2d} + 36\,Kd^2\,\epsilon^2\,\Lambda^{2d})}\Big),$

$\dfrac{1}{2\,Kd\,(8+n)}\Lambda^{-d}\Big(16\,Kd\,\Lambda^d + 2\,Kd\,n\,\Lambda^d - 10\,Kd\,\epsilon\,\Lambda^d - 2\,Kd\,n\,\epsilon\,\Lambda^d + \sqrt{(-A\,K0^3\,\epsilon^3\,\Lambda^6 + 256\,Kd^2\,\Lambda^{2d} + 64\,Kd^2\,n\,\Lambda^{2d} + 4\,Kd^2\,n^2\,\Lambda^{2d} + 192\,Kd^2\,\epsilon\,\Lambda^{2d} + 24\,Kd^2\,n\,\epsilon\,\Lambda^{2d} + 36\,Kd^2\,\epsilon^2\,\Lambda^{2d})}\Big)\Big\}$

Let's just keep go with $J = \begin{pmatrix} 2 - \frac{(n+2)\varepsilon}{n+8} & 0 \\ 0 & -\varepsilon \end{pmatrix}.$

y_t and y_u

Thus $y_t = 2 - \frac{n+2}{n+8}\varepsilon$, $y_u = -\varepsilon$

From the $\mathcal{F}$ form we know $h\phi$ is d-dimensional.

From above discussion we know $y_t = 2 - \frac{n+2}{n+8}\varepsilon$.

Also, $[T] + 2[\phi] = \underbrace{d}_{4-\varepsilon}. \Rightarrow [\phi] = 2 - \frac{6}{n+8}\varepsilon$.

$[h] + [\phi] = d = 4 - \varepsilon. \Rightarrow [h] = \underbrace{2 - \frac{n+2}{n+8}\varepsilon}_{}.$ y_h

$$\nu = \frac{1}{y_t} = \left(2 - \frac{n+2}{n+8}\varepsilon\right)^{-1}. \qquad \alpha = 2 - d\nu = 2 - (4-\varepsilon)\left(2 - \frac{n+2}{n+8}\varepsilon\right)^{-1}.$$

$$\beta = \frac{d - y_h}{y_t} = \frac{2 - \varepsilon + \frac{n+2}{n+8}\varepsilon}{2 - \frac{n+2}{n+8}\varepsilon} = \frac{2 - \frac{6}{n+8}\varepsilon}{2 - \frac{n+2}{n+8}\varepsilon}.$$

$$\gamma = \frac{2y_h - d}{y_t} = \frac{\varepsilon - 2\frac{n+2}{n+8}\varepsilon}{2 - \frac{n+2}{n+8}\varepsilon}.$$

Not relevant if ε is small. (u^4 term suppressed)

6. *The nonlinear σ model*

The differential recursion relations for this $n-$ component model in d dimensions were derived in lectures, and take the form (for temperature T and magnetic field h)

$$\frac{dT}{dl} = -(d-2)T + (n-2)K_d\Lambda^{d-2}T^2 \quad (4)$$

$$\frac{dh}{dl} = h\left(d - \frac{(n-1)}{2}TK_d\Lambda^{d-2}\right) \quad (5)$$

where Λ is a cutoff, and $K_d = S_d/(2\pi)^d$, with S_d the solid angle of a d-dimensional hypersphere.

(a) Find the fixed points T^*, h^* to leading order in $\epsilon = d-2$.

(b) Linearizing near the fixed point, find the scaling dimensions y_t and y_h, and thereby the critical exponents α, ν, and η

(c) In the case of $d = 2$, and $n > 2$, how does the correlation length behave as $T \to 0$?

(a) $\frac{dT}{dl} = 0$, $\frac{dh}{dl} = 0$. $\Rightarrow$ $h = 0$ or $T = \frac{2d}{(n-1)K_d}\Lambda^{2-d}$.

or $T = 0$ or $T = \frac{d-2}{(n-2)K_d}\Lambda^{2-d}$.

(b) We have $(T^*, h^*) = (0, 0)$ or $\left(\frac{2(\varepsilon+2)}{(n-1)K_d}\Lambda^{2-d}, 0\right)$ or $\left(\frac{\varepsilon}{(n-2)K_d}\Lambda^{2-d}, 0\right)$.

$$\frac{d}{dl}\begin{pmatrix}\delta T\\ \delta h\end{pmatrix} = \begin{pmatrix} J \end{pmatrix}\begin{pmatrix}\delta T\\ \delta h\end{pmatrix}. \qquad J = \begin{pmatrix} -\varepsilon + 2(n-2)K_d\Lambda^{d-2}T^* & 0 \\ 0 & d - \frac{(n-1)}{2}T^*K_d\Lambda^{d-2}\end{pmatrix}$$

$K_d\Lambda^{d-2}T^* = \frac{2(\varepsilon+2)}{n-1}$ or $\frac{\varepsilon}{n-2}$. We have

$$J = \begin{pmatrix} 3\varepsilon + \frac{4\varepsilon+8n}{n-1} & 0 \\ 0 & 0 \end{pmatrix} \quad \text{or} \quad J = \begin{pmatrix} \varepsilon & 0 \\ 0 & \frac{\varepsilon}{2} + \frac{2n-5}{n-2}\end{pmatrix}$$

We have $y_t, y_h = \frac{3n+1}{n-1}\varepsilon + \frac{8n}{n-1}, \; 0$

or $\varepsilon, \; \frac{\varepsilon}{2} + \frac{2n-5}{n-2}$

$$\nu = \frac{1}{y_t} = \frac{n-1}{(3n+1)\varepsilon + 8n} \quad \text{or} \quad \frac{1}{\varepsilon}.$$

$$\alpha = 2 - d\nu = 2 - (\varepsilon+2)\nu = 2 - (\varepsilon+2)\frac{n-1}{(3n+1)\varepsilon + 8n} = \frac{(5n+3)\varepsilon + 14n + 2}{(3n+1)\varepsilon + 8n}.$$

$$\text{or} \quad 2 - \frac{\varepsilon+2}{\varepsilon} = 1 - \frac{2}{\varepsilon}.$$

$$\eta = \frac{n+2}{n+8}\frac{\varepsilon^2}{54}. \quad ?$$

(c) $G = \langle \phi(x)\phi(0) \rangle \sim \frac{1}{|x|^{d-2+\eta}}$

$d-2+\eta \sim 0$ as $d=2$ and $n>0$.

correlation length tend to infinity.

Advanced Statistical Mechanics   Pset 3   yuxiang@uchicago.edu

1. *This problem introduces the idea of pinned elastic media, relevant for models of friction and forced flow, earthquakes, and sandpiles.*

A charge density wave (CDW) is a modulational instability in a solid that gives rise to an almost periodic electronic charge density

$$\rho(\mathbf{r}) = \rho_c + \rho_o \cos(\mathbf{Q}\cdot\mathbf{r} + \phi(\mathbf{r},\mathbf{t}))$$

Here ρ_c is the collective uniform average electronic charge density in the solid, and ρ_o ($\ll \rho_c$) the amplitude of the CDW, assumed constant in space. The phase variable $\phi(\mathbf{r},\mathbf{t})$ describes spatial and temporal elastic fluctuations of the CDW. Interactions with impurities at random positions $\mathbf{R}_i$ with a low concentration c give rise to a Hamiltonian for the phase variable ϕ

$$H = \int d^d\mathbf{r} \left[\frac{K}{2}(\nabla\phi)^2 + \sum_i V(\mathbf{r}-\mathbf{R_i})\rho(\mathbf{r}) + \rho_c E\phi\right] \tag{1}$$

Here E is a uniform electric field that drives the CDW. (Notice that $\rho_c\phi$ has the form of a polarization density, becuse ϕ corresponds to a displacement of the CDW) The impurities may be assumed to be described by a weak, short range potential $V = V_o\delta(\mathbf{r}-\mathbf{R_i})$. The dynamics are overdamped, described by

$$\gamma\dot{\phi} = -\frac{\delta H}{\delta\phi}$$

where $\dot{\phi}$ is the partial time derivative.

(a) Identifying the impurities as a weak random field, estimate the correlation length of the phase variable ϕ in d–dimensions.

(b) Noting that a stationary solution (minimum) of H implies multiple solutions under translation by $\phi \to \phi + 2n\pi$ where n is an integer, estimate the typical size of the barrier between metastable states.

(a) $H = \int d^d\vec{r}\left[\frac{K}{2}(\nabla\phi)^2 + \sum_i V(\vec{r}-\vec{R}_i)\rho(\vec{r}) + \rho_c E\phi\right]$

$\delta H = \int d^d r\left[\underline{K\nabla\phi\cdot\delta(\nabla\phi)} + \sum_i V(r-R_i)\delta\rho + \rho_c E\delta\phi\right]$

$\underbrace{\int\nabla\phi\cdot\delta\phi}_{=0} - \delta\phi\,\nabla\cdot(\nabla\phi)$

$$= \int d^d\vec{r}\left[-\kappa\,\nabla\cdot(\nabla\phi)\,\delta\phi + \sum_i V(r-R_i)\,\delta\rho + \rho_c E\,\delta\phi\right].$$

plug in $\gamma\dot{\phi} = -\frac{\delta H}{\delta\phi}$ to get

$$\gamma\dot{\phi} = -\kappa\nabla^2\phi + \underline{\sum_i V(r-R_i)\frac{\delta\rho}{\delta\phi}} + \rho_c E.$$

$$-\sum_i V_0\,\delta(\vec{r}-\vec{R}_i)\cdot\rho_0\sin(\vec{Q}\cdot\vec{r}+\phi)$$

$$\begin{cases} \vec{r}\neq\vec{R}_i, & \gamma\dot{\phi} = -\kappa\nabla^2\phi + \rho_c E. \\ \vec{r}=\vec{R}_i, & \gamma\dot{\phi} = -\kappa\nabla^2\phi - \underbrace{V_0\rho_0}_{\text{weak}}\sin(\vec{Q}\cdot\vec{R}_i+\phi) + \rho_c E. \end{cases}$$

To get correlation length let's focus on $\vec{r}\neq\vec{R}_i$ regions.

$\kappa\nabla^2\phi = \rho_c E$ has point-source solution $\phi = -\frac{\rho_c E}{\kappa}G_d(r)$.

We should find $\phi(\vec{r},t) = \phi(k)\,e^{i(\vec{k}\cdot\vec{r}-\omega t)}$. transform to k space:

$$\kappa k^2\phi_k - \rho_c\tilde{E} = -i\gamma\omega\phi_k. \quad\Rightarrow\quad \phi_k = \frac{\rho_c\tilde{E}}{\kappa k^2 + i\gamma\omega}$$

$$G(r,r') = \int\frac{d^dk}{(2\pi)^d}\langle\phi^*(k)\phi(k)\rangle e^{i\vec{k}\vec{r}} = \int\frac{d^dk}{(2\pi)^d}\frac{(\rho_c\tilde{E})^2}{\kappa^2k^4+\gamma^2\omega^2}\,\underline{e^{ikr}}.$$

suppressed

$$d^dk = \Omega_d\,k^{d-1}\,dk. \qquad\longrightarrow\qquad \frac{(\rho_c\tilde{E})^2}{\kappa^2}\int\frac{d^dk}{(2\pi)^d}\frac{1}{\left(k^2+\left(\frac{\gamma\omega}{\kappa}\right)^2\right)^2} \to \Delta$$

$$\propto \int_0^{\Lambda} dk\,\Omega_d\,\frac{k^{d-1}}{(k^2+\Delta)^2}$$

$\xrightarrow{u=k^2}$ $$= \Omega_d \int_0^{\Lambda^2} \frac{du}{2} \frac{u^{\frac{d-2}{2}}}{(u+\Delta)^2}$$ $$d\sqrt{u} = \frac{du}{2\sqrt{u}}$$

$$\sim \int_0^{\Lambda^2} -u^{\frac{d-2}{2}} d\left(\frac{1}{u+\Delta}\right) = -\frac{u^{\frac{d-2}{2}}}{u+\Delta}\Big|_0^{\Lambda^2} + \frac{d-2}{2}\int_0^{\Lambda^2} \frac{u^{\frac{d-4}{2}}}{u+\Delta} du$$

$$= \frac{d-2}{2}\int_0^{\Lambda^2} \frac{u^{\frac{d-4}{2}}}{u+\Delta} du - \frac{\Lambda^{d-2}}{\Lambda^2+\Delta}$$

this is at large k?

$$\sim \frac{d-2}{2}\int_0^{\Lambda^2} u^{\frac{d-6}{2}} du - \frac{\Lambda^{d-2}}{\Lambda^2+\Delta}$$

here at large r,

$$= \frac{d-2}{2(d-4)}\Lambda^{d-4} - \frac{\Lambda^{d-2}}{\Lambda^2+\Delta}.$$

small k contribute

Coming back to

$$G(r,r') = \int \frac{d^dk}{(2\pi)^d} \frac{(\ell_c \tilde{E})^2}{\kappa^2 k^4 + \gamma^2\omega^2} e^{ik\cdot r}$$

the temporal term : $$\int \frac{d\omega}{2\pi} \frac{(\ell_c \tilde{E})^2}{\kappa^2 k^4 + \gamma^2\omega^2} e^{-i\omega t}$$

$$\int \frac{1}{\omega^2 + \frac{\kappa^2 k^4}{\gamma^2}} e^{-i\omega t} d\omega$$

$$\Rightarrow \frac{\kappa}{2\gamma k^2} e^{-\frac{\gamma k^2}{\kappa}t}.$$

$$\Rightarrow k_c = \sqrt{\frac{\gamma}{\kappa}}.$$

$$\tau = \frac{\kappa}{\gamma k^2}.$$

$$G(r,r') = \int \frac{d^dk}{(2\pi)^d} \frac{(\ell_c \tilde{E})^2}{\kappa^2 k^4 + \gamma^2\omega^2} e^{ik\cdot r}$$

$$d^dk = \Omega_d k^{d-1} dk. \longrightarrow \frac{(\ell_c \tilde{E})^2}{\kappa^2} \int \frac{d^dk}{(2\pi)^d} \frac{k^{d-1}}{\left(k^2 + \left(\frac{\gamma\omega}{\kappa}\right)^2\right)^2} e^{i\vec{k}\cdot\vec{r}}$$

Δ

$$\propto \int dk\, \Omega_d \frac{k^{d-1}}{(k^2+\Delta)^2} e^{i\vec{k}\cdot\vec{r}}$$

For small k, $\sim \int dk \, \frac{k^{d-1}}{\Delta^2}$. large k, $\sim \int dk \, k^{d-5}$.

$$\int d\theta_1 \, e^{ikr\cos\theta_1} \sim (kr)^{-\frac{d-2}{2}} J_{\frac{d-2}{2}}(kr). \quad \leftarrow \text{from internet}$$

$$\Rightarrow G(r) \sim \int dk \frac{k^{d-1}}{(k^2+\Delta)^2} (kr)^{-\frac{d-2}{2}} \underbrace{J_{\frac{d-2}{2}}(kr)}. \sim \sqrt{\frac{\pi}{2kr}}\, e^{-kr} \quad (kr \gg 1)$$

$$\Rightarrow \quad G(r) \sim \int dk \frac{k^{\frac{d-1}{2}}}{(k^2+\Delta)^2} \frac{e^{-kr}}{r^{\frac{d-1}{2}}}$$

As shown above, contribution to this integral is mostly from

$k \sim$

... we should get something from the Δ term.

should I actually start from $G(r,t)$?

$k_0 = \ell_c E$. $\quad \Delta = \left(\frac{\gamma\omega}{k}\right)^2 \sim \left(\frac{\gamma^2 k^2}{k^2}\right)^2 \sim \left(\frac{\gamma^2 \ell_c^2 E^2}{k^2}\right)^2$.

$$\xi \sim \frac{1}{\sqrt{\Delta}} \sim \frac{k^2}{\gamma^2 \ell_c^2 E^2}. \qquad \times$$

$d=2$ &

critical dimension $d=4$.

with $\xi = \sqrt{\frac{k}{\gamma \ell_c E}}$

(b) $\kappa \nabla^2 \phi = \rho_c E$ has point-source solution $\phi_0 = -\frac{\rho_c E}{\kappa} G_d(r)$.

$$G_d(r) = -\frac{1}{(d-2)\Omega_d} \cdot \frac{1}{r^{d-2}} \quad (d \neq 2)$$

$$\phi(r) = \frac{1}{(d-2)\Omega_d} \cdot \frac{\rho_c E}{\kappa} \underline{\int \frac{1}{(\vec{r}-\vec{r}')^{d-2}} d^d \vec{r}'} , \quad \sim \frac{\Omega_d}{d-2} \frac{1}{r^{d-2}}$$

$$\phi \to \phi + 2n\pi .$$

thus $2\pi \sim \frac{\rho_c E}{\kappa} \frac{1}{r_0^{d-2}} \cdot \frac{1}{(d-2)^2}$

size of barrier $r_0 \sim \left(\frac{\rho_c E}{2\pi \kappa} \cdot \frac{1}{(d-2)^2} \right)^{\frac{1}{d-2}} \propto \left(\frac{\rho_c E}{\kappa} \right)^{\frac{1}{d-2}} \quad (d \neq 2)$

For $d=2$. $G_2(r) = \frac{1}{2\pi} \ln\left(\frac{1}{r}\right)$. $\phi(r) \propto \ln r$.

$2\pi \sim \frac{\rho_c E}{\kappa} \cdot \ln\left(\frac{r_0}{a}\right)$. $\quad r_0 \sim a e^{-\frac{\kappa}{\rho_c E}}$.

(c) Below the lower critical dimension that you have identified, explain why temperature is an unimportant, but not irrelevant, variable

(d) Estimate the value of the electric field E_T above which there are no stationary solutions.

(e) As a function of electric field, describe qualitatively the behavior of the static responses: $\frac{d\langle\phi\rangle}{dE}$ and $\frac{d\langle\dot{\phi}\rangle}{dE}$

(f) If in addition to a large DC field, a small ac component is added: $E(t) = E_o + \delta E \cos(\omega t)$ qualitatively discuss the frequency response as $E \to E_T$ from below.

(c) Temperature affects the γ term, but not really on other terms.

It might also introduce extra noise term on our equation.

So it is relevant by affecting the γ term in expression of the correlation length ξ.

(d) $\gamma \dot{\phi} = -K \nabla^2 \phi + \rho_c E$. always have steady-state solution.

how about $\gamma \dot{\phi} = -K \nabla^2 \phi - V_0 \rho_0 \sin(\vec{Q} \cdot \vec{R}_i + \phi) + \rho_c E$?

The criterion for existence of steady-state solution is

...

No, actually our solution is confined by the periodic property & boundary condition ?

maybe already went wrong from (a). "weak random field" could be treated as correlated noise term in dynamical equation ?

So $\gamma \dot{\phi} = -K \nabla^2 \phi + \rho_c E + F(\vec{r})$

$\langle F(\vec{r}) \rangle = 0$. $\langle F(\vec{r}) F(\vec{r} + \frac{2\pi}{\vec{Q}}) \rangle = V_0^2 \rho_0^2$.

in this way, ϕ has the noise term induced correlation

$$\propto \frac{V_0^2 \rho_0^2}{2\gamma} e^{-\alpha r} \, ? \quad \left(\text{from } \int_0^r e^{-\gamma(r-x)} F(x)\,dx \right)$$

should satisfy $|r-r'| \sim \frac{2\pi}{Q} \leq \frac{\gamma}{K}$?

how about E ?

Should start over and solve

$$\dot{\phi} = -\frac{k}{\gamma} \nabla^2 \phi + \frac{\rho_c E}{\gamma} + \frac{F(\vec{r})}{\gamma}$$

spatial correlation

the solution should have form

$\phi(r,t) = \phi(r)\phi(t)$

$$\nabla^2 \phi(r) = \frac{\rho_c E}{K} + \frac{\langle F(\vec{r}) F(\vec{r}+\frac{2\pi}{Q})\rangle}{K}.$$

no, can not really separate

$$\frac{d\phi(t)}{dt} =$$

how to solve such equation ?

$$\phi(r) = \frac{1}{(d-2)\Omega_d} \cdot \frac{\rho_c E}{K} \frac{\Omega_d}{d-2} \frac{1}{r^{d-2}}$$

(previous solution)

plus $\phi'(r)$ from spatial correlated term.

let $K \nabla^2 \phi' = F(\vec{r})$. $\langle F(\vec{r}) F(\vec{r}+\frac{2\pi}{Q})\rangle = V_0^2 \rho_0^2$.

$$\phi' = \frac{1}{(d-2)^2} \cdot \frac{V_0^2 \rho_0^2}{K} \frac{1}{r^{d-2}} \cdots$$

and $\langle \phi'(0)\, \phi'(\vec{r})\rangle \propto$

(e) $\gamma \langle \dot{\phi} \rangle = -K \nabla^2 \langle \phi \rangle + \rho_c E.$

For $\frac{d\langle \phi \rangle}{dE}$. assume stable solution we have $\frac{d\langle \nabla^2 \phi \rangle}{dE} = \frac{\rho_c}{K}$.

For $\frac{d\langle \dot{\phi} \rangle}{dE}$. assume spatial uniform we have $\frac{d\langle \dot{\phi} \rangle}{dE} = \frac{\rho_c}{\gamma}$.

both response is proportional to base charge density ρ_c.

and inversely affected by coupling K or decay γ.

(f) $E(t) = E_0 + \delta E \cos \omega t.$

induce temporal oscillation.

2. *The random energy model*

In this model, there are 2^N configurations (just as for a set of N Ising spins), but the energy is chosen stochastically. Denoting indices for the states by $i, j, ... = \{1, ...2^N\}$, the energies E_i are independent random variables drawn from the distribution

$$P(E) = \frac{1}{\sqrt{\pi N}} e^{-\frac{E^2}{N}}$$

and the partition function is defined as usual

$$Z = \sum_{j=1}^{2^N} exp[-\beta E_j]$$

The results here are to be computed in the large N limit.

(a) Show that the number of energy levels $A(\epsilon, \epsilon+\delta)$ in the interval $[N\epsilon, N(\epsilon+\delta)]$ has the expectation

$$\bar{A} = 2^N \int_{\epsilon}^{\epsilon+\delta} \sqrt{\frac{N}{\pi}} e^{-Nx^2} dx = e^{N \max_{x\in[\epsilon,\epsilon+\delta]} s_Q(x)}$$

where $s_Q(x) = \log 2 - x^2$

Explain what this means for the likelihood of finding a configuration outside the range $[-\epsilon^*, \epsilon^*]$ where $\epsilon^* = \sqrt{\log 2}$

(a) $$P_A = \int P(E) dE = \int_{N\varepsilon}^{N(\varepsilon+\delta)} \frac{1}{\sqrt{\pi N}} e^{-\frac{E^2}{N}} dE$$

$$\xrightarrow{x=\frac{E}{N}} = \int_{\varepsilon}^{\varepsilon+\delta} \sqrt{\frac{N}{\pi}} e^{-Nx^2} dx$$

$$\bar{A} = 2^N P_A = \int_{\varepsilon}^{\varepsilon+\delta} \underline{\sqrt{\frac{N}{\pi}} 2^N} e^{-Nx^2} dx$$

$$= e^{N\ln 2 + \frac{1}{2}\ln\frac{N}{\pi}} \int_{\varepsilon}^{\varepsilon+\delta} e^{-Nx^2} dx$$

$$= e^{N\ln 2} \cdot \sqrt{\frac{N}{\pi}} \cdot e^{-Nx^2} \cdot \delta \cdot \sqrt{\frac{\pi}{N}}$$

portion of Gauss integral

?

$$= \delta e^{N(\ln 2 - x^2)}$$

$S_Q(x)$ X not right

$$\Rightarrow e^{N\ln 2} \sqrt{\frac{N}{\pi}} \int_{\varepsilon}^{\varepsilon+\delta} e^{-Nx^2} dx$$

$$\Rightarrow e^{N\ln 2} \sqrt{\frac{N}{\pi}} \int_{\sqrt{N}\varepsilon}^{\sqrt{N}(\varepsilon+\delta)} e^{-y^2} \frac{dy}{\sqrt{N}}$$

$y = \sqrt{N} x$

$$\Rightarrow A = e^{N\ln 2} \cdot \frac{1}{\sqrt{\pi}} \int_{\sqrt{N}\varepsilon}^{\sqrt{N}(\varepsilon+\delta)} e^{-y^2} dy \approx \sqrt{N}\delta e^{-y_0^2}.$$

with x_0 set

$$= \sqrt{\frac{N}{\pi}} \delta e^{N\ln 2} \cdot e^{-Nx_0^2} = \sqrt{\frac{N}{\pi}} \delta e^{N(\ln 2 - x_0^2)}$$

$\max(\ln 2 - x_0^2)$. $S_Q(x)$.

Outside the range $[-\varepsilon^*, \varepsilon^*]$, the function $S_Q(x) = \max(\ln 2 - x^2)$

is no longer meaningful and we have to set the probability to 0.

(b) With the definition

$s(x) = s_Q(x)$, for $x \in [-\epsilon^*, \epsilon^*]$, and

$s(x) = 0$ otherwise,

show that the partition function is

$$Z = \int_{-\infty}^{\infty} dx e^{N(s(x) - \beta x)}$$

(c) Hence show that the free energy density is

$$f(\beta) = \lim_{N\to\infty} \frac{F}{N} = \frac{\beta}{4} - \frac{\log 2}{\beta} \text{ if } \beta < \beta_c \quad (2)$$

$$= -\sqrt{\log 2} \text{ if } \beta > \beta_c \quad (3)$$

Give an interpretation of this result.

(b) $Z = \sum_{j=1}^{2^N} e^{-\beta E_j}$

For symmetric region $[-\varepsilon^*, \varepsilon^*]$. max $S_Q(x)$ reduces to $S_Q(0) = \log 2$.

As we already showed, most energy levels falls within $[-\varepsilon^*, \varepsilon^*]$. $\bar{A} = 2^N$.

If we focus on energy levels E rather than individual energy E_j.

it gains a probability weight. set $x = \frac{E}{N}$.

we know $P_x = e^{NS(x)}$. Thus

$$Z = \sum_{j=1}^{2^N} e^{-\beta E_j} = \sum_{i=1}^{N} P_i e^{-\beta E_i} = \int_{-\infty}^{+\infty} dx\, e^{NS(x)} e^{-\beta N x} = \int_{-\infty}^{+\infty} dx\, e^{N(S(x)-\beta x)}$$

(c) $F = \frac{-\ln Z}{\beta}$. Use saddle point approx we have

$$\ln Z \approx N(S(x_0) - \beta x_0) + \ln\sqrt{\frac{2\pi}{-NS''(x_0)}}.$$ x_0 make $S(x_0) - \beta x_0$ maximum.

$$= \begin{cases} \log 2 - x^2 - \beta x + \ln\sqrt{\frac{\pi}{N}}. \quad x \in [-\sqrt{\log 2}, \sqrt{\log 2}]. & \Rightarrow x_0 = -\frac{\beta}{2}. \quad \max \log 2 + \frac{\beta^2}{4}. \\ -\beta x. \quad \text{other}. & \Rightarrow x_0 = -\log 2. \quad \max \beta\sqrt{\log 2}. \end{cases}$$

$$f = \lim_{N\to\infty} \frac{F}{N} = \lim_{N\to\infty} \frac{-\ln Z}{\beta N} = \begin{cases} -\frac{\beta}{4} - \frac{\log 2}{\beta}. & x \in [-\varepsilon^*, \varepsilon^*] \text{ corresponds to } \beta < \beta_c. \\ -\sqrt{\log 2}. & \text{other} \quad \beta > \beta_c. \end{cases}$$

meaning (1-order) phase transition at β_c.

3. *The roughening transition*

Consider a continuum interface model in three dimensions, described by fluctuations of a two-dimensional height $h(\mathbf{x}$, viz.

$$H_o = \frac{K}{2}\int d^2\mathbf{x}(\nabla h)^2$$

(a) Calculate the correlation function $\langle h(x)h(0)\rangle_o$ for large x and show that the interface is rough at any non-zero temperature. The subscript o refers to averaging with the Hamiltonian H_o

(b) Show that the following thermal average satisfies

$$\langle \exp[i\sum_i q_i h(\mathbf{x_i})]\rangle_o = \exp\left[\frac{1}{K}\sum_{i<j} q_i q_j C(\mathbf{x}_i - \mathbf{x}_j)\right] \qquad (4)$$

where $C(x) = \log x/2\pi$ is the two-dimensional Coulomb interaction, and $\sum_i q_i = 0$. *Hint: the constraint follows from the translational invariance of* H_o

(c) Prove that for k small or for separations large

$$\langle |(h(\mathbf{x}) - h(\mathbf{y}))|^2\rangle = -\frac{d^2}{dk^2}G_k(\mathbf{x}-\mathbf{y})$$

where $G_k(\mathbf{x}-\mathbf{y}) = \langle \exp[ik(h(\mathbf{x}) - h(\mathbf{y}))]\rangle$

(d) Show that

$$G_k(\mathbf{x}-\mathbf{y})_o = \left(\frac{|\mathbf{x}-\mathbf{y}|}{a}\right)^{-k^2/2\pi K}$$

where a is a short length scale cutoff.

(a) To $\vec{q}$ space, $\quad h(\vec{x}) = \int \frac{d^2 q}{(2\pi)^2}\tilde{h}(\vec{q})\, e^{i\vec{q}\cdot\vec{x}}$.

$$H_o = \frac{K}{2}\int d^2\vec{x}\,(\nabla h)^2 = \frac{K}{2}\int \frac{d^2 q}{(2\pi)^2} q^2 |h(\vec{q})|^2 .$$

$$Z = e^{-\beta H}. \quad \Rightarrow \quad \langle h(\vec{q})h(\vec{q}^{*})\rangle = \frac{1}{\beta K q^2}\delta^2(\vec{q}-\vec{q}')$$

$$G(\vec{x},0) = \langle h(x)h(0)\rangle_0 = \int \frac{d^2\vec{q}}{(2\pi)^2}\frac{d^2\vec{q}'}{(2\pi)^2}\langle \tilde{h}(\vec{q})\tilde{h}(\vec{q}')\rangle e^{i(\vec{q}-\vec{q}')\vec{x}}$$

$2\pi J_0(qx)$

$$= \int \frac{d^2\vec{q}}{(2\pi)^2}\frac{1}{\beta K\vec{q}^2} e^{i\vec{q}\cdot\vec{x}} \xrightarrow{\vec{q}=(q\cos\theta,\, q\sin\theta)} \int_0^\infty \frac{1}{(2\pi)^2}\frac{1}{\beta K q^2} q\,dq \underline{\int_0^{2\pi} e^{iqx\cos\theta} d\theta}$$

$$= \frac{1}{2\pi\beta K}\int_0^\infty dq\, \frac{J_0(qx)}{q} \xrightarrow{\text{large } x} \frac{1}{2\pi\beta K}\int_0^\infty \frac{1}{q}\sqrt{\frac{2}{\pi q x}}\, \underline{\cos\left(qx-\frac{\pi}{4}\right)}$$

this fluctuating term makes surface rough

(b) $$\langle e^{i\sum_i q_i h(\vec{x}_i)}\rangle = \langle \prod_i e^{iq_i h(\vec{x}_i)}\rangle = \prod_i e^{\langle iq_i h(\vec{x}_i)\rangle}$$

$$= \prod_i \left(1 + iq_i h(\vec{x}_i) + \frac{1}{2}(q_i h(\vec{x}_i))^2 + \cdots\right)$$

For Gaussian $h(\vec{x}_i)$. $$e^{\langle iq_i h(\vec{x}_i)\rangle} = e^{-\frac{1}{2}\langle q_i^2 h(\vec{x}_i)^2\rangle}$$ $h(\vec{x}) = \frac{1}{K}\frac{\log(x_i - x_j)}{2\pi}$?

$$\Rightarrow e^{-\frac{1}{2}\sum_{ij} q_i q_j C(\vec{x}_i - \vec{x}_j)/K} = e^{-\frac{1}{K}\sum_{i<j} q_i q_j C(\vec{x}_i - \vec{x}_j)}$$

(c) As used in (b). $e^{\langle A\rangle} = e^{\frac{1}{2}\langle A^2\rangle}$. $A \in$ Gaussian distribution.

$$\langle e^{ik(h(\vec{x})-h(\vec{y}))}\rangle = e^{-\frac{k^2}{2}(h(\vec{x})-h(\vec{y}))^2}.$$

$$\frac{dG_k}{dk} = -k\langle (h(\vec{x})-h(\vec{y}))^2\rangle e^{-\frac{k^2}{2}(h(\vec{x})-h(\vec{y}))^2}$$

vanish for small k

$$\frac{d^2G_k}{dk^2} = -\langle (h(\vec{x})-h(\vec{y}))^2\rangle \underbrace{e^{-\frac{k^2}{2}(h(\vec{x})-h(\vec{y}))^2}}_{\approx 1} + \underline{k^2}\langle (h(\vec{x})-h(\vec{y}))^2\rangle^2 e^{-\frac{k^2}{2}(h(\vec{x})-h(\vec{y}))^2}$$

$$\Rightarrow \quad \langle |h(\vec{x}) - h(\vec{y})|^2 \rangle = -\frac{d^2 G_k}{dk^2}.$$

(d) Let's calculate the exact form of correlation function following from (a). We already have

suppressed for large x

$$G(x,0) = \frac{1}{2\pi\beta K} \int_0^\Lambda dq \, \frac{1}{q} \underbrace{\sqrt{\frac{2}{\pi q x}} \cos\left(qx - \frac{\pi}{4}\right)} \sim \frac{1}{2\pi\beta K} \ln\left(\frac{\Lambda}{r}\right).$$

again indicates surface is rough

$$G_k(\vec{x}-\vec{y})_0 = \langle e^{ik(h(\vec{x})-h(\vec{y}))} \rangle_0 = e^{-\frac{k^2}{2}\langle |h(\vec{x})-h(\vec{y})|^2 \rangle_0}.$$

while $\langle |h(\vec{x}) - h(\vec{y})|^2 \rangle = \langle | \underline{h(\vec{x})^2} + \underline{h(\vec{y})^2} - 2h(\vec{x})h(\vec{y}) | \rangle_0$

$$= -2\langle h(\vec{x})\, h(\vec{y}) \rangle = \frac{1}{\pi\beta K} \ln \frac{|\vec{x}-\vec{y}|}{a}.$$

$$\Rightarrow \quad G_k(\vec{x}-\vec{y})_0 = \frac{|\vec{x}-\vec{y}|}{a}\, e^{-\frac{k^2}{2\pi\beta K}}.$$

(d) $H_0 = \frac{K}{2}\int d^2\vec{x}\, (\nabla h)^2.$ $\qquad U = y_0 \int d^2\vec{x} \cos(2\pi h).$

$H = H_0 + U.$ $\qquad h(\vec{x}) = \sum_{\vec{q}} h_{\vec{q}}\, e^{i\vec{q}\cdot\vec{x}}.$ $\qquad H_0 = \frac{K}{2}\sum_{\vec{q}} q^2 |h_{\vec{q}}|^2.$

$E_0 = \sum_q (n_{\vec{q}} + \frac{1}{2})\sqrt{K}\, q.$ $\qquad$ Find $E = E_0 + E_1 + E_2.$

(d) Crystals have facets owing to a natural lattice periodicity, which can be modeled by including a perturbation, here assumed small,

$$U = y_o \int d^2\mathbf{x} \cos(2\pi h) = \frac{y_o}{2} \int d^2x \left[e^{2i\pi h} + e^{-2i\pi h}\right]$$

where the lattice constant has been set to unity.

Compute a perturbation expansion to order y_o^2, and show that to second order

$$G_k(\mathbf{x}-\mathbf{y}) = G_k(\mathbf{x}-\mathbf{y})_o \times \left[1 + \frac{\pi^3 k^2}{K^2} y_o^2 C(\mathbf{x}-\mathbf{y}) \int_a^\infty dr r^3 \exp(-\frac{2\pi \log(r/a)}{k})\right]$$

By re-exponentiating, show that this result gives an effective coupling constant K_{eff}

(e) Recast these results into renormalization group equations, by scaling the cutoff from a to ae^l to obtain

$$\frac{dK}{dl} = \pi^3 a^4 y_o^2 + \ldots \quad (5)$$

$$\frac{dy_o}{dl} = (2 - \frac{\pi}{K}) y_o + \ldots \quad (6)$$

(f) Using these results, explain the phase diagram of the model

$$E_1 = \langle \psi_0 | U | \psi_0 \rangle = \langle \psi_0 | \, y_0 \int d^2\vec{x} \cos(2\pi h(\vec{x})) \, | \psi_0 \rangle = 0 .$$

$$E_2 = \frac{|\langle \psi_1 | U | \psi_0 \rangle|^2}{E_0 - E_1} + \frac{|\langle \psi_2 | U | \psi_0 \rangle|^2}{E_0 - E_2} \ \ldots$$

how this affect $G_k(\vec{x}-\vec{y})$?

by modify $h(\vec{x})$. ?

(missing derivation)

$$G_k(\vec{x}-\vec{y}) = G_k(\vec{x}-\vec{y})_0 \times \left[1 + \frac{\pi^3 k^2}{k^2} y_0^2 \, C(\vec{x}-\vec{y}) \int_a^\infty dr \, r^3 \exp\left(-\frac{2\pi \log(r/a)}{k}\right)\right]$$

$$G_k(\vec{x}-\vec{y}) \sim \left(G_k(\vec{x}-\vec{y})_0\right)^{\frac{k}{K_{eff}}} \qquad K_{eff} = k \ln\left(\frac{G_k(\vec{x}-\vec{y})_0}{G_k(\vec{x}-\vec{y})}\right) \approx -\frac{\pi^3 k^2}{k} y_0^2 \, C(\vec{x}-\vec{y}) \int_a^\infty dr \, r^3 \exp\left(-\frac{2\pi \log(r/a)}{k}\right)$$

4. *One step replica symmetry breaking in the p-spin spherical model*

The free energy of the infinite range, p-spin spherical model in the large-N limit is given by

$$F = \lim_{n\to 0} -\frac{1}{2\beta n}\left[\frac{\beta^2}{2}\sum_{a,b} Q_{ab}^p + \log\det Q\right] \tag{7}$$

We propose a replica-symmetry-broken solution of the form of an $n \times n$ matrix

$$Q = \begin{pmatrix} 1 & q_1 & q_1 & & & & & \\ q_1 & 1 & 1 & & q_0 & & q_0 & \\ q_1 & q_1 & 1 & & & & & \\ & & & 1 & q_1 & q_1 & & \\ & q_0 & & q_1 & 1 & 1 & q_0 & \\ & & & q_1 & q_1 & 1 & & \\ & & & & & & \cdot & \\ & q_0 & & & q_0 & & \cdot & \\ & & & & & & & \cdot \end{pmatrix} \tag{8}$$

where we have broken up the matrix into $m \times m$ blocks (here 3×3), with the offdiagonal blocks all having the overlap q_0 (on every element) and the diagonal blocks have self-overlap unity, other elements q_1. We expect on physical grounds that $1 > q_1 > q_0$.

(a) Compute the overlap distribution as a function of n, m, q_0, q_1

$$P(q) = \frac{2}{n(n-1)}\sum_{a,b}\delta(q - Q_{ab})$$

(a) number of q_1 = $(m^2 - m)\cdot n$

q_0 : $\left[\left(\frac{n}{m}\right)^2 - \frac{n}{m}\right]\cdot m^2$ $\quad \frac{n}{m} \quad \frac{n-m}{m}$

$$\Rightarrow P(q) = \frac{2}{n(n-1)}\left(\delta(q-q_1)\, n_{q_1} + \delta(q-q_0)\, n_{q_0}\right) = \frac{2}{n(n-1)}\left((m-1)nm\,\delta(q-q_1) + n(n-m)\,\delta(q-q_0)\right)$$

$$= \frac{2m(m-1)}{n-1}\delta(q-q_1) + \frac{2n(n-m)}{n-1}\delta(q-q_0)$$

Show that your calculation leads to a physically sensible result in the limit $n \to 0$, provided $0 \le m \le 1$.

(b) Show that the matrix Q has the following eigenvalues and degeneracies

$$\lambda_1 = (1-q_1) \qquad d_1 = n - n/m \tag{9}$$
$$\lambda_2 = (1-q_1) + m(q_1 - q_0) \qquad d_2 = n/m - 1 \tag{10}$$
$$\lambda_3 = (1-q_1) + m(q_1 - q_0) + nq_0 \qquad d_3 = 1 \tag{11}$$

(c) Hence obtain the free energy (taking the limits $n \to 0$ and keeping $0 \le m \le 1$)

$$\begin{aligned} -2\beta F_{1RSB} &= \frac{\beta^2}{2}\left[1 + (m-1)q_1^p - mq_0^p\right] \\ &+ \frac{(m-1)}{m}\log(1-q_1) + \frac{1}{m}\log\left[m(q_1-q_0) + (1-q_1)\right] \\ &+ \frac{q_0}{m(q_1-q_0)+(1-q_1)} \end{aligned} \tag{12}$$

(d) Minimise F_{1RSB} to obtain three saddle point equations, for m, q_0, q_1. Solve these equations, numerically or graphically, and explain the results.

Hint. You should find that $q_o = 0$, and at high enough temperatures q_1 is zero, while m is undetermined. On lowering temperature, find the first temperature $T = T_s$ where q_1 is nonzero while $m = 1$.

with $n \to 0$, $0 \le m \le 1$. $P(q) = 2m(1-m)\delta(q-q_1)$.

(b) For general $n \times n$ matrix $\begin{pmatrix} A & & \\ & \ddots & \\ & & A \end{pmatrix}$ other elements B.

$M = I(A-B) + (1 \cdots 1)B$.

It has eigenvector $(1, \cdots 1)^T$ and $\vec{v}_{i\perp}$. $(i = 2, \cdots n)$.

Has eigenvalues $\lambda_{1\sim n-1} = A - B$, $\lambda_n = A - B + nB$.

For our case, $A = \begin{pmatrix} 1 & & q_1 \\ & \ddots & \\ q_1 & & 1 \end{pmatrix}_{m\times m}$. $B = \begin{pmatrix} & q_0 & \end{pmatrix}_{m\times m}$. $n \to \frac{n}{m}$.

A has eigenvalues $\lambda_{(1\cdots m-1)_A} = 1 - q_1$, $\lambda_{m_A} = 1 - q_1 + m q_1$.

$\Rightarrow \lambda_1 = 1 - q_1$. number: $(m-1)\cdot\frac{n}{m} = n - \frac{n}{m}$.

$(A-B)$ $\lambda_2 = 1 - q_1 + m q_1 - m q_0$. number: $\frac{n}{m} - 1$.

$(A-B+nB)$ $\lambda_3 = 1 - q_1 + m(q_1 - q_0) + n q_0$. number: 1.

(c) $F = \lim\limits_{n\to 0} -\frac{1}{2\beta n}\left[\frac{\beta^2}{2}\sum_{ab} Q_{ab}^p + \log\det Q\right]$.

As from above, $\det Q = \lambda_1^{d_1}\lambda_2^{d_2}\lambda_3^{d_3}$. (each subspace diagonal)

And $\sum_{ab} Q_{ab}^p = \frac{n}{m}\cdot m(m-1)q_1^p + m^2\cdot\frac{n}{m}\left(\frac{n}{m}-1\right)q_0^p + n\cdot 1^p$

$$-2\beta F = \lim_{n\to 0}\left\{\frac{\beta^2}{2n}\left[n + n(m-1)q_1^p + n(n-m)q_0^p\right] + \frac{1}{n}\left(n - \frac{n}{m}\right)\log(1-q_1)\right.$$
$$\left. + \frac{1}{n}\left(\frac{n}{m}-1\right)\log\left[(1-q_1) + m(q_1-q_0)\right] + \frac{1}{n}\log\left[(1-q_1) + m(q_1-q_0) + nq_0\right]\right\}$$

$$= \lim_{n\to 0}\left\{\frac{\beta^2}{2}\left[1 + (m-1)q_1^p + (n-m)q_0^p\right] + \frac{m-1}{m}\log(1-q_1)\right.$$
$$\left. + \left(\frac{1}{m}-\frac{1}{n}\right)\log\left[m(q_1-q_0) + (1-q_1)\right] + \frac{1}{n}\log\left[m(q_1-q_0) + 1 - q_1 + nq_0\right]\right\}$$

$$= \frac{\beta^2}{2}\left[1 + (m-1) q_1^p - m q_0^p\right] + \frac{m-1}{m}\log(1-q_1)$$

$$+ \frac{1}{m}\log\left[m(q_1-q_0) + (1-q_1)\right] + \frac{q_0}{m(q_1-q_0)+1-q_1}.$$

(d) Minimize F_{RSB}. $\partial F/\partial(m, q_0, q_1)$.

$$\frac{\partial F}{\partial m} = \frac{\beta^2}{2}\left(q_1^p - q_0^p\right) + \frac{1}{m^2}\log(1-q_1) - \frac{1}{m^2}\log\left(m(q_1-q_0)+(1-q_1)\right)$$

$$+ \frac{1}{m}\frac{(q_1-q_0)}{m(q_1-q_0)+(1-q_1)} + \frac{q_1-q_0}{\left(m(q_1-q_0)+(1-q_1)\right)^2} = 0.$$

$$\frac{\partial F}{\partial q_1} = \frac{\beta^2}{2}(m-1)p q_1^{p-1} - \frac{1-m}{m}\frac{1}{(1-q_1)} - \frac{1}{m}\frac{m-1}{m(q_1-q_0)+(1-q_1)} + \frac{(m-1)q_0}{\left(m(q_1-q_0)+(1-q_1)\right)^2} = 0.$$

$$\frac{\partial F}{\partial q_0} = -\frac{\beta^2}{2} m p q_0^{p-1} - \frac{m}{m(q_1-q_0)+(1-q_1)} + \frac{1}{m(q_1-q_0)+(1-q_1)} - \frac{m q_0}{\left(m(q_1-q_0)+(1-q_1)\right)^2}$$

$$= 0.$$

numerics : . . .