

基本物理量是时空变换的生成元

非相对论情形下, Galileo 时空变换有 10 个生成元.

量子态表示为 Hilbert 空间中的向量, 变换和生成元具有算子的形式.

范数

对任意 $\psi \in V$, 定义实数 $\|\psi\|$.

• 对所有 $\psi \neq 0$, $\|\psi\| > 0$. 当且仅当 $\psi = 0$, $\|\psi\| = 0$.

• 用 a 数乘 ψ 得 $a\psi$, 范数为 $\|a\psi\| = |a|\|\psi\|$.

子空间, 正交子空间

M 为 H 的一个子空间.

$$H = M \oplus M^\perp.$$

$$(\psi, \varphi) = 0, \quad \forall \psi \in M, \varphi \in M^\perp.$$

算子 T 的范数

$$\|T\| = \sup_{\|\psi\|=1} \|T(\psi)\|_{H'}.$$

属于 H 的向量 ψ 变换到 H' 中.

范数为 $\|T(\psi)\|_{H'}$.

考虑所有属于 H 的单位向量, 最大的 $\|T(\psi)\|_{H'}$ 即是 T 的范数.

线性算子 T 将 H 中的向量变为 H' 中的.

这些线性算子构成的空间记作 $L(H, H')$.

若 $H = H'$, $L \rightarrow L(H)$, T 为 H 上的线性算子.

$L(H, \mathbb{C})$. H 中的向量变成 $\mathbb{C}$ 复数, H 上的线性泛函.

H 对偶空间, $H^\top$.

Riesz 引理

对每一个 $T \in \mathcal{H}^+$, 有唯一的 $q_T \in \mathcal{H}$, 使对所有的 $\psi \in \mathcal{H}$, 有 $T(\psi) = (q_T, \psi)$.

向量变为数的变换最终落实到内积形式

对 $\mathcal{H}$ 中的每一个向量 q 总可定义一个连续的线性泛函 $T_q \in \mathcal{H}^+$.

$$T_q(\psi) = (q, \psi).$$

厄米共轭的存在性.

内积 $(q, X\psi)$ 可表示为 $T_q(X\psi)$.

$$\underline{T_q(X\psi)} = (T_q \circ X)\psi.$$

$$\text{对比 } T_q(\psi) = (q, \psi).$$

给出一个新的泛函 $F_q(\psi) = (q, X\psi)$.

对于 F_q , 一定存在 $\chi \in \mathcal{H}$, 使

$$F_q(\psi) = (\chi, \psi) = T_\chi(\psi).$$

厄米共轭存在

X 给定时, χ 由 q 唯一确定. 因此一定存在线性算子 $X^\dagger$ 使 $\chi = X^\dagger q$.

于是 $F_q = T_{X^\dagger q}$. 即

$$(q, X\psi) = F_q(\psi) = T_{X^\dagger q}(\psi) = (X^\dagger q, \psi).$$

算子的表示

$$X = \sum_{i,j=1}^N x_{ij} |e_i\rangle\langle e_j|.$$

$$|\psi\rangle = \sum_i |e_i\rangle \langle e_i, \psi \rangle = \sum_i \psi_i |e_i\rangle.$$

$$\langle \varphi | = \langle \varphi | \sum_i |e_i\rangle\langle e_i| = \sum_i \varphi_i^* \langle e_i|.$$

求迹运算

$$\text{Tr}(X) = \sum_{i=1}^N x_{ii}$$

$$\text{Tr}(X) = \sum_{i=1}^N \langle e_i | X | e_i \rangle = \sum_{i=1}^N \langle e_i | \left(\sum_{m,n=1}^N x_{mn} | e_m \rangle \langle e_n | \right) | e_i \rangle$$

转置

$$\begin{aligned} X^T &= \left(\sum_{ij} x_{ij} | e_i \rangle \langle e_j | \right)^T \\ &= \sum_{ij} x_{ij} | e_j \rangle \langle e_i | = \sum_{ij} x_{ji} | e_i \rangle \langle e_j | \end{aligned}$$

厄米共轭

$$\langle X | \psi \rangle = \langle \psi | X \rangle^* = \langle \psi | X^\dagger | \varphi \rangle^* = \langle \varphi | X | \psi \rangle$$

$$| \chi \rangle = X^\dagger | \varphi \rangle$$

厄米矩阵 A

本征基 $A = \sum_i a_i | \alpha_i \rangle \langle \alpha_i |$

在 A 表象, 任何向量 $|\psi\rangle$ 可展开为

自然基 $A = \sum_{ij} a_{ij} | e_i \rangle \langle e_j |$

$$|\psi\rangle = \sum_i c_i | \alpha_i \rangle, \quad c_i = \langle \alpha_i | \psi \rangle$$

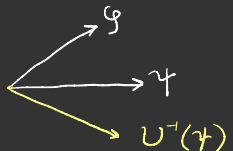
B 表象

$$B = \sum_j b_j | \beta_j \rangle \langle \beta_j |, \quad |\psi\rangle = \sum_j d_j | \beta_j \rangle, \quad d_j = \langle \beta_j | \psi \rangle$$

酉变换

$\mathbb{C}^n$ 上的酉变换 对 $\psi, \varphi \in \mathbb{C}^n$ 及 $U \in \mathcal{U}(\mathbb{C})$ 若

酉变换 $(U(\varphi), \psi) = (\varphi, U^{-1}(\psi))$



或 $UU^\dagger = U^\dagger U = 1$ 的矩阵为酉矩阵.

本征值模为1. 行/列向量正交归一. $|\det(U)| = 1$.

若 $\mathcal{U}$ 为有限维空间 $\mathbb{C}^n$ 上酉矩阵的集合, 则

• 对 $c \in \mathbb{C}$, $|c|=1$ 和 $U \in \mathcal{U}$, 有 $cU \in \mathcal{U}$.

• 若 $U, V \in \mathcal{U}$, $UV \in \mathcal{U}$.

• 若 $U \in \mathcal{U}$, $U^{-1} \in \mathcal{U}$.

集合 $\mathcal{U}$ 具有群结构, 是 $U(n)$ 群.

$\mathbb{C}^2$ 上的酉矩阵 $U(1)$ 群

$$U = \underline{e^{i\tau}} \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix}$$

$$U(2) \simeq U(1) \times SU(2)$$

$$U \in SU(2), \det(U) = 1$$

Wigner 定理.

$$\text{对 } |\psi\rangle = U|\varphi\rangle, \quad |\varphi\rangle = U|\psi\rangle$$

保内积变换是酉/反酉的.

若 $|\langle\psi|\varphi\rangle| = |\langle\psi|\varphi\rangle|$, 则 U 是酉或反酉的.

表象变换.

$$\{|\alpha_i\rangle\} \quad \{|\beta_j\rangle\}$$

$$|\beta_j\rangle = \sum_i u_{ij} |\alpha_i\rangle, \quad U = (u_{ij})$$

A 表象 B 表象

$$u_{ij} = \langle\alpha_i|\beta_j\rangle, \quad \underline{U = \sum_i |\beta_j\rangle\langle\alpha_i|}$$

$$\text{或写为: } U|\alpha_j\rangle = |\beta_j\rangle$$

将 $|\alpha_i\rangle$ 映射为 $|\beta_i\rangle$.

$$u_{ij} = \langle\alpha_i|U|\alpha_j\rangle = \langle\alpha_i|\beta_j\rangle$$

$$\begin{pmatrix} & \\ & \\ & \\ 1 & \end{pmatrix} \text{--- } j \text{ 行}$$

$$U|e_j\rangle = \begin{pmatrix} u_{1j} \\ u_{2j} \\ \vdots \\ u_{nj} \end{pmatrix} = \sum_i u_{ij} |e_i\rangle = |\beta_j\rangle$$

注意：此处 U 是对基向量的变换。

新表象中向量分量的变换由 $U^\dagger$ 决定。

$$\begin{aligned} |\psi\rangle &= \sum_i c_i |\alpha_i\rangle = \sum_{ij} c_i |\beta_j\rangle \langle\beta_j|\alpha_i\rangle \\ &= \sum_j \left(\sum_i U_{ij}^* c_i \right) |\beta_j\rangle = \sum_j \left[\sum_i (U^\dagger)_{ji} c_i \right] |\beta_j\rangle \end{aligned}$$

一个例子。

$$\sigma_z \rightarrow \sigma_x \text{ 表象. } |\chi+\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad |\chi-\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

变换矩阵

$$U = |\chi+\rangle\langle z+| + |\chi-\rangle\langle z-| = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}. \quad \text{Hadamard matrix.}$$

σ_x 中向量的分量由 $U^\dagger$ 决定。

$$|\psi\rangle = |z+\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad U^\dagger \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}} (|\chi+\rangle + |\chi-\rangle)$$

写为列向量时注意表象

$$|\psi\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}_{\sigma_z} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}_{\sigma_x}.$$

力学量的变换

$$A \text{ 表象中. } X = \sum_{ij} x_{ij} |\alpha_i\rangle\langle\alpha_j| = X^{(A)}.$$

$$\begin{aligned} B \text{ 表象中. } X &= \sum_{i,j} x_{ij} |\beta_m\rangle\langle\beta_n| \langle\alpha_i|\alpha_j\rangle |\beta_m\rangle\langle\beta_n| \\ &= \sum_{i,j} U_{im}^* x_{ij} U_{jn} |\beta_m\rangle\langle\beta_n|. \end{aligned}$$

$$X^{(B)} = U^\dagger X^{(A)} U.$$

$$\text{例: } \sigma_z \text{ 表象中的 } \sigma_x. \quad \sigma_x^{(x)} = U^\dagger \sigma_x^{(z)} U = \text{diag}(1, -1).$$

在自身表象中是对角的。厄米矩阵对角化。

一般形式的矩阵. 酉变换未必使其对角化.

分解形式. 1. polar decomposition. X 为任意方阵

$$X = \underbrace{PU}_{\text{半正厄米矩阵, 总可通过酉变换对角化}}$$

2. Singular value decomposition.

$$X = V \underbrace{\Lambda}_{\text{对角的实数矩阵}} W^\dagger. \quad V, W \text{ 酉}$$

设量子系统处于 A 的本征态 $|\alpha_k\rangle$. 希望知道 B 的测量结果几率分布.

在 B 表象中表示 $|\alpha_k\rangle$.

$$|\alpha_k\rangle = \sum_i \langle \beta_i | \alpha_k \rangle |\beta_i\rangle.$$

若 $[A, B] \neq 0$. 一般情况下 $|\beta_i\rangle$ 不是 A 的本征态. 对 B 的测量结果不确定.

投影算子

非简并情形下定义一维投影算符 $\pi_i = |\alpha_i\rangle\langle\alpha_i|$.

厄米 $\pi_i = \pi_i^\dagger$. 正交 $\pi_i \pi_j = \delta_{ij} \pi_i$.

$\pi_i^2 = \pi_i$. 本征值为 1, 0. 完备 $\sum_i \pi_i = 1$.

有酉变换将自然基向量 $|e_i\rangle$ 变为 $|\alpha_i\rangle$.

$$V|e_i\rangle = |\alpha_i\rangle. \quad \text{有 } \sum_i \pi_i = \sum_i |\alpha_i\rangle\langle\alpha_i| = V \left(\sum_i |e_i\rangle\langle e_i| \right) V^\dagger = V V^\dagger = 1.$$

π_j 作用于 $|\psi\rangle$.

$$\pi_j |\psi\rangle = |\alpha_j\rangle\langle\alpha_j|\psi\rangle = c_j |\alpha_j\rangle. \quad (\text{未归一化})$$

$$p_j = |c_j|^2 = \langle\psi|\pi_j|\psi\rangle = \text{Tr}[\pi_j(|\psi\rangle\langle\psi|)]$$

几率 $p_j = \pi_j$ 在 $|\psi\rangle$ 中的期望值.

子空间上的投影算子

$\mathbb{C}^N$ 中的子空间 S 的基向量是 $|\alpha_j\rangle, j=1,2,\dots,m$.

子空间上的投影算子为 $\Pi^S = \sum_{j=1}^m |\alpha_j\rangle\langle\alpha_j|$.

$|\alpha_i\rangle\langle\alpha_i|$

Π_i 作用于 $|\psi\rangle$ 得 $c_i|\alpha_i\rangle$. 并不是完整的测量过程.

未归一化的态矢量, 仍属 Hilbert 空间.

量子测量: 不同类型的测量不能对易. 测量结果不能共存.

对易力学量可以有相同的本征向量.

$[A, B] = 0$ 时, 可以在同一表象中描述这两个力学量. 可在相同实验环境测量. 相容的.

观测量的期望值

$$\begin{aligned}\langle A \rangle_\psi &= \sum_i a_i |\langle \alpha_i | \psi \rangle|^2 = \sum_i a_i \langle \alpha_i | \psi \rangle \langle \psi | \alpha_i \rangle \\ &= \sum_i \langle \psi | \alpha_i \rangle a_i \langle \alpha_i | \psi \rangle = \langle \psi | A | \psi \rangle = \text{Tr}(\psi^\dagger A \psi)\end{aligned}$$

通过力学量期望值引入密度矩阵.

对纯态系综, 设空间基向量为 $|\varphi_i\rangle$.

$$\begin{aligned}\langle A \rangle_\psi &= \langle \psi | A | \psi \rangle = \sum_i \langle \psi | \varphi_i \rangle \langle \varphi_i | A | \psi \rangle \\ &= \sum_i \langle \varphi_i | A | \psi \rangle \langle \psi | \varphi_i \rangle = \text{Tr}(\psi^\dagger A \psi) = \text{Tr}(\psi A)\end{aligned}$$

对混合系综, $\mathcal{E} = \{p_i, \psi_i\}$.

$$\langle A \rangle_{\mathcal{E}} = \sum_i p_i \langle \psi_i | A | \psi_i \rangle = \sum_i p_i \text{Tr}(\psi_i^\dagger A) = \text{Tr}\left(\sum_i p_i \psi_i^\dagger A\right)$$

可表示为 $\mathcal{E} = \{p_i, \psi_i\} \rightarrow \sum_i p_i |\psi_i\rangle\langle\psi_i|$.

定义密度算符

对混合系综 $\mathcal{E} = \{p_i, \psi_i\}$.

$$\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|, \quad p_i \geq 0, \quad \sum_i p_i = 1.$$

系综的平均量子态

1. $\text{Tr}(\rho) = 1.$

$\text{Tr}(\rho) = 1$ 是由什么保证的?

2. $\rho = \rho^\dagger$

半正定

$$\sum_i p_i = 1.$$

3. $\forall |\phi\rangle \in \mathcal{H}, \langle\phi|\rho|\phi\rangle \geq 0.$

设 $\mathcal{H}$ 有限维, 基向量为 $\{|\psi_i\rangle\}$.

$$\text{Tr}(\rho) = \sum_i \langle\psi_i|\rho|\psi_i\rangle = \sum_{i,k} p_k \langle\psi_i|\psi_k\rangle\langle\psi_k|\psi_i\rangle$$

$$= \sum_k p_k \underbrace{\sum_i |\langle\psi_i|\psi_k\rangle|^2}_{\text{波函数的归一化}} = \sum_k p_k = 1.$$

波函数的归一化

对任意 $|\psi\rangle \in \mathcal{H}$, 有

$$\langle\psi|\rho|\psi\rangle = \sum_k p_k \langle\psi|\psi_k\rangle\langle\psi_k|\psi\rangle \geq 0.$$

最大混合态

$\mathbb{C}^n$ 中, 定义如下形式的混合态

$$\rho = \frac{1}{n} \text{diag}(1, 1, \dots, 1).$$

纯度

对于纯态 $|\psi\rangle\langle\psi|$, $\text{Tr}\psi = \text{Tr}\psi^2 = 1.$

对于混合态 $\text{Tr}\rho = 1, \quad \text{Tr}\rho^2 < 1.$

引入混态的意义

· 实验需要

· 量子态的几何结构

· 开放量子系统, 量子关联

· 将正交的经典信息植入非正交的量子态

混态下的 Born 规则

$$\rho = \sum_k p_k \psi_k \quad \text{被测力学量 } A = \sum_i a_i |\alpha_i\rangle\langle\alpha_i|$$

测量得结果 a_i 的几率为 $\rightarrow$ 处于 ψ_k 时测得 a_i 的几率, 而处于 ψ_k 几率为 p_k .

$$\begin{aligned} p_i &= \sum_k p_k \underbrace{|\langle\alpha_i|\psi_k\rangle|^2}_{|\alpha_i\rangle\langle\alpha_i|} = \sum_k p_k \langle\alpha_i|\psi_k\rangle\langle\psi_k|\alpha_i\rangle \\ &= \langle\alpha_i| \left[\sum_k p_k \psi_k \langle\psi_k| \right] |\alpha_i\rangle = \langle\alpha_i|\rho|\alpha_i\rangle = \text{Tr}(\rho \pi_i) \end{aligned}$$

对比普通情况 $p_j = |c_j|^2 = \langle\psi|\pi_j|\psi\rangle = \text{Tr}[\pi_j(|\psi\rangle\langle\psi|)]$

几率 = π_j 在 $|\psi\rangle$ 中的期望值.

$$\rho \rightarrow \pi_i \rho \pi_i^\dagger = \pi_i \rho \pi_i = p_i |\alpha_i\rangle\langle\alpha_i|$$

$$\rho \rightarrow \sum_i \pi_i \rho \pi_i = \sum_i p_i |\alpha_i\rangle\langle\alpha_i|$$

$$p_i = |\langle\alpha_i|\psi\rangle|^2 = \text{Tr}(\psi \pi_i)$$

纯态 $\psi = |\psi\rangle\langle\psi| \rightarrow \sum_i \pi_i \psi \pi_i = \sum_i p_i |\alpha_i\rangle\langle\alpha_i|$

保迹过程 $\text{Tr} \rho = 1 \quad \text{Tr} \left(\sum_i \pi_i \rho \pi_i \right) = 1$

正交变换 完全正交变换

测量前后期望值不变

$$\text{Tr} \left(A \sum_i p_i |\alpha_i\rangle\langle\alpha_i| \right) = \sum_i a_i p_i = \text{Tr}(A\rho)$$

双值量子系统

纯态在基向量上 $|\psi\rangle = c_0|0\rangle + c_1|1\rangle$, $|c_0|^2 + |c_1|^2 = 1$.

$$|\psi\rangle = e^{i\phi} \begin{pmatrix} \cos\frac{\theta}{2} e^{-i\frac{\phi}{2}} \\ \sin\frac{\theta}{2} e^{i\frac{\phi}{2}} \end{pmatrix} \quad \text{是 } \sigma_n \text{ 本征态, 本征值 } 1.$$

密度矩阵形式

$$\rho = |\psi\rangle\langle\psi| = \begin{pmatrix} \cos^2\frac{\theta}{2} & \sin\frac{\theta}{2}\cos\frac{\theta}{2}e^{-i\phi} \\ \sin\frac{\theta}{2}\cos\frac{\theta}{2}e^{i\phi} & \sin^2\frac{\theta}{2} \end{pmatrix}$$

用 Pauli 矩阵展开

$$\rho = \frac{1}{2} (1 + \sigma_x \sin\theta \cos\phi + \sigma_y \sin\theta \sin\phi + \sigma_z \cos\theta)$$

$$= \frac{1 + \vec{\sigma} \cdot \vec{n}}{2}$$

$$\vec{n} = (\sin\theta \cos\phi, \sin\theta \sin\phi, \cos\theta)$$

$$\sigma_n |\psi\rangle = (2\rho - 1) |\psi\rangle = |\psi\rangle$$

与 $|\psi\rangle$ 相差相因子

也可直接计算 σ_n 本征向量

$$\sigma_n |n\pm\rangle = \pm |n\pm\rangle$$

$$|n+\rangle = \begin{pmatrix} \cos\frac{\theta}{2} e^{-i\frac{\phi}{2}} \\ \sin\frac{\theta}{2} e^{i\frac{\phi}{2}} \end{pmatrix}, \quad |n-\rangle = \begin{pmatrix} -\sin\frac{\theta}{2} e^{-i\frac{\phi}{2}} \\ \cos\frac{\theta}{2} e^{i\frac{\phi}{2}} \end{pmatrix}$$

$$\sigma_n = |n+\rangle\langle n+| - |n-\rangle\langle n-|$$

$$|n\pm\rangle\langle n\pm| = \frac{1 \pm \sigma_n}{2}$$

$$\text{又由 } 1 = |n+\rangle\langle n+| + |n-\rangle\langle n-|$$

$\mathbb{C}^2$ 中两个正交的向量对应 $\mathbb{C}^3$ 两个相反的方向。

密度矩阵与 Bloch 向量

$$\mathbb{C}^2 \text{ 中任意量子态可表示为 } \rho = \frac{1}{2}(1 + \vec{\sigma} \cdot \vec{r}), \quad \vec{r} \in \mathbb{R}^3, \quad |\vec{r}| \leq 1.$$

$\vec{r}$: Bloch 向量. 分布在 $\mathbb{R}^3$ 球的表面及内部.

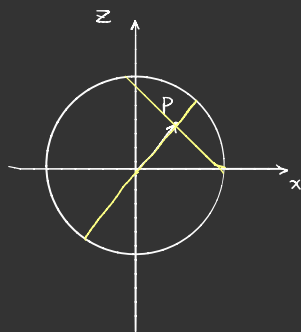
球心: 最大混合态 $\frac{1}{2}$.

$\mathbb{C}^2$ 中的纯态 混合态

$$r_x = \text{Tr}(\rho \sigma_x) \quad r_y = \text{Tr}(\rho \sigma_y) \quad r_z = \text{Tr}(\rho \sigma_z)$$

有

$$\begin{aligned} \rho \sigma_x &= \frac{1}{2}(\sigma_x + r_x \sigma_x^2 + r_y \sigma_y \sigma_x + r_z \sigma_z \sigma_x) \\ &= \frac{1}{2}(\sigma_x + r_x - i r_y \sigma_z + i r_z \sigma_y) \end{aligned}$$



$$\rho = \frac{1}{2} \mathbb{1} + \frac{1}{2} \vec{r} \cdot \vec{\sigma} \quad \vec{r} = r \vec{n} \quad \sigma_n = \vec{\sigma} \cdot \vec{n} \quad |\vec{r}| \text{ 也标示着纯度}$$

本征值为 $\frac{1+r}{2}, \frac{1-r}{2}$.

$$\rho = \frac{1+r}{2} |n+\rangle\langle n+| + \frac{1-r}{2} |n-\rangle\langle n-|$$

本征向量 $|n+\rangle, |n-\rangle$.

对 $\mathbb{C}^2$ 上的两个纯态 ψ_1, ψ_2 .

$$|\langle \psi_1 | \psi_2 \rangle|^2 = \text{Tr}(\psi_1 \psi_2)$$

两态的重叠程度

$$= \text{Tr} \left[\frac{1}{2}(\mathbb{1} + \vec{\sigma} \cdot \vec{n}_1) \frac{1}{2}(\mathbb{1} + \vec{\sigma} \cdot \vec{n}_2) \right]$$

$$= \frac{1}{2} + \frac{1}{4} \text{Tr} \left[\underbrace{(\vec{\sigma} \cdot \vec{n}_1)(\vec{\sigma} \cdot \vec{n}_2)} \right] = \frac{1}{2} + \frac{1}{2} \vec{n}_1 \cdot \vec{n}_2$$

$$\mathbb{1} \vec{n}_1 \cdot \vec{n}_2 + i \vec{\sigma} \cdot (\vec{n}_1 \times \vec{n}_2)$$

对两个混合态

$$\text{Tr}(\rho_1 \rho_2) = \frac{1}{2} + \frac{1}{2} \vec{r}_1 \cdot \vec{r}_2 \quad \text{不可能正交}$$

对应于两个矩阵的 Hilbert-Schmidt (HS) 范数 (Frobenius 范数) 及内积

$$\|A\|_2 = \|A\|_{\text{HS}} = \sqrt{\text{Tr}(A^\dagger A)}$$

$$(A, B) = \text{Tr}(A^\dagger B)$$

酉矩阵

$\mathbb{C}^N$ 上的酉矩阵构成酉群 $U(N)$. 可分为两个单群 $U(1)$, $SU(N)$.

$SU(2)$ 群酉矩阵的一般形式

$$U = \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix}, \quad a, b \in \mathbb{C}, \quad |a|^2 + |b|^2 = 1.$$

3个独立实参数

单参数酉群

$U(s)$ 单参数 Lie 群. $U(0) = 1$. $U(s_1)U(s_2) = U(s_1 + s_2)$. $U^\dagger(s) = U(-s) = U^\dagger(s)$.

考虑无穷小变换 $s \rightarrow 0$.

$$U(s) = 1 + \left. \frac{dU}{ds} \right|_{s=0} s + \dots$$

$\therefore \left. \frac{dU}{ds} \right|_{s=0}$ 是反厄米的.

$$UU^\dagger = 1 + \left[\left. \frac{dU}{ds} + \frac{dU^\dagger}{ds} \right] \right|_{s=0} s + \dots$$

$$\left. \frac{dU}{ds} \right|_{s=0} = iK, \quad K = K^\dagger.$$

K 是 $U(s)$ 的生成元. 有限大小的酉变换表示为

$$U(s) = e^{isk}.$$

$SU(2)$ 的生成元

$$U = \begin{pmatrix} \cos \alpha e^{i\xi} & \sin \alpha e^{i\eta} \\ -\sin \alpha e^{-i\eta} & \cos \alpha e^{-i\xi} \end{pmatrix}, \quad \alpha \in [0, \frac{\pi}{2}].$$

两体量子系统

自由度: 不能在同一个 Hilbert 空间中描述的状态

直积空间中的向量

$$\dim(\mathcal{H}^A) = d^A \quad \text{基向量组 } \{|i\rangle\} \quad i = 0, 1, \dots, d^A - 1.$$

$$\dim(\mathcal{H}^B) = d^B \quad \{|\mu\rangle\} \quad \mu = 0, 1, \dots, d^B - 1.$$

$\mathcal{H}$ 中任意态矢 $|\psi\rangle$ 可在 $\{|\mu\rangle\}$ 上展开.

$$|\psi\rangle = \sum_{i=0}^{d^A-1} \sum_{\mu=0}^{d^B-1} c_{i\mu} |i\rangle |\mu\rangle \quad c_{i\mu} = \langle i\mu | \psi \rangle.$$

$d^A \otimes d^B$ 系统.

Schmidt 分解.

对 $|\psi\rangle \in \mathcal{H}$ 总能找到 $\mathcal{H}^A$ 和 $\mathcal{H}^B$ 的特定基向量. 记为 $\{|e_i\rangle\}$, $\{|f_j\rangle\}$.

$$\text{使} \quad |\psi\rangle = \sum_{j=0}^d c_j |e_j\rangle |f_j\rangle$$

其中 $d = \min\{d^A, d^B\} - 1$ 且每一个 c_j 为正.

直积空间上的矩阵

$d^A d^B$ 维空间上的算符 X . 在自然基向量 $|i\rangle |\mu\rangle$ 上.

$$X = \sum_{ij\mu\nu} \underbrace{x_{\mu\nu}}_{ij} |i\mu\rangle \langle j\nu|.$$

$$\langle i\mu | X | j\nu \rangle = \text{Tr}(X |j\nu\rangle \langle i\mu|).$$

还可表示为

$$X = \sum_{ij\mu\nu} x_{\mu\nu} |i\rangle \langle j| \otimes |\mu\rangle \langle \nu|.$$

$\mathcal{H}$ 上的任意矩阵不一定总能表示为 $\mathcal{H}^A$ 上矩阵与 $\mathcal{H}^B$ 上矩阵的直积.

但总能表示为

$$X = \sum_i M_i \otimes N_i$$

若 A, B 分属不同的 Hilbert 空间, 它们一定对易

$$[A, B] = [A \otimes 1, 1 \otimes B] = 0$$

对矩阵的部分变换

$$X^T = \sum_{ij\mu\nu} x_{i\mu,j\nu} |j\nu\rangle\langle i\mu| \quad \text{或} \quad X^T = \sum_{ij\mu\nu} x_{j\nu,i\mu} |i\mu\rangle\langle j\nu|$$

$$X^\dagger = \sum_{ij\mu\nu} x_{i\mu,j\nu}^* |j\nu\rangle\langle i\mu| \quad \text{或} \quad X^\dagger = \sum_{ij\mu\nu} x_{j\nu,i\mu}^* |i\mu\rangle\langle j\nu|$$

$$\text{Tr}(X) = \sum_{i\mu} \langle i\mu | X | i\mu \rangle = \sum_{i\mu} \sum_{j\nu} x_{i\mu,j\nu} \langle i\mu | i\mu \rangle \langle j\nu | j\nu \rangle = \sum_{i\mu} x_{i\mu,i\mu}$$

部分转置

$$X^{TA} = \sum_{ij\mu\nu} x_{i\mu,j\nu} |j\nu\rangle\langle i\nu| = \sum_{ij\mu\nu} x_{i\mu,j\nu} |j\rangle\langle i| \otimes |\nu\rangle\langle \nu|$$

$$X^{TB} = \sum_{ij\mu\nu} x_{i\mu,j\nu} |i\nu\rangle\langle j\mu| = \sum_{ij\mu\nu} x_{i\mu,j\nu} |i\rangle\langle j| \otimes |\nu\rangle\langle \mu|$$

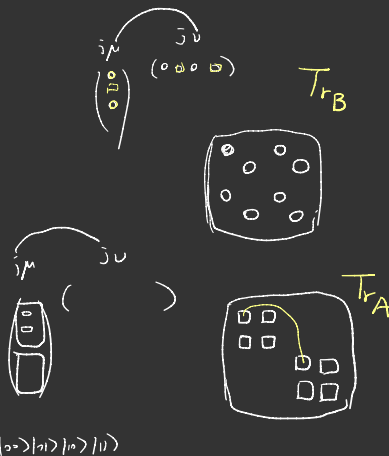
部分迹

$$X^A = \text{Tr}_B(X) = \sum_{j\nu} x_{i\mu,j\nu} |i\rangle\langle j|$$

$$X^B = \text{Tr}_A(X) = \sum_{i\mu} x_{i\mu,j\nu} |\mu\rangle\langle \nu|$$

$$\text{Tr}(X) = \text{Tr}_A[\text{Tr}_B(X)] = \text{Tr}_B[\text{Tr}_A(X)]$$

约化矩阵



$$|00\rangle|11\rangle|00\rangle|11\rangle$$

设 $A \in \mathcal{L}(\mathcal{H}^A)$, $C \in \mathcal{L}(\mathcal{H}^A \otimes \mathcal{H}^B)$. 有

$$\text{Tr}(A \text{Tr}_B C) = \text{Tr}[(A \otimes \mathbf{1})C] \quad \text{从右往左理解更好些}$$

$$\Delta \quad C = \sum_{\mu, \nu} \underline{c_{\mu\nu}} \otimes |\mu\rangle\langle\nu|, \quad \mathcal{H}^A \text{ 上的矩阵}$$

$$\text{Tr}_B C = \sum_{\mu} c_{\mu\mu} \quad \begin{pmatrix} c_{00} & c_{01} \\ c_{10} & c_{11} \end{pmatrix}$$

$$\text{Tr}(A \text{Tr}_B C) = \text{Tr}(A \sum_{\mu} c_{\mu\mu})$$

$$\begin{aligned} \Rightarrow \text{Tr}_B[(A \otimes \mathbf{1})C] &= \sum_{\mu, \nu} \text{Tr}_B[(A \otimes \mathbf{1})(c_{\mu\nu} \otimes |\mu\rangle\langle\nu|)] \\ &= \sum_{\mu, \nu} \text{Tr}_B[A c_{\mu\nu} \otimes |\mu\rangle\langle\nu|] = \sum_{\mu} A c_{\mu\mu} \end{aligned}$$

$$\text{Tr}[(A \otimes \mathbf{1})C] = \text{Tr}_A[\text{Tr}_B[(A \otimes \mathbf{1})C]] = \text{Tr}_A(\sum_{\mu} A c_{\mu\mu}) \quad \checkmark \quad \text{本身就在 } \mathcal{H}^A$$

设 $|\psi\rangle, |\varphi\rangle \in \mathcal{H}^A$, $C \in \mathcal{L}(\mathcal{H}^A \otimes \mathcal{H}^B)$. 有 $\langle\psi|C|\varphi\rangle \in \mathcal{L}(\mathcal{H}^B)$ 且

$$\langle\psi|C|\varphi\rangle = \text{Tr}_A[(|\varphi\rangle\langle\psi| \otimes \mathbf{1})C] \quad \text{若 } \mathcal{H}^B = \mathbb{C}$$

$$\Delta \quad \langle\psi|C|\varphi\rangle = \sum_{\mu, \nu} \langle\psi|c_{\mu\nu}|\varphi\rangle |\mu\rangle\langle\nu|$$

$$\langle\psi|C|\varphi\rangle = \text{Tr}(|\varphi\rangle\langle\psi|C)$$

$$\begin{aligned} \text{Tr}_A[(|\varphi\rangle\langle\psi| \otimes \mathbf{1})C] &= \sum_{\mu, \nu} \text{Tr}_A[(|\varphi\rangle\langle\psi| \otimes \mathbf{1})(c_{\mu\nu} \otimes |\mu\rangle\langle\nu|)] \\ &= \sum_{\mu, \nu} \text{Tr}_A(\underline{c_{\mu\nu} |\varphi\rangle\langle\psi|} \otimes |\mu\rangle\langle\nu|) \\ &\quad \downarrow ? \\ &= \sum_{\mu, \nu} \langle\psi|c_{\mu\nu}|\varphi\rangle \otimes |\mu\rangle\langle\nu| = \langle\psi|C|\varphi\rangle \end{aligned}$$

约化密度矩阵

$\mathcal{H} = \mathcal{H}^A \otimes \mathcal{H}^B$ 中的任意一个混合态密度矩阵可在自然基上表示为

$$\rho = \sum_{ij\mu\nu} \rho_{ij\mu\nu} |i\rangle\langle j| \otimes |\mu\rangle\langle\nu|.$$

力学量 X , (两体系统整体的力学量)

$$\langle X \rangle_\rho = \text{Tr}(X\rho) = \sum_{ij\mu\nu} \langle i|\mu|\rho|j\nu\rangle \langle j\nu|X|i\mu\rangle$$

仅属于子系统 A 的力学量 A .

$$\begin{aligned}\langle A \rangle_\rho &= \text{Tr}[(A \otimes \mathbb{1}^B)\rho] \\&= \text{Tr}\left[(A \otimes \mathbb{1}^B) \sum_{ij\mu\nu} \rho_{ij\mu\nu} |i\rangle\langle j| \otimes |\mu\rangle\langle\nu|\right] \\&= \sum_{ij\mu\nu} \rho_{ij\mu\nu} \underbrace{\text{Tr}[A|i\rangle\langle j| \otimes |\mu\rangle\langle\nu|]}_{\text{Tr}_A(T_{r_B})} = \text{Tr}_A(T_{r_B}) \rightarrow \mu=\nu \\&= \text{Tr}_A\left[A \sum_{ij\mu} \rho_{ij\mu\mu} |i\rangle\langle j|\right] \\&= \text{Tr}_A[A\rho^A].\end{aligned}$$

$$\rho^A = \text{Tr}_B \rho = \sum_{ij\mu} \rho_{ij\mu\mu} |i\rangle\langle j|.$$

$$\text{Tr}(\rho^A) = 1, \quad \rho^A = (\rho^A)^\dagger.$$

$$\forall |x\rangle \in \mathcal{H}^A, \quad \langle x|\rho^A|x\rangle = \text{Tr}_A(\rho^A|x\rangle\langle x|) = \text{Tr}[\rho(|x\rangle\langle x| \otimes \mathbb{1}^B)].$$

$$= \text{Tr}\left[\rho(|x\rangle\langle x| \otimes \sum_\mu |\mu\rangle\langle\mu|)\right]$$

$$= \sum_\mu \text{Tr}[\rho(|x\rangle\langle x| \otimes |\mu\rangle\langle\mu|) (\langle x|\otimes\langle\mu|)]$$

$$= \sum_\mu (\langle x|\otimes\langle\mu|)\rho(|x\rangle\otimes|\mu\rangle) \geq 0.$$

满足密度矩阵的
性质

求迹 —— 测量过程/退相干过程?

$$p(a) = \text{Tr}[\rho(\pi_a^A \otimes \mathbf{1}^B)] = \text{Tr}_A[\rho^A \pi_a^A]$$

局部测量结果的概率分布

$$p(b) = \text{Tr}[\rho(\mathbf{1}^A \otimes \pi_b^B)] = \text{Tr}_B[\rho^B \pi_b^B]$$

取决于相应的约化密度算符

... 定域性.

开放量子系统的演化

系统 $\mathcal{Q}$ 仪器 $\mathcal{M}$

$$|\psi(0)\rangle = |\psi\rangle \otimes |\varphi\rangle \in \mathcal{H}^{\mathcal{Q}} \otimes \mathcal{H}^{\mathcal{M}}$$

$\mathcal{H}^{\mathcal{Q}}$

$\mathcal{H}^{\mathcal{M}}$

$$|\psi(t)\rangle = U(t)|\psi(0)\rangle = U(t)(|\psi\rangle \otimes |\varphi\rangle)$$

密度矩阵

$$\rho^{\mathcal{Q}}(t) = U(t) \rho^{\mathcal{Q}}(0) U^\dagger(t) = U(t)(\psi \otimes \varphi) U^\dagger(t).$$

$$\rho^{\mathcal{Q}}(t) = \text{Tr}_{\mathcal{M}} \psi(t).$$

设 $\mathcal{H}^{\mathcal{M}}$ 基向量为 $|\mu\rangle$, $\mu = 0, 1, \dots, n-1$.

$$U(t) = \sum_{\mu, \nu=0}^{n-1} \underline{U_{\mu\nu}} \otimes |\mu\rangle\langle\nu|.$$

$\mathcal{H}^{\mathcal{Q}}$ 上的 $n \times n$ 矩阵

$$\text{有 } \sum_{\nu} U_{\mu\nu} U_{\mu'\nu}^\dagger = \delta_{\mu\mu'} \mathbf{1}^{\mathcal{Q}}, \quad \sum_{\mu} U_{\mu\nu}^\dagger U_{\mu\nu} = \delta_{\nu\nu'} \mathbf{1}^{\mathcal{Q}}.$$

$$\psi(t) = U(t)(\psi \otimes \varphi) U^\dagger(t)$$

$$= \sum_{\mu, \mu', \nu} (U_{\mu\nu} \psi U_{\mu'\nu}^\dagger) \otimes (|\mu\rangle\langle\nu| \varphi \langle\nu|\mu'|)$$

利用

$$\langle\psi|C|\varphi\rangle = \text{Tr}_{\mathcal{A}} [(|\varphi\rangle\langle\psi| \otimes \mathbf{1}) C]$$

$\Downarrow$

$\mathcal{H}^{\mathcal{M}}$ 上求迹

$$\text{Tr}_{\mathcal{M}} \psi(t) = \sum_{\mu, \nu} (\underline{U_{\mu\nu}} \psi U_{\mu\nu}^\dagger) \underline{\langle\nu|\varphi\rangle\langle\varphi|\mu\rangle}$$

$$\text{因 } \text{Tr}_{\mathcal{M}} [U(\mathbf{1} \otimes |\varphi\rangle\langle\mu|)]$$

$$= \text{Tr}_{\mathcal{M}} \left[\sum_{\mu', \nu'} (U_{\mu'\nu'} \otimes |\mu\rangle\langle\nu|) (\mathbf{1} \otimes |\varphi\rangle\langle\mu|) \right]$$

$$\text{令 } K_{\mu} = \sum_{\nu} U_{\mu\nu} \langle\nu|\varphi\rangle = \text{Tr}_{\mathcal{M}} [U(\mathbf{1} \otimes |\varphi\rangle\langle\mu|)]$$

$$= \sum_{\nu} U_{\mu\nu} \langle\nu|\varphi\rangle = \sum_{\nu} U_{\mu\nu} \langle\nu|\varphi\rangle.$$

$$= \langle\mu|U|\varphi\rangle.$$

$$K_{\mu}^\dagger = \sum_{\nu} U_{\mu\nu}^\dagger \langle\varphi|\nu\rangle = \text{Tr}_{\mathcal{M}} [U^\dagger(\mathbf{1} \otimes |\mu\rangle\langle\varphi|)] = \langle\varphi|U^\dagger|\mu\rangle.$$

初态 $\psi = \rho^{\mathcal{Q}}(0)$.

$$\rho^S(t) = \sum_{\mu} K_{\mu}(t) \rho^S(0) K_{\mu}^{\dagger}(t)$$

K_{μ}

$K_{\mu}^{\dagger}$

$$\text{Tr}_{\mathcal{H}} \rho(t) = \sum_{\mu\nu\nu'} (u_{\mu\nu} u_{\mu\nu'}^{\dagger}) \langle \nu | \rho \rangle \langle \nu' | \nu \rangle$$

$$\sum_{\mu} K_{\mu}^{\dagger} K_{\mu} = \sum_{\mu\nu\nu'} u_{\mu\nu'}^{\dagger} u_{\mu\nu} \langle \nu' | \nu \rangle \langle \nu | \nu \rangle = \sum_{\nu\nu'} \mathbf{1}^S \delta_{\nu\nu'} \langle \nu' | \nu \rangle \langle \nu | \nu \rangle = \mathbf{1}^S$$

演化中守恒

$$\text{Tr} \rho^S(t) = \text{Tr} \sum_{\mu} K_{\mu}(t) \rho^S(0) K_{\mu}^{\dagger}(t) = \text{Tr} \sum_{\mu} K_{\mu}^{\dagger}(t) K_{\mu}(t) \rho^S(0) = \text{Tr} \rho^S(0)$$

注: 1. 系统与环境的初态是直积态.

2. 适用于系统初态是混态

3. 设 $\mathcal{H}^M$ 的另一组基向量是 $\{|\alpha\rangle\}$ 得另一组 Kraus 算符

$$J_{\alpha} = \langle \alpha | U | \psi \rangle$$

两组基向量之间关系是某酉矩阵 V . $V_{\alpha\mu} = \langle \alpha | \mu \rangle$

$$|\mu\rangle = \sum_{\alpha} V_{\alpha\mu} |\alpha\rangle$$

$$\text{代入得 } K_{\mu} = \sum_{\alpha} V_{\alpha\mu}^* \langle \alpha | U | \psi \rangle = \sum_{\alpha} V_{\alpha\mu}^* J_{\alpha}$$

$$\rho^S(t) = \sum_{\mu} K_{\mu} \rho^S(0) K_{\mu}^{\dagger} = \sum_{\mu, \alpha, \alpha'} V_{\alpha\mu}^* V_{\alpha'\mu} J_{\alpha} \rho^S(0) J_{\alpha'}^{\dagger}$$

$$= \sum_{\alpha, \alpha'} \left(\sum_{\mu} V_{\alpha\mu}^* V_{\alpha'\mu} \right) J_{\alpha} \rho^S(0) J_{\alpha'}^{\dagger}$$

$$= \sum_{\alpha, \alpha'} \delta_{\alpha\alpha'} J_{\alpha} \rho^S(0) J_{\alpha'}^{\dagger} = \sum_{\alpha} J_{\alpha} \rho^S(0) J_{\alpha}^{\dagger}$$

4. 酉演化. 此时所有 K_{μ} 等于某个作用于系统的酉变换 $U^S(t)$.

$$\rho^S(t) = U^S(t) \rho^S(0) U^{S\dagger}(t)$$

投影测量. Kraus 算符为投影算符. $\Pi_i = |\alpha_i\rangle \langle \alpha_i|$

$$\rho^S(t) = \sum_i \Pi_i \rho^S(0) \Pi_i = \sum_{\mu} \langle \alpha_i | \rho^S(0) | \alpha_i \rangle |\alpha_i\rangle \langle \alpha_i|$$

等距变换

$$|\psi\rangle \otimes |\phi\rangle \longrightarrow U(|\psi\rangle \otimes |\phi\rangle).$$

对量子态 $|\psi\rangle \longrightarrow U(|\psi\rangle \otimes |\phi\rangle).$

$$\mathcal{H}^a \longrightarrow \mathcal{H}^a \otimes \mathcal{H}^M.$$

记为 V 变换

$$|\psi\rangle \xrightarrow{V} |\Omega_i\rangle \equiv U(|\psi\rangle \otimes |\phi\rangle) \in \mathcal{H}.$$

变换 V 保证变换前后量子态正交归一性

$$\langle i | j \rangle = \delta_{ij} \xrightarrow{V} \langle \Omega_i | \Omega_j \rangle = \delta_{ij}.$$

矩阵形式 $V = \sum_i |\Omega_i\rangle \langle i|$. $d^{\Omega M} \times d^{\Omega}$ 矩阵.

$$V^\dagger V = \mathbb{1}^{\Omega}. \quad V \text{ 是等距变换.}$$

系统量子态演化

$$\psi \rightarrow \psi = V \psi V^\dagger. \quad \mathcal{H}^M \text{ 基向量 } |\mu\rangle.$$

$$\rho = \sum_{\mu} \langle \mu | V \psi V^\dagger | \mu \rangle.$$

Kraus 算符 $K_{\mu} = \langle \mu | V = \sum_i \langle \mu | \Omega_i \rangle |i\rangle.$

$$\sum_{\mu} K_{\mu} K_{\mu}^\dagger = \mathbb{1}^{\Omega}.$$

广义量子测量

初态设为纯态 $\psi = |\psi\rangle\langle\psi|$ 保迹演化

$$\psi \rightarrow \rho = \sum_{\mu} K_{\mu} \psi K_{\mu}^{\dagger} \quad \sum_{\mu} K_{\mu} K_{\mu}^{\dagger} = \mathbb{1}.$$

视为操作过程

操作算符和效果算符

仪器 $M = \sum_{\mu} m_{\mu} |\mu\rangle\langle\mu|$

得到 $|\mu\rangle$ 时, 用投影算符 $\Pi_{\mu}^M = |\mu\rangle\langle\mu|$ 作用于仪器.

整体上, 用 $\mathbb{1}^{\otimes} \otimes \Pi_{\mu}^M$ 作用于 $\psi(t)$, 有

$$\begin{aligned} \psi(t) &\longrightarrow (\mathbb{1}^{\otimes} \otimes \Pi_{\mu}^M) \psi(t) (\mathbb{1}^{\otimes} \otimes \Pi_{\mu}^M) \\ &= \sum_{\nu\nu'} (U_{\mu\nu} \psi U_{\mu\nu}^{\dagger}) \otimes \langle\nu|\phi|\nu'\rangle |\mu\rangle\langle\mu| \\ &= K_{\mu} \psi K_{\mu}^{\dagger} \otimes |\mu\rangle\langle\mu|. \end{aligned}$$

仪器上得到结果 m_{μ} 的几率是

$$p_{\mu} = \text{Tr} \left[(\mathbb{1}^{\otimes} \otimes \Pi_{\mu}^M) \psi(t) (\mathbb{1}^{\otimes} \otimes \Pi_{\mu}^M) \right] = \text{Tr} (K_{\mu} \psi K_{\mu}^{\dagger}) = \text{Tr} (\underbrace{K_{\mu}^{\dagger} K_{\mu}}_{\text{效果算符}} \psi).$$

令 $e_{\mu} = \frac{1}{p_{\mu}} K_{\mu} \psi K_{\mu}^{\dagger}$.

系综的平均量子态 $\rho = \sum_{\mu} p_{\mu} e_{\mu} = \sum_{\mu} K_{\mu} \psi K_{\mu}^{\dagger}$ 操作算符

广义测量

用一组效果算符 $\{E_\mu\}$ 描述. E_μ 半正定. $E_\mu \geq 0$. $\sum_\mu E_\mu = 1$.

对应测量结果记 m_μ . 几率 $\text{Tr}(E_\mu \rho)$.

$\{E_\mu\}$ 构成了单位算符 I 的正值算符分解. POVM.

开放系统演化与退相干

量子纯态变换. 确定纯态演化或映射的操作 — 算符

混态

— 超算符

量子 Liouville eqn

$$\rho(t) = U(t) \rho(0) U^\dagger(t)$$

若 H 不显含 t .

$$\rho(t) = U(t) \rho(0) U^\dagger(t)$$

$$i\hbar \frac{d\rho(t)}{dt} = [H(t), \rho(t)]$$

$$U(t) = e^{-iHt}$$

转入 interaction picture

$$H = H_0 + H_I \quad \rho_I(t) = U_0(t) \rho(t) U_0^\dagger(t)$$

求得

$$i\hbar \frac{d\rho_I(t)}{dt} = [H_I^{(I)}(t), \rho_I(t)]$$

$$H_I^{(I)}(t) = U_0(t) H_I U_0^\dagger(t)$$

期望值的演化方程

$$\text{对算符 } \Omega, \quad \overline{\Omega(t)} = \text{Tr}[\rho(t) \Omega] \quad \frac{\partial \overline{\Omega(t)}}{\partial t}$$

$$i\hbar \frac{\partial}{\partial t} \overline{\Omega(t)} = i\hbar \text{Tr} \left[\rho(t) \frac{\partial \Omega(t)}{\partial t} \right] + i\hbar \text{Tr} \left[\frac{\partial \rho(t)}{\partial t} \Omega(t) \right]$$

$$\text{有 } i\hbar \frac{\partial}{\partial t} \overline{\Omega(t)} = i\hbar \frac{\partial \overline{\Omega(t)}}{\partial t} + \text{Tr} \{ [H(t), \rho(t)] \Omega(t) \}$$

密度矩阵的演化 — 超算符的映射

$$\text{初态 } \rho_A(0) \otimes |\sigma_E\rangle\langle\sigma_E|$$

$$\text{演化 } \rho_{AE}(t) = U_{AE}(t) \{ \rho_A(0) \otimes |\sigma_E\rangle\langle\sigma_E| \} U_{AE}^\dagger(t)$$

$$\rho_A(t) = \text{Tr} \{ U_{AE}(t) [\rho_A(0) \otimes |\sigma_E\rangle\langle\sigma_E|] U_{AE}^\dagger(t) \}$$

$$= \sum_{\mu} \langle \mu |_E \left[U_{AE}(t) \rho_A(0) \otimes |\sigma_E\rangle\langle\sigma_E| U_{AE}^\dagger(t) \right] | \mu \rangle_E$$

令 $\underbrace{\langle \mu |_E U_{AE}(t) | 0 \rangle_E}_{\text{环境的基础}} = M_\mu(t)$ $\underbrace{| 0 \rangle_E}_{\text{初始环境}}$

表示环境 E 由基态向 $|\mu\rangle_E$ 的跳变
对系统 A 的影响。

可将 A 任一初始混态 $\rho_A(0)$ 演化过程表示为

$$\rho_A(t) = \underbrace{\mathcal{E}(\rho_A(0))}_{\text{超算符的 Kraus 表示}} = \sum_\mu M_\mu(t) \rho_A(0) M_\mu^\dagger(t)$$

$$\begin{aligned} \sum_\mu M_\mu^\dagger(t) M_\mu(t) &= \sum_\mu \langle 0 |_E U_{AE}^\dagger(t) |\mu\rangle_E \langle \mu |_E U_{AE}(t) | 0 \rangle_E \\ &= \langle 0 |_E U_{AE}^\dagger(t) U_{AE}(t) | 0 \rangle_E = 1 \end{aligned}$$

超算符中是子系统密度矩阵的时间映射算符，提供了演化的普遍描述。

特例: $\rho_A \rightarrow \rho_A' = \sum_i E_i \rho_A E_i$ $\sum_i E_i = 1$ Von Neumann 正交投影测量

$\rho_A \rightarrow \rho_A'' = \sum_\mu \sqrt{F_\mu} \rho_A \sqrt{F_\mu}$ $\sum_\mu F_\mu = 1$ POVM

超算符的性质

i) 所有超算符只构成动力学半群

超算符必定保持内积，即必为等距算符。

超算符若可逆必为么正，反之亦然。

$\{M_\mu\}$ 中只有一个线性独立的
并且有逆存在。

$M_\mu M_\nu^\dagger = C_{\mu\nu}$ 是常数，且 $\text{Tr} C = 1$ 。

ii) Kraus 定理。

设 $\mathcal{E}$ 为子系统中对密度矩阵的某个映射，如果它满足以下 4 个条件

① 线性的 ② Hermitian ③ 保迹的 ④ 完全正的。

就必定可将它表示为算符求和形式。

亲属态方法

设伴随空间 H_B 的维数不小于所研究空间的维数 $\dim H_B \geq \dim H_A = N$.

给定 H_A 中任一算符 M_A . 现打算通过描述算符 $M_A \otimes I_B$ 如何对 $H_A \otimes H_B$ 中的一个最大纠缠态作用, 来完全表征 M_A 在 H_A 中的作用.

考虑最大纠缠态

$$|\psi\rangle_{AB} = \sum_{i=1}^N |i\rangle_A |i\rangle_B.$$

H_A 中的任一纯态

$$|\varphi\rangle_A = \sum_i \alpha_i |i\rangle_A.$$

映射到 H_B 中相应纯态

$$\underline{|\varphi\rangle_A} = \langle \varphi^* |_B |\psi\rangle_{AB}, \quad \underline{|\varphi^*\rangle_B} = \sum_i \alpha_i^* |i\rangle_B.$$

relative state

index state

$$|\varphi\rangle_A \longleftrightarrow |\varphi^*\rangle_B \quad \text{反线性的么正映射}$$

一般可能只涉及 H_B 某个子空间 ($n \leq N$).

将算符 $M_A \otimes I_B$ 作用在态 $|\psi\rangle_{AB}$ 上给出

$$(M_A \otimes I_B) |\psi\rangle_{AB} = \sum_i M_A |i\rangle_A \otimes |i\rangle_B.$$

M_A 对亲属态作用的表示:

$$M_A |\varphi\rangle_A = \langle \varphi^* |_B (M_A \otimes I_B) |\psi\rangle_{AB}.$$

H_B 中测量结果为 $|\varphi^*\rangle_B$ 时, 便在 H_A 中制备了态 $|\varphi\rangle_A$.

先制备态再施加算符 先施加算符再制备态

将算符态方法应用于超算符

$\$A$ 完全正. $\$A \otimes I_B$ 为正.

将 $(\$A \otimes I_B)$ 作用于 $\rho_{AB} = \frac{1}{N} |\psi\rangle_{AB} \langle \psi|_{AB}$. 得 ρ'_{AB} . (可展开为一个纯态系综)

$$(\$A \otimes I_B) \left(\frac{1}{N} |\psi\rangle_{AB} \langle \psi|_{AB} \right) = \sum_{\mu} p_{\mu} |\Phi_{\mu}\rangle_{AB} \langle \Phi_{\mu}|_{AB} \quad p_{\mu} > 0, \quad \sum_{\mu} p_{\mu} = 1, \quad \langle \Phi_{\mu} | \Phi_{\mu} \rangle = 1.$$

$$\begin{aligned} \$A(|\psi\rangle_A \langle \psi|_A) &= (\$A \otimes I_B) \left[\langle \psi^* |_B \psi \rangle_{AB} \langle \psi|_{AB} \psi^* \rangle_B \right] \\ &= \langle \psi^* |_B (\$A \otimes I_B) (|\psi\rangle_{AB} \langle \psi|_{AB}) |\psi^* \rangle_B \\ &= N \sum_{\mu} p_{\mu} \langle \psi^* | \Phi_{\mu} \rangle_{AB} \langle \Phi_{\mu} |_{AB} \psi^* \rangle_B \\ &\equiv \sum_{\mu} M_{\mu} |\psi\rangle_A \langle \psi|_A M_{\mu}^{\dagger} \quad \forall |\psi\rangle_A \in \mathcal{H}_A. \end{aligned}$$

算符系列 M_{μ} :

$$|\psi\rangle_A \rightarrow \sqrt{p_{\mu} N_B} \langle \psi^* | \Phi_{\mu} \rangle_{AB} \equiv M_{\mu} |\psi\rangle_A. \quad M_{\mu} \text{ 关于 } |\psi\rangle_A \text{ 线性}$$

此时对任意混态有

$$\$A(\rho_A) = \sum_{\mu} M_{\mu} \rho_A M_{\mu}^{\dagger}, \quad \forall \rho_A \in \mathcal{H}_A. \quad \text{因任意 } \rho_A \text{ 可展开为一个纯态系综.}$$

而 $\$A$ 为线性的. 于是

$$\begin{aligned} \rho_A = \sum_i p_i |\psi_i\rangle_A \langle \psi_i|_A &\Rightarrow \$A(\rho_A) = \sum_i p_i \$A(|\psi_i\rangle_A \langle \psi_i|_A) \\ &= \sum_i p_i M_{\mu} |\psi_i\rangle_A \langle \psi_i|_A M_{\mu}^{\dagger} = \sum_{\mu} M_{\mu} \rho_A M_{\mu}^{\dagger}. \end{aligned}$$

$$\sum_{\mu} M_{\mu}^{\dagger} M_{\mu} = 1. \quad \text{因 } \$A \text{ 对任何 } \rho_A \text{ 是保迹的.}$$

总结. 因 $\$A$ 是完全正的. $\$A \otimes I_B$ 将 $\mathcal{H}_A \otimes \mathcal{H}_B$ 上一个最大纠缠态的 ρ 变为另一个 ρ' . 它能表达成一个纯态系综. 每个态都关联着求和中的每一项.

系綜表示 $\longleftrightarrow$ 算符求和表示

$$\hat{\phi}_A(|i\rangle_A \langle j|) = \sum_{\mu} M_{\mu} |i\rangle_A \langle j| M_{\mu}^{\dagger}$$

$$\begin{aligned} (\hat{\phi}_A \otimes I_B) \left(\frac{1}{N} |\Psi_{AB}\rangle \langle \Psi_{AB}| \right) &= (\hat{\phi}_A \otimes I_B) \frac{1}{N} \sum_{i,j} |i\rangle_A |i'\rangle_B \langle j|_A \langle j| \\ &= \sum_{i,j,\mu} M_{\mu} |i\rangle_A |i'\rangle_B \langle j|_A \langle j| M_{\mu}^{\dagger} \\ &= \sum_{\mu} \sum_{i,j} (M_{\mu} |i\rangle_A |i'\rangle_B) (\langle j|_B \langle j|_A M_{\mu}^{\dagger}) \\ &= \sum_{\mu} \varrho_{\mu} |\Phi_{\mu}\rangle_{AB} \langle \Phi_{\mu}|_{AB} \end{aligned}$$

此处 $\sum_i M_{\mu} |i\rangle_A |i'\rangle_B \equiv \sqrt{\varrho_{\mu}} |\Phi_{\mu}\rangle_{AB}$.

主方程推导

Pauli-Zwanzig 方法, Louisell 方法, 超算符方法, *

孤立系 A 么正演化下的密度矩阵演化

$$\frac{d\rho_A(t)}{dt} = e^{-iH_A t/\hbar} \rho_A(0) e^{iH_A t/\hbar} \quad \rho_A(0) \longleftrightarrow \frac{d\rho_A(t)}{dt}$$

假定 $\frac{d\rho_A(t)}{dt} = \text{Tr}_E \left[U_{EA}(t) \rho_A(0) \otimes \rho_E(0) U_{EA}^\dagger(t) \right] \equiv \underline{L}(t) [\rho_A(0)]$

考虑 $0 \rightarrow dt$ 演化

$$\begin{cases} \rho_A(dt) \equiv \phi(\rho_A(0)) = \sum_{\mu} M_{\mu}(dt) \rho_A(0) M_{\mu}^\dagger(dt) \\ M_{\mu}(dt) = \langle \mu |_E U_{AE}(dt) | 0 \rangle_E \end{cases}$$

$$\text{令 } \begin{cases} U_{AE}(dt) = I_{AE} - \frac{i}{\hbar} H dt \\ M_0(dt) = \langle 0 |_E U_{AE}(dt) | 0 \rangle_E = I_A + \left(-\frac{i}{\hbar} H_A + K_A \right) dt \\ M_{\mu}(dt) = \langle \mu |_E U_{AE}(dt) | 0 \rangle_E = -i L_{A,\mu} \sqrt{dt} \end{cases}$$

代入得

$$\begin{aligned} \rho_A(dt) &= M_0(dt) \rho_A(0) M_0^\dagger(dt) + \sum_{\mu \neq 0} M_{\mu}(dt) \rho_A(0) M_{\mu}^\dagger(dt) \\ &= \left[1 + \left(-\frac{i}{\hbar} H_A + K_A \right) dt \right] \rho_A(0) \left[1 + \left(\frac{i}{\hbar} H_A + K_A \right) dt \right] \\ &\quad + \sum_{\mu} (-i L_{A,\mu} \sqrt{dt}) \rho_A(0) (i L_{A,\mu}^\dagger \sqrt{dt}) \end{aligned}$$

$$\cong \varrho_A(0) + \left(-\frac{i}{\hbar} H_A + K_A\right) \varrho_A(0) dt + \varrho_A(0) \left(\frac{i}{\hbar} H_A + K_A\right) dt \\ + \sum_{\mu>0} L_{A,\mu} \varrho_A(0) L_{A,\mu}^\dagger dt.$$

$$\left. \frac{d\varrho_A(t)}{dt} \right|_{t=0} = \left(-\frac{i}{\hbar} H_A + K_A\right) \varrho_A(0) + \varrho_A(0) \left(\frac{i}{\hbar} H_A + K_A\right) + \sum_{\mu>0} L_{A,\mu} \varrho_A(0) L_{A,\mu}^\dagger.$$

时间原点改为 t 得

$$\frac{d\varrho_A(t)}{dt} = -\frac{i}{\hbar} [H_A, \varrho_A(t)] + K_A \varrho_A(t) + \varrho_A(t) K_A + \sum_{\mu>0} L_{A,\mu} \varrho_A(t) L_{A,\mu}^\dagger. \quad (*)$$

注意到

$$I_A = \sum_{\mu=0} M_\mu^\dagger(dt) M_\mu(dt) \\ = \left[I_A + \left(\frac{i}{\hbar} H_A + K_A\right) dt \right] \left[I_A + \left(-\frac{i}{\hbar} H_A + K_A\right) dt \right] + \sum_{\mu>0} L_{A,\mu}^\dagger L_{A,\mu} dt \\ = I_A + 2K_A dt + \sum_{\mu>0} L_{A,\mu}^\dagger L_{A,\mu} dt$$

$$\text{有 } K_A = -\frac{1}{2} \sum_{\mu>0} L_{A,\mu}^\dagger L_{A,\mu}$$

代回(*)得

$$\frac{d\varrho(t)}{dt} = -\frac{i}{\hbar} [H, \varrho(t)] + \sum_{\mu>0} \left\{ L_\mu \varrho(t) L_\mu^\dagger - \frac{1}{2} L_\mu^\dagger L_\mu \varrho(t) - \frac{1}{2} \varrho(t) L_\mu^\dagger L_\mu \right\}.$$

另推导

$$\hat{H} = \hat{H}_S \otimes \hat{I}_B + \overset{\text{small}}{\hat{I}_S} \otimes \hat{H}_B + \lambda \hat{H}_{SB}$$

$$\frac{d\rho(t)}{dt} = \frac{1}{i\hbar} [\hat{H}, \rho(t)]$$

⇒ Interaction Picture

$$\hat{H}_0 = \hat{H}_S \otimes \hat{I}_B + \hat{I}_S \otimes \hat{H}_B \quad U = e^{-i\hat{H}_0 t/\hbar}$$

$$\tilde{\rho}(t) = U^\dagger(t) \rho(t) U(t)$$

$$\tilde{H}_I(t) = U^\dagger(t) \hat{H}_{SB} U(t)$$

$$\frac{d\tilde{\rho}(t)}{dt} = \frac{1}{i\hbar} [\tilde{H}_I(t), \tilde{\rho}(t)] + U(t) \frac{d\rho}{dt} U^\dagger(t)$$

$$\frac{d\tilde{\rho}(t)}{dt} = \frac{1}{i\hbar} [\tilde{H}_I(t), \tilde{\rho}(t)]$$

形式解为

$$\begin{aligned} \tilde{\rho}(t) &= \tilde{\rho}(0) + \frac{\lambda}{i\hbar} \int_0^t ds [\tilde{H}_I(s), \tilde{\rho}(s)] \\ &\quad - \frac{\lambda^2}{\hbar^2} \int_0^t ds_1 \int_0^{s_1} ds_2 [\tilde{H}_I(s_1), [\tilde{H}_I(s_2), \tilde{\rho}(s_2)]] + O(\lambda^3) \end{aligned}$$

$$\text{假设 } \tilde{\rho}(0) = \tilde{\rho}_S(0) \otimes \tilde{\rho}_B$$

$$\text{得 } \dot{\tilde{\rho}}_S(t) = \frac{\lambda}{i\hbar} \text{Tr}_B [\tilde{H}_I(t), \tilde{\rho}(0)] - \frac{\lambda^2}{\hbar^2} \int_0^t ds \text{Tr}_B [\tilde{H}_I(t), [\tilde{H}_I(s), \tilde{\rho}(s)]]$$

$\tilde{H}_I$ 可写为直积形式

$$\tilde{H}_I = \sum_{\alpha} \hat{S}_{\alpha} \otimes \hat{B}_{\alpha}$$

环境初始处于热平衡态, $\tilde{\rho}_B$ 只含对角项.

总可通过重新定义 $\hat{H}_s$ 与 $\hat{H}_b$ 使所有 $\hat{B}_\alpha$ 不含对角项.

使 $\text{Tr}[\hat{B}_\alpha \hat{B}_\alpha] = 0$.

主方程的求解

Evolution of a single qubit

$$\dot{\rho} = -\frac{i}{\hbar} [H, \rho] + \frac{1}{2} \sum_{i,j=1}^3 \gamma_{ij} ([\sigma_i, \rho \sigma_j] + [\sigma_i \rho, \sigma_j])$$

Solution $\dot{\vec{P}}(t) = \mathbf{O}(t) \cdot \vec{P}(t) + \mathbf{C}$

Consider $H = \frac{\hbar}{2} (\omega \cdot \sigma_3 + \lambda_1 \sigma_1 + \lambda_2 \sigma_2)$ $\gamma = \begin{bmatrix} \lambda & & \\ & \lambda & \\ & & \lambda \end{bmatrix}$

① Hamiltonian limit. ignore γ_{ij}

② ignore Hamiltonian.

① Bloch sphere polarization vector

$$\vec{P} = \langle \vec{\sigma} \rangle = \text{Tr}(\rho \vec{\sigma}) = (\rho_{12} + \rho_{21}, i(\rho_{12} - \rho_{21}), \rho_{11} - \rho_{22})$$

$$\dot{\vec{P}} = \text{Tr}(\dot{\rho} \vec{\sigma}) = -\frac{i}{\hbar} \text{Tr}([H, \rho] \vec{\sigma})$$

$$[\sigma_3, \rho] = \begin{pmatrix} 2\rho_{12} & \\ & -2\rho_{21} \end{pmatrix}$$

$$[H, \rho] = H\rho - \rho H$$

$$[\sigma_1, \rho] = \begin{pmatrix} \rho_{21} - \rho_{12} & \rho_{22} - \rho_{11} \\ \rho_{11} - \rho_{22} & \rho_{12} - \rho_{21} \end{pmatrix}$$

$$= \frac{\hbar}{2} \begin{pmatrix} \lambda_1(\rho_{21} - \rho_{12}) & 2\omega \cdot \rho_{12} + \lambda_1(\rho_{22} - \rho_{11}) \\ -i\lambda_2(\rho_{12} + \rho_{21}) & +i\lambda_2(\rho_{11} - \rho_{22}) \\ -2\omega \cdot \rho_{21} + \lambda_1(\rho_{11} - \rho_{22}) & \lambda_1(\rho_{11} - \rho_{22}) \\ +i\lambda_2(\rho_{11} - \rho_{22}) & +i\lambda_2(\rho_{12} + \rho_{21}) \end{pmatrix}$$

$$[\sigma_2, \rho] = \begin{pmatrix} -i\rho_{11} - i\rho_{22} & i\rho_{11} - i\rho_{22} \\ i\rho_{11} - i\rho_{22} & i\rho_{12} + i\rho_{21} \end{pmatrix}$$

$$= \frac{\hbar}{2} \begin{pmatrix} i\lambda_1 p_2 - i\lambda_2 p_1 & 2\omega_0 p_2 - \lambda_1 p_3 + i\lambda_2 p_3 \\ -2\omega_0 p_1 + \lambda_1 p_3 + i\lambda_2 p_3 & -i\lambda_1 p_2 + i\lambda_2 p_1 \end{pmatrix} \quad \begin{aligned} p_2 &= (p_1 - ip_2)/2 \\ p_1 &= (p_1 + ip_2)/2 \end{aligned}$$

$$= \frac{\hbar}{2} \begin{pmatrix} i(\lambda_1 p_y - \lambda_2 p_x) & \omega_0 (p_x - ip_y) + (i\lambda_2 - \lambda_1) p_z \\ -\omega_0 (p_x + ip_y) + (\lambda_1 + i\lambda_2) p_z & -i(\lambda_1 p_y - \lambda_2 p_x) \end{pmatrix}$$

$$\text{Then } \text{Tr}([\hat{H}, \rho] \sigma_x) = -\hbar i \omega_0 p_y + i\hbar \lambda_2 p_z$$

$$\text{Tr}([\hat{H}, \rho] \sigma_y) = i\hbar \omega_0 p_x - i\hbar \lambda_1 p_z$$

$$\text{Tr}([\hat{H}, \rho] \sigma_z) = i\hbar (\lambda_1 p_y - \lambda_2 p_x)$$

$$\dot{\vec{P}}(t) = O(t) \vec{P}(t) + C. = -\frac{i}{\hbar} \text{Tr}([\hat{H}, \rho] \sigma)$$

$$O = \begin{pmatrix} -\omega_0 & \lambda_2 \\ \omega_0 & -\lambda_1 \\ \lambda_1 & -\lambda_2 \end{pmatrix} \rightarrow \text{eigenvalues} \quad \begin{aligned} 0 \\ i\sqrt{\lambda_1^2 + \lambda_2^2 + \omega_0^2} \\ -i\sqrt{\lambda_1^2 + \lambda_2^2 + \omega_0^2} \end{aligned}$$

equation of motion of $\vec{P}$ can be written as

$$\dot{\vec{P}} = \vec{\omega} \times \vec{P}. \quad \vec{\omega} = (\lambda_1, \lambda_2, \omega_0)$$

evolution is unitary. polarization rotates without changing magnitude.

$$p_i = a_{1i} + a_{2i} e^{i\sqrt{\lambda_1^2 + \lambda_2^2 + \omega_0^2} t} + a_{3i} e^{-i\sqrt{\lambda_1^2 + \lambda_2^2 + \omega_0^2} t}$$

$$\begin{aligned}
\textcircled{2} \quad \dot{\rho} &= \frac{1}{2} \sum_{i,j=1}^3 \gamma_{ij} \left([\sigma_i, \rho \sigma_j] + [\sigma_i \rho, \sigma_j] \right) \\
&= \frac{1}{2} \left\{ \gamma_{11} \left([\sigma_1, \rho \sigma_1] + [\sigma_1 \rho, \sigma_1] \right) + \gamma_{22} \left([\sigma_2, \rho \sigma_2] + [\sigma_2 \rho, \sigma_2] \right) \right. \\
&\quad \left. + \gamma_{33} \left([\sigma_3, \rho \sigma_3] + [\sigma_3 \rho, \sigma_3] \right) \right\} \\
&= \frac{1}{2} \left[\gamma_{11} (2\sigma_1 \rho \sigma_1 - 2\rho) + \gamma_{22} (2\sigma_2 \rho \sigma_2 - 2\rho) + \gamma_{33} (2\sigma_3 \rho \sigma_3 - 2\rho) \right] \\
&= \lambda (\sigma_1 \rho \sigma_1 + \sigma_2 \rho \sigma_2 + \sigma_3 \rho \sigma_3) - 3\lambda \rho.
\end{aligned}$$

$$\dot{\rho} = 2\lambda \begin{pmatrix} \rho_{22} - \rho_{11} & -2\rho_{12} \\ -2\rho_{21} & \rho_{11} - \rho_{22} \end{pmatrix} = 2\lambda \begin{pmatrix} -p_3 & i p_2 - p_1 \\ -p_1 - i p_2 & p_3 \end{pmatrix}$$

$$\dot{\vec{p}} = \text{Tr}(\dot{\rho} \vec{\sigma}) = -4\lambda p_1 \hat{x} - 4\lambda p_2 \hat{y} - 4\lambda p_3 \hat{z} = -4\lambda \vec{p}.$$

$$\text{So} \quad \vec{p} = \vec{p}_0 e^{-4\lambda t}.$$

When damping dominates, polarization vector length $\rightarrow 0$ as it evolves.

Quantum trajectory of a state

$$\frac{d\rho(t)}{dt} = -\frac{i}{\hbar} [H, \rho(t)] + \sum_{\mu>0} \left\{ L_{\mu} \rho(t) L_{\mu}^{\dagger} - \frac{1}{2} L_{\mu}^{\dagger} L_{\mu} \rho(t) - \frac{1}{2} \rho(t) L_{\mu}^{\dagger} L_{\mu} \right\}$$

$$\dot{\rho} = -\frac{i}{\hbar} [H, \rho] - \frac{\gamma}{2} (\sigma_+ \sigma_- \rho + \rho \sigma_+ \sigma_-) + \gamma \sigma_- \rho \sigma_+$$

$$H = -\frac{1}{2} \hbar \omega_0 \sigma_z$$

$$L_1 = \sqrt{\gamma} \sigma_-$$

Suppose at $t=t_0$, system is in pure state $|\phi(t_0)\rangle = \alpha|0\rangle + \beta|1\rangle$.

Probability of quantum jump:

$$\begin{aligned} dp &= \langle \phi(t_0) | L_1^{\dagger} L_1 | \phi(t_0) \rangle dt \\ &= (\alpha^* \langle 0| + \beta^* \langle 1|) \gamma \sigma_+ \sigma_- (\alpha |0\rangle + \beta |1\rangle) dt = \gamma |\beta|^2 dt \end{aligned}$$

When there is no quantum jump

$$\begin{aligned} |\phi_0(t+dt)\rangle &= [I + (-iH + K) dt] |\phi_0(t_0)\rangle \\ &= \left[I + i\frac{\omega_0 dt}{2} \sigma_z - \frac{\gamma}{2} dt \sigma_+ \sigma_- \right] (\alpha |0\rangle + \beta |1\rangle) \\ &= \alpha \left(1 - i\frac{\omega_0 dt}{2} \right) |0\rangle + \beta \left(1 + i\frac{\omega_0 dt}{2} - \frac{\gamma dt}{2} \right) |1\rangle \end{aligned}$$

$$\dot{\alpha} = -\frac{i\omega_0}{2} \alpha \quad \dot{\beta} = \frac{i\omega_0 - \gamma}{2} \beta$$

$$\alpha(t_0+t) = \alpha(t_0) e^{-\frac{i\omega_0}{2}t}$$

$$\beta(t_0+t) = \beta(t_0) e^{i(\frac{\omega_0}{2} - \frac{\gamma}{2})t}$$

open system evolves with quantum channels

$$\rho(t) \xrightarrow{\mathcal{E}(dt)} \rho(t+dt)$$

$$\mathcal{E}(dt) = 1 + \mathcal{L} dt$$

$$\rho(t+dt) = \sum_a M_a \rho(t) M_a^\dagger = \rho(t) + \dots dt$$

$$M_0 = 1 + \dots dt, \quad M_a \sim \sqrt{dt}, \quad \text{for } a > 0.$$

parameterize

$$M_0 = 1 + \frac{1}{\hbar}(-iH + K) dt, \quad M_a = L_a \sqrt{dt}$$

$$\sum_a M_a^\dagger M_a = 1 \rightarrow \frac{2}{\hbar} K dt + \sum_a L_a^\dagger L_a dt + O(dt^2) = 0$$

$$K = -\frac{\hbar}{2} \sum_a L_a^\dagger L_a$$

$$\dot{\rho} = \frac{1}{dt} [\rho(t+dt) - \rho(t)]$$

$$\dot{\rho} = -\frac{i}{\hbar} [H, \rho] + \sum_a \left(L_a \rho L_a^\dagger - \frac{1}{2} L_a^\dagger L_a \rho - \frac{1}{2} \rho L_a^\dagger L_a \right)$$

Another form of master eqn

$$\dot{\rho} = -\frac{i}{\hbar} [H, \rho] + \frac{1}{2} \sum_{i,j=1}^{N-1} a_{ij} \left\{ [\sigma_i, \rho \sigma_j^\dagger] + [\sigma_j^\dagger \rho, \sigma_i] \right\}$$

$\sigma_i, i=1, \dots, N-1, \sigma_0 = \frac{1}{\sqrt{N}}$. Basis for $N \times N$ Hamiltonian

$$\text{Tr } \sigma_i = 0 \text{ for } i \neq 0. \quad \text{Tr}(\sigma_i, \sigma_j) = \delta_{ij}.$$

We can diagonalize a_{ij} with $S a S^\dagger = \tilde{a}$, eigenvalues λ_i .

$$\vec{\sigma} = \begin{pmatrix} \sigma_1 \\ \vdots \\ \sigma_{N-1} \end{pmatrix} \quad \vec{\omega} = S \vec{\sigma}, \quad \vec{\omega} = \begin{pmatrix} \omega_1 \\ \vdots \\ \omega_{N-1} \end{pmatrix}.$$

We require a positive $\rightarrow \lambda_i > 0$.

$$\text{define } \vec{L} = \begin{pmatrix} L_1 \\ \vdots \\ L_{N-1} \end{pmatrix} = \begin{pmatrix} \sqrt{\lambda_1} \omega_1 \\ \vdots \\ \sqrt{\lambda_{N-1}} \omega_{N-1} \end{pmatrix}.$$

Heisenberg picture

ρ fixed. Operator evolves.

$$\dot{A} = i[H, A] + \sum_{a>0} \left(L_a^\dagger A L_a - \frac{1}{2} L_a^\dagger L_a A - \frac{1}{2} A L_a^\dagger L_a \right).$$

局域测量 — POVM

1) 直和子空间解释

设 H_A 为 $H = H_A \oplus H_A^\perp$ 的一部分.

$$\text{基 } \{|i\rangle\} \quad \{|\mu\rangle\} \quad \langle i|\mu\rangle = 0.$$

H 有正交基 $\{|u_\alpha\rangle\}$. 设 M_A 为 H_A 中的一个可观测量. 有

$$M_A |\psi^\perp\rangle = \langle \psi^\perp | M_A = 0.$$

$$|u_\alpha\rangle = \underbrace{|\tilde{\psi}_\alpha\rangle}_{\in H_A} + |\tilde{\psi}_\alpha^\perp\rangle.$$

从子空间 H_A 中看不一定正交归一.

由 $\langle u_\alpha | u_\alpha \rangle = 1$. 记 $\lambda_\alpha = 1 - \langle \tilde{\psi}_\alpha^\perp | \tilde{\psi}_\alpha^\perp \rangle$.

$$\text{令 } |\tilde{\psi}_\alpha\rangle = \sqrt{\lambda_\alpha} |\psi_\alpha\rangle, \quad \text{已归一.}$$

$$E_\alpha = |u_\alpha\rangle\langle u_\alpha|$$

设在 H 中对 H_A 中的一个态 ρ_A 执行向 $\{|u_\alpha\rangle\}$ 的正交投影测量.

$$H_A \text{ 中看来, } \text{Prob}(\alpha) = \langle u_\alpha | \rho_A | u_\alpha \rangle = \langle \tilde{\psi}_\alpha | \rho_A | \tilde{\psi}_\alpha \rangle.$$

$$= \lambda_\alpha \langle \psi_\alpha | \rho_A | \psi_\alpha \rangle.$$

定义 H_A 中的一组算符

$$F_\alpha \equiv E_A E_\alpha E_A = |\tilde{\psi}_\alpha\rangle\langle \tilde{\psi}_\alpha| = \lambda_\alpha |\psi_\alpha\rangle\langle \psi_\alpha|.$$

$$\text{Prob}(\alpha) = \langle \tilde{\psi}_\alpha | \rho_A | \tilde{\psi}_\alpha \rangle = \lambda_\alpha \langle \psi_\alpha | \rho_A | \psi_\alpha \rangle \equiv \text{Tr}(F_\alpha \rho_A).$$

F_α : 厄米, 非负, 迹不一定为1, 不一定正交. 而

$$\sum_\alpha F_\alpha = E_A \sum_\alpha E_\alpha E_A = E_A = I_A.$$

定义 系统 A 的一种 POVM 是一组能分解 I_A 的非负厄米算符系列

$$\{F_\alpha, \alpha=1,2,\dots,n; F_\alpha^\dagger = F_\alpha; {}_A\langle\psi|\psi\rangle_A \geq 0, \sum_\alpha F_\alpha = I_A\}$$

不一定彼此正交的正值算符之和对于空间单位算符的分解方案。

对投影算符 $P^2 = P, \sqrt{P} = P, |\psi\rangle\langle\psi|$ 是投影算符。

广义测量前后态的改变是

$$\rho_A \longrightarrow \rho'_A = \sum_\alpha [\lambda_\alpha \langle\psi_\alpha|\rho_A|\psi_\alpha\rangle] \cdot |\psi_\alpha\rangle\langle\psi_\alpha| = \sum_\alpha \sqrt{F_\alpha} \rho_A \sqrt{F_\alpha}$$

对比: $\rho_A \longrightarrow \rho'_A = \sum_\alpha E_\alpha \rho_A E_\alpha$ 正交投影

$$\dim(H_A) \leq \text{number of } F_\alpha \leq \text{number of } E_\alpha = \dim(H_A) + \dim(H_B)$$

ii) 直积子空间解释。

N 维系统 A 处于态 ρ_A 。另有辅助系统 B 处于态 ρ_B 。

$\rho_{AB} = \rho_A \otimes \rho_B$ 进行正交投影测量 $\{E_\mu\}$; $\sum_\mu E_\mu = I_{AB}$ 。

$$\text{Prob}(\mu) = \text{Tr}_{AB}(E_\mu \cdot \rho_A \otimes \rho_B) = \sum_{m,n} \sum_{r,s} (E_\mu)_{mr,ns} (\rho_A)_{mn} (\rho_B)_{sr}$$

$$= \sum_{m,n} (F_\mu)_{mn} (\rho_A)_{mn} = \text{Tr}_A(F_\mu \rho_A)$$

有 $\text{Prob}(\mu) = \text{Tr}_{AB}(E_\mu \cdot \rho_A \otimes \rho_B) = \text{Tr}_A(F_\mu \rho_A)$ 。

$$(F_\mu)_{mn} = \sum_{r,s} (E_\mu)_{mr,ns} (\rho_B)_{sr} = \left(\text{Tr}_B [E_\mu (I_A \otimes \rho_B)] \right)_{mn}$$

$$\begin{aligned}
 D[\hat{J}]\hat{e} &= \hat{J}\hat{e}\hat{J}^\dagger - \frac{1}{2}\{\hat{J}^\dagger\hat{J}, \hat{e}\} \\
 &= (\hat{a}-\alpha)\hat{e}(\hat{a}^\dagger-\alpha^*) - \frac{1}{2}\{(\hat{a}^\dagger-\alpha^*)(\hat{a}-\alpha), \hat{e}\} \\
 &=
 \end{aligned}$$

$$\begin{aligned}
 \underbrace{a e a^\dagger} &= -\frac{1}{2}\alpha^* a e - \frac{1}{2}\alpha e a^\dagger \\
 &+ \frac{1}{2}\alpha^* e a + \frac{1}{2}\alpha a^\dagger e \\
 &\underbrace{-\frac{1}{2}a^\dagger a e - \frac{1}{2}e a^\dagger a}
 \end{aligned}$$

$$(M_A \otimes I_B) |\psi\rangle_{AB} = \sum_i M_A |i\rangle_A \otimes |i'\rangle_B$$

$$\begin{aligned}
 \langle \varphi^* \rangle_B (M_A \otimes I_B) |\psi\rangle_{AB} &= \sum_i M_A |i\rangle_A \otimes \langle \varphi^* |_B |i'\rangle_B \\
 &= M_A \sum_i \langle \varphi^* |_B |i\rangle_A |i'\rangle_B \\
 &\quad \hookrightarrow \sum_i \alpha_i \langle i' |_B \\
 &= M_A \sum_i \alpha_i |i\rangle_A = M_A |\varphi\rangle_A
 \end{aligned}$$

$$(E_A \otimes I_B) \langle \varphi_i^* | \sum_{j,j'} |j\rangle_A \langle j'|_A \underbrace{|j\rangle_B \langle j'|_B}_{\text{wavy}} \varphi_i^* \rangle$$

$$E_A \sum_{j,j'} |j\rangle_A \langle j'|_A \langle \varphi_i^* | j \rangle_B \langle j'|_B \varphi_i^* \rangle$$

$$\langle j'|_B \sum \alpha_n |n\rangle \quad \alpha_i \cdot \alpha_i \cdot |i\rangle \langle i|$$

量子力学重学

有限维复空间一定是完备的内积空间, 因此, 在有限维情形下, 可以不提完备性. 但是, 在无限维情形下, 只有完备的内积空间才是 Hilbert 空间.

谱分解定理?

定理2.1 (谱分解定理) 当且仅当一个算符是正规的, 它才是可对角化的且具有正交归一的本征基矢.

- 动量值假设: 当粒子处于由动量算子的一个本征态所描述的状态时, 对该粒子进行动量测量, 会给出相应本征值之动量。

这一类假设, 有时称之为谱分解原理, 可以直接推广到其他可观测量。

$$\langle \psi | A | \psi \rangle = \text{Tr}(A |\psi\rangle\langle\psi|)$$

一直用得 unnatural, 如何理解.

信息与能量的关系.



一个例子太少. 更多的例子?

这样，与前面讨论过的位置情况类似，一般而言，我们不能预言粒子有确切的动量，而只能预言测量到不同动量值的几率。这一点多少令人困惑与疑虑，尤其是考虑到动量是一个守恒量。在我们的直觉中，守恒量多少意味着系统的某种固有性质。事实上，在所有已完成的实验中，动量与能量守恒都是（在实验精度内）精确成立的。这一点与上述动量的不确定性如何协调，是量子力学中最为微妙的问题之一，在此我们仅仅提及，以后会再回到这一问题。

一些技巧

$$V(x) \text{ 中一维运动粒子 } \hat{H}_0 = \frac{\hat{p}^2}{2\mu} + V(x), \quad E_n^{(0)},$$

$$\text{求 } \hat{H} = \hat{H}_0 + \frac{\lambda}{\mu} \hat{p} \text{ 本征值.}$$

$$\textcircled{1} \text{ 由 F-H 定理, } \frac{\partial E_n}{\partial \lambda} = \langle n | \frac{\partial \hat{H}}{\partial \lambda} | n \rangle = \frac{1}{\mu} \langle n | \hat{p} | n \rangle.$$

$$\begin{aligned} \text{在 } |n\rangle \text{ 上, } \langle n | [\hat{x}, \hat{H}] | n \rangle &= \langle n | \hat{x} \hat{H} - \hat{H} \hat{x} | n \rangle \\ &= (E_n - E_n) \langle n | \hat{x} | n \rangle = 0. \end{aligned}$$

$$\text{又 } [\hat{x}, \hat{H}] = \left[\hat{x}, \frac{\hat{p}^2}{2\mu} + V(x) + \frac{\lambda}{\mu} \hat{p} \right] = \frac{i\hbar}{\mu} (\hat{p} + \lambda).$$

$$\langle n | \hat{p} | n \rangle = -\lambda \Rightarrow \frac{\partial E_n}{\partial \lambda} = -\frac{\lambda}{\mu}.$$

$$\Rightarrow E_n = -\frac{\lambda^2}{\mu} + C.$$

$$\lambda = 0, \quad E_n = C = E_n^{(0)}, \quad \Rightarrow E_n = E_n^{(0)} - \frac{\lambda^2}{\mu}.$$

② p 表象的 $\hat{H}_0$ 定态方程

$$\left[\frac{\hat{p}^2}{2\mu} + V(\hat{x} = i\hbar \frac{\partial}{\partial p}) \right] \psi_n^{(0)}(p) = E_n^{(0)} \psi_n^{(0)}(p).$$

$$\hat{H} = \hat{H}_0 + \frac{\lambda}{\mu} \hat{p} = \frac{p^2}{2\mu} + \frac{\lambda}{\mu} p + V(\hat{x} = i\hbar \frac{\partial}{\partial p})$$

$$= \frac{(p+\lambda)^2}{2\mu} + V(\hat{x} = i\hbar \frac{\partial}{\partial p}) - \frac{\lambda^2}{2\mu}$$

$$p' = p + \lambda, \quad p \rightarrow p'$$

$$\hat{H} = \frac{p'^2}{2\mu} + V(\hat{x} = i\hbar \frac{\partial}{\partial p'}) - \frac{\lambda^2}{2\mu}.$$

$$\left[\frac{p'^2}{2\mu} + V(\hat{x} = i\hbar \frac{\partial}{\partial p'}) - \frac{\lambda^2}{2\mu} \right] \varphi(p') = E \varphi(p')$$

$$\left[\frac{p'^2}{2\mu} + V(\hat{x} = i\hbar \frac{\partial}{\partial p'}) \right] \varphi(p') = \left(E + \frac{\lambda^2}{2\mu} \right) \varphi(p').$$

$$E = E_n^{(0)} - \frac{\lambda^2}{2\mu}.$$